International Journal of Pure and Applied Mathematics
Volume 79 No. 2 2012, 309-328
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED DERIVATIVE OPERATOR

N.M. Mustafa[1], M. Darus[2 §]

[1,2]School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600, Bangi, Selangor, MALAYSIA

Abstract: Differential subordination and superordination results are obtained for analytic functions in the open unit disk which are associated with the generalized derivative operator $D_p^{\alpha,\delta}(\mu, q, \lambda)f(z)$. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained. Some of the results established in this paper would provide extensions to those given in the earlier works.

AMS Subject Classification: 30C45
Key Words: analytic function, differential subordination, superordination

1. Definition and Preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}\ and\ |z| < 1\}$ and let $\mathcal{H}[a, n]$ denote the subclass of $\mathcal{H}(\mathbb{U})$ of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, with $\mathcal{H}_0 = \mathcal{H}[0, 1]$ and $\mathcal{H} = \mathcal{H}[1, 1]$. If f, g are members of $\mathcal{H}(\mathbb{U})$ we say that a function f is subordinate to a

Received: June 5, 2012

§Correspondence author

function g or g is said to be superordinate to f if there exists a function ω with $\omega(0)=0, \quad |\omega(z)|<1$ for all $(z\in\mathbb{U})$, such that $f(z)=g(w(z))$. In such a case we write $f\prec g$. Further, if the function g is univalent in $\mathbb{U}$ then we have the following equivalent$f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$.

Let $A(p)$ denote the class of all analytic functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p}, \qquad (z\in\mathbb{U}, \quad p\in\mathbb{N}=1,2,...,). \tag{1}$$

Now, using the operator $D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)$, where $\lambda,q\geq 0,\ \mu,\delta,\alpha\in\mathbb{N}_0=\{0,1,2...\}$, introduced by authors (see [1]) for a function $f\in A(p)$ given by (1) as follows:

$$D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)=z^p+\sum_{k=1}^{\infty}(\frac{k+p}{p})^{\alpha}(\frac{p+q+\lambda k}{p+q})^{\mu}c(\delta,k)a_{k+p}z^{k+p}. \tag{2}$$

It is easily verified from (2) that

$$(p+q)D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)=\lambda z[D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)]'+(p+q-\lambda p)D_p^{\alpha,\delta}(\mu,q,\lambda)f(z). \tag{3}$$

To prove our results, we need the following definitions and Lemmas.

Denote by Q the set of functions q that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$, where

$$E(q)=\{\xi\in\partial\mathbb{U}:\lim_{z\to\xi}q(z)=\infty\},$$

and are such that $q'(\xi)\neq 0 \quad \xi\in\partial\mathbb{U}\setminus E(q)$. Further let the subclass of Q for which $q(0)=a$ be denoted by $Q(a), Q(0)\equiv Q_0$ and $Q(1)\equiv Q_1$.

Definition 1.1. (see [8]) Let Ω be a set in $\mathbb{C}$; $q\in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$ consists of those functions $\psi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ that satisfy the admissibility condition:

$$\psi(r,s,t;z)\notin\Omega,$$

whenever $r=q(\xi)$, $s=k\xi q'(\xi)$,

$$\Re\left\{\frac{t}{s}+1\right\}\geq k\Re\left\{1+\frac{\xi q'(\xi)}{q(\xi)}\right\},$$

where $z\in\mathbb{U}$, $\xi\in\partial\mathbb{U}\setminus E(q)$ and $k\geq n$. We write $\Psi_1[\Omega,q]=\Psi[\Omega,q]$.

In particular, when $q(z)=M\frac{Mz+a}{M+\bar{a}z}$, with $M>0$ and $|a|<M$, then $q(\mathbb{U})=\mathbb{U}_M=\{w:|w|<M\}$, $q(0)=a,E(q)=\phi$ and $q\in Q$. In this case, we set $\Psi_n[\Omega,M,a]=\Psi[\Omega,q]=\Psi_1[\Omega,q]=\Psi[\Omega,q]$, and in the special case when the set $Q=\mathbb{U}_M$, the class is simply denoted by $\Psi_n[M,a]$.

Definition 1.2. (see [9]) Let Ω be a set in $\mathbb{C}$, $q(z) \in H[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; \xi) \in \Omega,$$

whenever $r = q(z)$, $s = \frac{zq'(z)}{m}$,

$$\Re\left\{\frac{t}{s} + 1\right\} \geq \frac{1}{m}\Re\left\{1 + \frac{\xi q'(\xi)}{q(\xi)}\right\},$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U}$ and $m \geq k \geq 1$. In particular, we write $\Psi'_1[\Omega, q] = \Psi'[\Omega, q]$.

Lemma 1.3. (see [8]) *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$.If the analytic function $g(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$, satisfies*

$$\psi\left(g(z), zg'(z), z^2 g''(z); z\right) \in \Omega,$$

then

$$g(z) \prec q(z).$$

Lemma 1.4. (see [9]) *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$, $g \in Q(a)$ and*

$$\psi\left(g(z), zg'(z), z^2 g''(z), z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\psi\left(g(z), zg'(z), z^2 g''(z), z\right), \quad (z \in \mathbb{U})\right\}.$$

In the present investigation, the differential subordination result of Miller and Mocanu [8] is extended for functions associated with generalized derivative operator. A similar problem for analytic functions was studied by many authors for example see [2]-[7]. Additionally, the corresponding differential superordination problem is investigated, and several sandwich-type results are obtained.

2. Subordination Results Associated with Generalized Operator

Definition 2.1. Let Ω be a set in $\mathbb{C}$, $q \in Q_0 \cap \mathcal{H}[0, p]$. The class of admissible functions $\Phi_Q[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u = q(\xi), \quad v = \frac{k\xi q' + \frac{(p+q-\lambda p)}{\lambda} g(z)}{\frac{p+q}{\lambda}},$$

$$\Re\left\{ \frac{(\frac{p+q}{\lambda})^2 w - (\frac{p+q-\lambda p}{\lambda})^2 u}{\frac{(p+q)}{\lambda} v - \frac{(p+q-\lambda p)}{\lambda} u} - 2(\frac{p+q-\lambda p}{\lambda}) \right\} \geq k\Re\left\{ 1 + \frac{\xi q'(\xi)}{q(\xi)} \right\},$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$, $\lambda > 0, q \geq 0$, and $k \geq p$.

Theorem 2.2. *Let $\phi \in \Psi_Q[\Omega, q]$. If $f \in A(p)$ satisfies*

$$\phi\left(D_p^{\alpha,\delta}(\mu, q, \lambda) f(z), D_p^{\alpha,\delta}(\mu+1, q, \lambda) f(z), D_p^{\alpha,\delta}(\mu+2, q, \lambda) f(z) \right) \subset \Omega,$$

where $\lambda > 0, q \geq 0$, $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$D_p^{\alpha,\delta}(\mu, q, \lambda) f(z) \prec q(z), \quad (z \in \mathbb{U}).$$

Proof. Define the analytic function g in $\mathbb{U}$ by

$$D_p^{\alpha,\delta}(\mu, q, \lambda) f(z) = g(z). \quad (\lambda > 0, \; q \geq 0 \; \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}). \tag{4}$$

In view of the relation (3) from (4), we get

$$D_p^{\alpha,\delta}(\mu+1, q, \lambda) f(z) = \frac{zg'(z) + \frac{(p+q-\lambda p)}{\lambda} g(z)}{\frac{p+q}{\lambda}}. \tag{5}$$

Further computations show that

$$D_p^{\alpha,\delta}(\mu+2, q, \lambda) f(z) = \frac{z^2 g''(z) + (1 + 2\frac{(p+q-\lambda p)}{\lambda}) z g'(z) + (\frac{p+q-\lambda p}{\lambda})^2 g(z)}{(\frac{p+q}{\lambda})^2}. \tag{6}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s + \frac{(p+q-\lambda p)}{\lambda} r}{\frac{p+q}{\lambda}}, \quad w = \frac{t + (1 + 2\frac{(p+q-\lambda p)}{\lambda}) s + (\frac{p+q-\lambda p}{\lambda})^2 r}{(\frac{p+q}{\lambda})^2}. \tag{7}$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) =$$

$$\phi(r, \frac{s + \frac{(p+q-\lambda p)}{\lambda} r}{\frac{p+q}{\lambda}}, \frac{t + (1 + 2\frac{(p+q-\lambda p)}{\lambda}) s + (\frac{p+q-\lambda p}{\lambda})^2 r}{(\frac{p+q}{\lambda})^2}; z). \tag{8}$$

The proof shall make use of Lemma 1.3.Using equations (4), (5) and (6), from (8), we obtain

$$\psi\left(g(z), zg'(z), z^2g''(z); z\right)$$
$$= \phi\left(D_p^{\alpha,\delta}(\mu, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z); z\right), \quad (9)$$

where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}$.

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Psi_Q[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{(\frac{p+q}{\lambda})^2 w - (\frac{p+q-\lambda p}{\lambda})^2 u}{\frac{(p+q)}{\lambda} v - \frac{(p+q-\lambda p)}{\lambda} u} - 2(\frac{p+a-\lambda p}{\lambda}),$$

and hence $\psi \in \Psi_p[\Omega, q]$, By Lemma 1.3,

$$g(z) \prec q(z),$$

or

$$D_p^{\alpha,\delta}(\mu, q, \lambda)f(z) \prec q(z)(z \in \mathbb{U}).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω In this case the class $\Psi_Q[h(\mathbb{U}), q]$ is written as $\Psi_Q[h, q]$. The following result is an immediate consequence of Theorem 2.2.

Theorem 2.3. *Let $\phi \in \Psi_Q[h, q]$. If $f \in A(p)$ satisfies*

$$\phi\left(D_p^{\alpha,\delta}(\mu, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z)\right) \prec h(z),$$

where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$D_p^{\alpha,\delta}(\mu, q, \lambda)f(z) \prec q(z), \quad (z \in \mathbb{U}).$$

Our next result is an extension of Theorem 2.2 to the case where the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 2.1. *Let $\Omega \in \mathbb{C}$ and let q be univalent in $\mathbb{U}$, $q(0) = 0$. Let $\Psi_Q[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$.If $f(z) \in A(p)$ and*

$$\phi\left(D_p^{\alpha,\delta}(\mu, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z)\right) \in \Omega,$$

where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$D_p^{\alpha,\delta}(\mu, q, \lambda)f(z) \prec q(z), \quad (z \in \mathbb{U}).$$

Proof. Theorem 2.2 yields $D_p^{\alpha,\delta}(\mu,q,\lambda)f(z) \prec q(\rho z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

In the particular case $q(z) = Mz, M > 0$, and in view of the definition 1.2, the class of admissible functions $\Psi_Q[\Omega, q]$ denoted by $\Psi_Q[\Omega, M]$. is described below.

Definition 2.4. Let Ω be a set in $\mathbb{C}$, $M \geq 0$. The class of admissible functions $\Phi_Q[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi\left(Me^{i\theta}, \frac{N + \frac{(p+q-\lambda p)}{\lambda}}{\frac{p+q}{\lambda}} Me^{i\theta}, \frac{L + ((1 + 2\frac{(p+q-\lambda p)}{\lambda})N + (\frac{p+q-\lambda p}{\lambda})^2)Me^{i\theta}}{(\frac{p+q}{\lambda})^2}; z\right) \notin \Omega,$$

where $\lambda > 0, q \geq 0$, $\theta \in \mathbb{R}$, $\Re(Le^{i\theta}) \geq N(N-1)M$ for all real θ, $N \geq 1$, $z \in \mathbb{U}$ $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}$.

Corollary 2.2. *Let $\phi \in \Psi_Q[\Omega, q]$. If $f \in A(p)$ satisfies*

$$\phi\left(D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z); z\right) \in \Omega,$$

where $\lambda > 0, q \geq 0$, $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$\left|D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)\right| < M, \quad (z \in \mathbb{U}).$$

Proof. Theorem 2.2 gives

$$D_p^{\alpha,\delta}(\mu,q,\lambda)f(z) \prec q(z) = Mz,$$

$$D_p^{\alpha,\delta}(\mu,q,\lambda)f(z) \prec q(z) = Mw(z).$$

Hence $|D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)| < M$ where $|w(z)| < 1$.

In the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$ the class $\Psi_Q[\Omega, M]$ is simply denoted by $\Psi_Q[M]$.

Corollary 2.3. *Let $\phi \in \Psi_Q[\Omega, q]$. If $f \in A(p)$ satisfies*

$$\left|\phi\left(D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z); z\right)\right| < M,$$

where $\lambda > 0, q \geq 0$, $\lambda, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$\left|D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)\right| < M, \quad (z \in \mathbb{U}).$$

Definition 2.5. Let Ω be a set in $\mathbb{C}$; $q \in Q_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{Q,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} : \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$r = q(\xi), \quad s = \frac{k\xi q' + \frac{(1+q-\lambda)}{\lambda} g(z)}{\frac{1+q}{\lambda}},$$

$$\Re\left\{ \frac{(\frac{1+q}{\lambda})^2 w - (\frac{1+q-\lambda}{\lambda})^2 u}{\frac{(1+q)}{\lambda} v - \frac{(1+q-\lambda)}{\lambda} u} - 2(\frac{1+q-\lambda}{\lambda}) \right\} \geq k\Re\left\{1 + \frac{\xi q'(\xi)}{q(\xi)}\right\},$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$, $\lambda > 0, q \geq 0$, and $k \geq 1$.

Theorem 2.6. *Let $\phi \in \Psi_{Q,1}[\Omega, q]$. If $f \in A(p)$ satisfies*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu, q, \lambda) f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1, q, \lambda) f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2, q, \lambda) f(z)}{z^{p-1}}; z \right) \subset \Omega, \quad (10)$$

where $\lambda > 0, q \geq 0$, $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$\frac{D_p^{\alpha,\delta}(\mu, q, \lambda) f(z)}{z^{p-1}} \prec q(z), \quad (z \in \mathbb{U}).$$

Proof. Define the analytic function g in $\mathbb{U}$ by

$$\frac{D_p^{\alpha,\delta}(\mu, q, \lambda) f(z)}{z^{p-1}} = g(z), \quad (11)$$

where $\lambda > 0, q \geq 0$, $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

In view of the relation (3) from (11), we get

$$\frac{D_p^{\alpha,\delta}(\mu+1, q, \lambda) f(z)}{z^{p-1}} = \frac{zg'(z) + \frac{(1+q-\lambda)}{\lambda} g(z)}{\frac{1+q}{\lambda}}. \quad (12)$$

Further computations show that

$$\frac{D_p^{\alpha,\delta}(\mu+2, q, \lambda) f(z)}{z^{p-1}} = \frac{z^2 g''(z) + (1 + 2\frac{(1+q-\lambda)}{\lambda}) zg'(z) + (\frac{1+q-\lambda}{\lambda})^2 g(z)}{(\frac{1+q}{\lambda})^2}. \quad (13)$$

Further, let us define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s + \frac{(1+q-\lambda)}{\lambda} r}{\frac{1+q}{\lambda}},$$

$$w = \frac{t + (1 + 2\frac{(1+q-\lambda)}{\lambda})s + (\frac{1+q-\lambda}{\lambda})^2 r}{(\frac{1+q}{\lambda})^2}.$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z)$$

$$= \phi(r, \frac{s + \frac{(1+q-\lambda)}{\lambda}r}{\frac{1+q}{\lambda}}, \frac{t + (1 + 2\frac{(1+q-\lambda p)}{\lambda})s + (\frac{1+q-\lambda}{\lambda})^2 r}{(\frac{1+q}{\lambda})^2}; z). \qquad (14)$$

The proof shall make use of Lemma 1.3. Using equations (11), (12) and (13), from (14), we obtain

$$\psi(g(z), zg'(z), z^2 g''(z); z) =$$

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu, q, \lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z)}{z^{p-1}}; z\right), \qquad (15)$$

where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Hence (10) becomes

$$\psi(g(z), zg'(z), z^2 g''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Psi_{Q,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{(\frac{1+q}{\lambda})^2 w - (\frac{1+q-\lambda p}{\lambda})^2 u}{\frac{(1+q)}{\lambda}v - \frac{(1+q-\lambda)}{\lambda}u} - 2(\frac{1 + a - \lambda p}{\lambda}),$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Lemma 1.3,

$$g(z) \prec q(z) \quad or \quad \frac{D_p^{\alpha,\delta}(\mu, q, \lambda)f(z)}{z^{p-1}} \prec q(z), (z \in \mathbb{U}).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω In this case the class $\Psi_{Q,1}[h(\mathbb{U}), q]$.

In the particular case $q(z) = Mz$,$M > 0$, and in view of the definition 1.2, the class of admissible functions $\Psi_{Q,1}[\Omega, q]$ denoted by $\Psi_{Q,1}[\Omega, M]$. is described below.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 2.6.

Theorem 2.7. *Let $\phi \in \Psi_{Q,1}[h,q]$. If $f \in A(p)$ satisfies*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{z^{p-1}}; z\right) \prec h(z), \quad (16)$$

where $\lambda > 0, q \geq 0,\ \mu,\delta,\alpha \in \mathbb{N}_0 = \{0,1,2...\}, z \in \mathbb{U}$.

Then

$$\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}} \prec q(z), \quad (z \in \mathbb{U}).$$

Definition 2.8. Let Ω be a set in $\mathbb{C}$; $M \geq 0$. The class of admissible functions $\Phi_{Q,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi\left(Me^{i\theta}, \frac{N + \frac{(1+q-\lambda)}{\lambda}}{\frac{1+q}{\lambda}} Me^{i\theta}, \frac{L + ((1 + 2\frac{(1+q-\lambda p)}{\lambda})N + (\frac{1+q-\lambda}{\lambda})^2)Me^{i\theta}}{(\frac{1+q}{\lambda})^2}; z\right) \notin \Omega,$$

where $\lambda > 0, q \geq 0,\ \theta \in \mathbb{R},\ \Re(Le^{i\theta}) \geq N(N-1)M,\ N \geq 1,\ z \in \mathbb{U}$, for all real θ.

Corollary 2.4. *Let $\phi \in \Psi_{Q,1}[\Omega, q]$. If $f \in A(p)$ satisfies*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{z^{p-1}}; z\right) \in \Omega,$$

where $\lambda > 0, q \geq 0,\ \mu,\delta,\alpha \in \mathbb{N}_0 = \{0,1,2...\}, z \in \mathbb{U}$.

Then

$$\left|\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}\right| < M, \quad (z \in \mathbb{U}).$$

In the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$ the class $\Psi_{Q,1}[\Omega, M]$ is simply denoted by $\Psi_{Q,1}[M]$.

Corollary 2.5. *Let $\phi \in \Psi_{Q,1}[\Omega, q]$. If $f \in A(p)$ satisfies*

$$\left|\phi\left(\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{z^{p-1}}; z\right)\right| < M,$$

where $\lambda > 0, q \geq 0,\ \mu,\delta,\alpha \in \mathbb{N}_0 = \{0,1,2...\}, z \in \mathbb{U}$.

Then

$$\left|\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}\right| < M, \quad (z \in \mathbb{U}).$$

Definition 2.9. Let Ω be a set in $\mathbb{C}$; $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{Q,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u = q(\xi), \quad v = g(\xi) + \frac{\lambda}{p+q}\frac{k\xi g(\xi)}{\xi},$$

$$\Re\left\{\frac{(\frac{p+q}{\lambda})^2 w - (\frac{p+q-\lambda p}{\lambda})^2 u}{\frac{(p+q)}{\lambda}v - \frac{(p+q-\lambda p)}{\lambda}u} - 2(\frac{p+q-\lambda p}{\lambda})\right\} \geq k\Re\left\{1 + \frac{\xi q'(\xi)}{q(\xi)}\right\},$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$, $\lambda > 0, q \geq 0$, and $k \geq 1$.

Theorem 2.10. *Let $\phi \in \Psi_{Q,2}[\Omega, q]$. If $f \in A(p)$ satisfies*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}; z\right) \subset \Omega, \quad (17)$$

where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)} \prec q(z), \quad (z \in \mathbb{U}).$$

Proof. Define an analytic function g in $\mathbb{U}$ by

$$g(z) = \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}. \quad (18)$$

Using (18), we get

$$\frac{zg'(z)}{g(z)} = \frac{z\left(D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)\right)'}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)} - \frac{z\left(D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)\right)'}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}. \quad (19)$$

By making use of (2) in (19), we get

$$\frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)} = g(z) + \left(\frac{\lambda}{p+q}\right)\frac{zg'(z)}{g(z)}. \quad (20)$$

Further computations show that

$$\frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)} = \frac{1}{\frac{zg'}{\frac{p+q}{\lambda}g(z)}+g(z)}\left[\frac{z^2g''(z)}{(\frac{p+q}{\lambda})^2g(z)}+\frac{zg'(z)}{(\frac{p+q}{\lambda})^2g(z)}+\frac{3zg'(z)}{(\frac{p+q}{\lambda})}+g(z)\right]. \quad (21)$$

Define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r, \quad v=r+\frac{s}{\frac{p+q}{\lambda}r},$$

$$w=\frac{1}{\frac{s}{\frac{p+q}{\lambda}r}+r}\left[\frac{t}{(\frac{p+q}{\lambda})^2r}+\frac{s}{(\frac{p+q}{\lambda})^2r}+\frac{3s}{(\frac{p+q}{\lambda})}+r\right].$$

Let

$$\psi(r,s,t;z)=\phi\left(u,v,w;z\right))=$$

$$\phi\left(r,r+\frac{s}{\frac{p+q}{\lambda}r},\frac{1}{\frac{s}{\frac{p+q}{\lambda}r}+r}\left[\frac{t}{(\frac{p+q}{\lambda})^2r}+\frac{s}{(\frac{p+q}{\lambda})^2r}+\frac{3s}{(\frac{p+q}{\lambda})}+r\right];z\right). \quad (22)$$

The proof shall make use of Lemma 1.3. Using equations (18), (20) and (21), from (22), we obtain

$$\psi(g(z),zg'(z),z^2g''(z);z)=$$

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)},\frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)},\frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)};z\right), \quad (23)$$

where $\lambda>0, q\geq 0,\ \mu,\delta,\alpha\in\mathbb{N}_0=\{0,1,2...\}, z\in\mathbb{U}$.

Hence (17) becomes

$$\psi(g(z),zg'(z),z^2g''(z);z)\in\Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Psi_{Q,2}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s}+1=\left(\frac{(1+q)v}{\lambda(u-v)}\right)w-\frac{(3v-2u)\frac{p+q}{\lambda}}{u-v},$$

and hence $\psi\in\Psi_Q[\Omega,q]$. By Lemma 1.3,

$$g(z)\prec q(z) \quad \textit{or} \quad \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}\prec q(z), \quad (z\in\mathbb{U}).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω In this case the class $\Psi_{Q,2}[h(\mathbb{U}), q]$.

In the particular case $q(z) = Mz, M > 0$, and in view of the definition 1.2, the class of admissible functions $\Psi_{Q,2}[\Omega, q]$ denoted by $\Psi_{Q,2}[\Omega, M]$. is described below.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 2.10.

Theorem 2.11. *Let $\phi \in \Psi_{Q,2}[h, q]$. If $f \in A(p)$ satisfies*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}; z\right) \prec h(z),$$

where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)} \prec q(z), \quad (z \in \mathbb{U}).$$

Definition 2.12. Let Ω be a set in $\mathbb{C}$, $M \geq 0$. The class of admissible functions $\Phi_{Q,2}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z)) =$$
$$\phi\left(Me^{i\theta}, Me^{i\theta} + \frac{N}{\frac{p+q}{\lambda}}, \frac{1}{M^2e^{i\theta} + \frac{(p+q)MN}{\lambda}}\left[M^3e^{2i\theta} + \frac{3M^2Ne^{i\theta}}{\frac{p+q}{\lambda}} + \frac{Le^{-i\theta} + MN}{(\frac{p+q}{\lambda})^2}\right]; z\right),$$

where $\lambda > 0, q \geq 0,\ \theta \in \mathbb{R},\ \Re(Le^{i\theta}) \geq N(N-1)M,\ N \geq 1$ for all real θ.

Corollary 2.6. *Let $\phi \in \Psi_{Q,2}[\Omega, q]$. If $f \in A(p)$ satisfies*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}; z\right) \in \Omega,$$

where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$.

Then

$$\left|\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}\right| < M, \quad (z \in \mathbb{U}).$$

3. Superordination Results Associated with Generalized Operator

Definition 3.1. Let Ω be a set in $\mathbb{C}$; $q \in Q_0 \cap \mathcal{H}[0,p]$,$zq'(z) \neq 0$ The class of admissible functions $\Phi'_Q[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \xi) \notin \Omega,$$

whenever

$$u = q(\xi), \quad v = \frac{zq' + m\frac{(p+q-\lambda p)}{\lambda}g(z)}{m(\frac{p+q}{\lambda})},$$

$$\Re\left\{\frac{(\frac{p+q}{\lambda})^2 w - (\frac{p+q-\lambda p}{\lambda})^2 u}{\frac{(p+q)}{\lambda}v - \frac{(p+q-\lambda p)}{\lambda}u} - 2(\frac{p+q-\lambda p}{\lambda})\right\} \geq \frac{1}{m}\Re\left\{1 + \frac{zq'(z)}{q(z)}\right\},$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$, $\lambda > 0, q \geq 0$, and $m \geq p$.

Theorem 3.2. *Let $\phi \in \Psi'_Q[\Omega, q]$. If $f \in A(p)$, $D_p^{\alpha,\delta}(\mu, q, \lambda)f(z) \in \mathcal{H}_0$ and*

$$\phi\left(D_p^{\alpha,\delta}(\mu, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z)\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \phi\left(D_p^{\alpha,\delta}(\mu, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z)\right), \quad (24)$$

where $\lambda > 0, q \geq 0$, $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$, implies

$$q(z) \prec D_p^{\alpha,\delta}(\mu, q, \lambda)f(z), \quad (z \in \mathbb{U}).$$

Proof. From (9) and (24), we have

$$\Omega \subset \phi\left(D_p^{\alpha,\delta}(\mu, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z), D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z)\right), \quad (z \in \mathbb{U})$$

From (7), we see that the admissibility condition for $\phi \in \Psi'_Q[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence and by Lemma 1.4,we get $q(z) \prec g(z)$.

$$q(z) \prec D_p^{\alpha,\delta}(\mu, q, \lambda)f(z), \quad (z \in \mathbb{U}).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω In this case the class $\Psi'_Q[h(\mathbb{U}), q]$ is written as $\Psi'_Q[h, q]$. Proceeding similarly as in the previous section , the following result is an immediate consequence of Theorem 3.2.

Theorem 3.3. *Let $h(z)$ is analytic on $\mathbb{U}$ and $\phi \in \Psi'_Q[h,q]$. If $f \in A(p)$ $D_p^{\alpha,\delta}(\mu,q,\lambda)f(z) \in \mathcal{H}_0$ and*

$$\phi\left(D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)\right),$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)\right), \quad (25)$$

where $\lambda > 0, q \geq 0,\ \mu,\delta,\alpha \in \mathbb{N}_0 = \{0,1,2...\}, z \in \mathbb{U}$, implies

$$q(z) \prec D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), \quad (z \in \mathbb{U}).$$

Proof. From (25) we get

$$h(z) = \Omega \subset \phi\left(D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)\right),$$

and so by Theorem 3.2, we get

$$q(z) \prec D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), \quad (z \in \mathbb{U}).$$

Theorems 3.2 and 3.3, can only be used to obtain subordinations of differential superordination of the form (24) or (25). The following theorem proves the existence of the best subordinant of (25) for certain ϕ.

Combining Theorems 2.3 and 3.3, we obtain the following Sandwich-type theorem.

Corollary 3.1. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, $h_2(z)$ be univalent function in $\mathbb{U}$,$q_2(z) \in \mathcal{H}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Psi_Q[h_2,q_2] \cap \Psi_Q[h_2,q_2]$. If $f \in A(p)$, $D_p^{\alpha,\delta}(\mu,q,\lambda)f(z) \in \mathcal{H}[0,p] \cap \mathcal{H}_0$ and*

$$\phi\left(D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)\right),$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec \phi\left(D_p^{\alpha,\delta}(\mu,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z), D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)\right) \prec h_2(z), \quad (26)$$

where $\lambda > 0, q \geq 0,\ \mu,\delta,\alpha \in \mathbb{N}_0 = \{0,1,2...\}, z \in \mathbb{U}$, implies

$$q_1(z) \prec D_p^{\alpha,\delta}(\mu,q,\lambda)f(z) \prec q_1(z), \quad (z \in \mathbb{U}).$$

Definition 3.4. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}_0$, $zq'(z) \neq 0$ The class of admissible functions $\Phi'_{Q,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \xi) \notin \Omega,$$

whenever

$$r = q(z), \quad s = \frac{zq' + \frac{m(1+q-\lambda)}{\lambda} g(z)}{m(\frac{1+q}{\lambda})},$$

$$\Re\left\{ \frac{(\frac{1+q}{\lambda})^2 w - (\frac{1+q-\lambda}{\lambda})^2 u}{\frac{(1+q)}{\lambda} v - \frac{(1+q-\lambda)}{\lambda} u} - 2(\frac{1+q-\lambda}{\lambda}) \right\} \geq \frac{1}{m} \Re\left\{ 1 + \frac{\xi q'(\xi)}{q(\xi)} \right\},$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$, $\lambda > 0, q \geq 0$, and $m \geq 1$.

The following result is associated with Theorem 2.6.

Theorem 3.5. *Let $\phi \in \Psi'_{Q,1}[\Omega, q]$. If $f \in A(p)$ $\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}} \in \mathcal{H}_0$, and*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu, q, \lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z)}{z^{p-1}} \right),$$

where $\lambda > 0, q \geq 0$, $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$ is univalent in $\mathbb{U}$, then

$$\Omega \subset \phi\left(\frac{D_p^{\alpha,\delta}(\mu, q, \lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1, q, \lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2, q, \lambda)f(z)}{z^{p-1}} \right), \quad (27)$$

implies

$$q(z) \prec \frac{D_p^{\alpha,\delta}(\mu, q, \lambda)f(z)}{z^{p-1}}, \quad (z \in \mathbb{U}).$$

Proof. From (14) and (27), we have

$$\Omega \subset \psi(g(z), zg'(z), z^2 g''(z); z).$$

From (15), we see that the admissibility condition for $\phi \in \Psi'_{Q,1}[\Omega, q]$ is equivalent to the admissibility condition for Ψ as given in Definition 3. Hence $\psi \in \Psi'_{Q,1}[\Omega, q]$ and by Lemma1.2, we get $q(z) \prec g(z)$

$$q(z) \prec \frac{D_p^{\alpha,\delta}(\mu, q, \lambda)f(z)}{z^{p-1}}, \quad (z \in \mathbb{U}).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω In this case the class $\Psi'_Q[h(\mathbb{U}), q]$ is written as $\Psi'_Q[h, q]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.5.

Theorem 3.6. *Let $q \in H_0$, and let h be analytic on $\mathbb{U}$, and and $\phi \in \Psi'_{Q,1}[\Omega, q]$. If $f \in A(p)$ $\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}} \in \mathcal{H}_0$, and*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{z^{p-1}}\right),$$

is univalent in $\mathbb{U}$, where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{z^{p-1}}\right),$$

implies

$$h(z) \prec \frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}, \quad (z \in \mathbb{U}).$$

Combining Theorem 16 and 3.6, we obtain the following Sandwich-type theorem.

Corollary 3.2. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, $h_2(z)$ be univalent function in $\mathbb{U}$,$q_2(z) \in \mathcal{H}_0$, with $q_1(0) = q_2(0) = 0$, and $\phi \in \Psi_Q[h_2, q_2] \cap \Psi_Q[h_2, q_2]$. If $f \in A(p)$, $\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}} \in \mathcal{H}[0,p] \cap \mathcal{H}_0$, and*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{z^{p-1}}\right),$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec \phi\left(\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{z^{p-1}}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{z^{p-1}}\right) \prec h_2(z), \quad (28)$$

where $\lambda > 0, q \geq 0,\ \mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$, implies

$$q_1(z) \prec \frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}} \prec q_2(z), \quad (z \in \mathbb{U}).$$

Definition 3.7. Let Ω be a set in $\mathbb{C}$, $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{Q,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \xi) \notin \Omega,$$

whenever

$$u = q(z), \quad v = g(z) + \frac{\lambda}{m(p+q)} \frac{zg(z)}{g(z)},$$

$$\Re\left\{ \frac{(\frac{p+q}{\lambda})^2 w - (\frac{p+q-\lambda p}{\lambda})^2 u}{\frac{(p+q)}{\lambda} v - \frac{(p+q-\lambda p)}{\lambda} u} - 2(\frac{p+a-\lambda p}{\lambda}) \right\} \geq \frac{1}{m} \Re\left\{ 1 + \frac{zq'(z)}{q(z)} \right\},$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$, $\lambda > 0, q \geq 0$, and $m \geq 1$.

Now we will give the dual result of Theorem 2.10 for the differential superordination.

Theorem 3.8. *Let* $\phi \in \Psi'_{1,2}[\Omega, q]$. *If* $f \in A(p)$, $\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)} \in \mathcal{H}_0$, *and*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)} \right),$$

where $\lambda > 0, q \geq 0$, $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$, *is univalent in* $\mathbb{U}$, *then*

$$\Omega \subset \phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)} \right), \quad (29)$$

implies

$$q(z) \prec \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \quad (z \in \mathbb{U}).$$

Proof. From (22) and (29), we have

$$\Omega \subset \psi(g(z), zg'(z), z^2g''(z); z).$$

From (23), we see that the admissibility condition for $\phi \in \Psi'_{Q,2}[\Omega, q]$ is equivalent to the admissibility condition for Ψ as given in Definition 1.2.

Hence $\psi \in \Psi'_{Q,2}[\Omega, q]$ and by Lemma 1.4, we get $q(z) \prec g(z)$

$$q(z) \prec \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \quad (z \in \mathbb{U}).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω In this case the class $\Psi'_{Q,2}[h(\mathbb{U}), q]$, is written as $\Psi'_{Q,2}[h, q]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.8.

Theorem 3.9. *Let $q \in \mathbb{H}$, and let h be analytic on $\mathbb{U}$, and and $\phi \in \Psi'_{Q,2}[\Omega, q]$. If $f \in A(p)$, $\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)} \in \mathcal{H}_1$, and*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}\right)$$

where $\lambda, \mu, q \geq 0$, $\delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$, is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}\right),$$

implies

$$h(z) \prec \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \quad (z \in \mathbb{U}).$$

Combining Theorem 2.11 and 3.9, we obtain the following Sandwich-type theorem.

Corollary 3.3. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, $h_2(z)$ be univalent function in $\mathbb{U}$,$q_2(z) \in \mathcal{H}_0$ with $q_1(0) = q_2(0) = 0$, and $\phi \in \Psi_{Q,2}[h_2, q_2] \cap \Psi_{Q,2}[h_2, q_2]$. If $f \in A(p)$, $\frac{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}{z^{p-1}} \in \mathcal{H}[0, p] \cap \mathcal{H}_0$, and*

$$\phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}\right),$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec \phi\left(\frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}, \frac{D_p^{\alpha,\delta}(\mu+3,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu+2,q,\lambda)f(z)}\right) \prec h_2(z), \quad (30)$$

where $\lambda > 0, q \geq 0$, $\mu, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, z \in \mathbb{U}$, implies

$$q_1(z) \prec \frac{D_p^{\alpha,\delta}(\mu+1,q,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,q,\lambda)f(z)} \prec q_2(z), \quad (z \in \mathbb{U}).$$

Remarks. Other works related to differential subordination or superordination can be found in [10]-[19].

Acknowledgements

The work here is supported by MOHE: UKM-ST-06-FRGS0244-2010 and LRGS-TD-2011-UKM-ICT-03-02.

References

[1] N.M. Mustafa, M. Darus, Differential subordination and superordination for new linear derivative operator, *Int. J. Pure and Appl. Math.,* **70**, No. 6 (2011), 825-835.

[2] R. Aghalary, R.M. Ali, S.B. Joshi, V. Ravichandran, Inequalities for analytic functions defined by certain linear operator, *Int. J. Math. Sci.,* **4**, No. 2 (2005), 267-274.

[3] R.M. Ali, V. Ravichandran, N. Seenivasagan, Dierential subordination and superordination of analytic functions de.ned by the multiplier transformation, *Math. Inequal. Appl.,* **12**, No. 1 (2009), 123-139.

[4] M.K. Aouf, Inequalities involving certain integral operator, *J. Math. Inequal.,* **2**, No. 2 (2008), 537-547.

[5] M.K. Aouf, H.M. Hossen, A.Y. Lashin, An application of certain integral operators, *J. Math. Anal. Appl.,* **248**, No. 2 (2000), 475-481.

[6] M.K. Aouf, T.M. Seoudy, Dierential subordination and superordination of analytic functions de.ned by an integral operator, *European J. Pure Appl. Math.,* **3**, No. 1 (2010), 26-44.

[7] Y.C. Kim, H.M. Srivastava, Inequalities involving certain families of integral and convolution operators, *Math. Inequal. Appl.,* **7**, No. 2 (2004), 227-234.

[8] S.S. Miller, P.T. Mocanu, *Differential subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, **225**, Marcel Dekker, NewYork and Basel (2000).

[9] S. Miller, P.T. Mocanu, Subordination of differential superordinations, *Complex variables Theory Appl.,* **48**, No. 10 (2003), 815-826.

[10] R.W. Ibrahim, M. Darus, Subordination for meromorphic functions defined by an operator, *Int. J. Pure and Appl. Math.,* **69**, No. 4 (2011), 413-419.

[11] M.H. Mohd, M. Darus, Differential subordination and superordination for Srivastava-Attiya operator, *Int. J. Differential Equations*, **2011** (2011), Article ID 902830, 19 pages.

[12] S. Sivasubramanian, A. Mohammed and M. Darus, Certain subordination properties for subclasses of analytic functions involving complex order, *Abstract and Applied Analysis*, **2011** (2011), Article ID 375897, 8 pages.

[13] A. Mohammed, M. Darus, D. Breaz, On subordination, starlikeness and convexity of certain integral operators, *Mathatica*, Tome 53 **76**, No. 2 (2011), 165-170.

[14] M. Darus, I. Faisal, Differential subordination for meromorphic multivalent quasi-convex functions, *Facta Universitatis (NIS)*, Ser. Math. Inform., **26** (2011), 1-7.

[15] R.W. Ibrahim, M. Darus, Differential subordination for meromorphic multivalent quasi-convex functions, *Acta Mathematica Universitatis Comenianae*, **79**, No. 1 (2010), 39-45.

[16] M.H. Al-Abbadi, M. Darus, Differential Subordination Defined by New Generalised Derivative Operator for Analytic Functions, *Int. J. Math. Math. Sci.*, **2010** (2010), Article ID 369078, 15 pages.

[17] M. Darus, K. Al-Shaqsi, On Subordinations for certain analytic functions associated with generalized integral operator, *Lobachevskii Journal of Mathematics*, **29**, No. 2, 90-97.

[18] R.W. Ibrahim, M. Darus, Subordination and superordination for univalent solutions for fractional differential equations, *J. Math. Anal. Appl.*, **345**, No. 2 (2008), 871-879.

[19] R.W. Ibrahim, M. Darus, On subordination theorems for new classes of normalize analytic functions, *Appl. Math. Sci.*, **2**, No. 56 (2008), 2785-2794.