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FINITE SETS AND POSTULATION: FURTHER REMARKS

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Abstract: A typical statement proved here. Fix integer $r \ge 4$ and $k \ge 2$. Let $S \subset \mathbb{P}^r$ be a finite set such that $d := \sharp(S) \le rk+1$ and $\sharp(S \cap V) \le r-1$ for each $(r-2)$-dimensional linear subspace $V \subset \mathbb{P}^r$. We have $h^1(\mathcal{I}_S(k)) > 0$ if and only if there is a hyperplane $H \subset \mathbb{P}^r$ such that $h^1(H, \mathcal{I}_{S\cap H,H}(k)) > 0$. If there is such a hyperplane H, then it is unique and H is the unique hyperplane containing the maximal number of points of S. If $d \le (r-1)k+1$, then $h^1(\mathcal{I}_S(k)) = 0$.

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1. Introduction

We first prove the following result.

Theorem 1. *Fix integer $r \ge 4$ and $k \ge 2$. Let $S \subset \mathbb{P}^r$ be a finite set such that $d := \sharp(S) \le rk+1$ and $\sharp(S \cap V) \le r-1$ for each $(r-2)$-dimensional linear subspace $V \subset \mathbb{P}^r$.*

(i) We have $h^1(\mathcal{I}_S(k)) > 0$ if and only if there is a hyperplane $H \subset \mathbb{P}^r$ such that $h^1(H, \mathcal{I}_{S\cap H,H}(k)) > 0$. If there is such a hyperplane H, then it is unique and H is the unique hyperplane containing the maximal number of points of S.

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(ii) If $d \le (r-1)k+1$, then $h^1(\mathcal{I}_S(k)) = 0$.

Then we prove several results in which we assume that S has a weak form of the uniform position property, e.g. it has the uniform position property with respect to forms of a prescribed degree.

In the last section we extend [1], Theorem 1, to the case $\sharp(S) = 4m-1$ and prove the following result (it extends just by one [1], Theorem 1, but it does not reach the case $\sharp(S) = 4m$ in which new examples occur; is the guess of [1] correct, i.e. are the only new examples with $\sharp(S) = 4m$ the complete intersection of two quadric surfaces and a degree m surface?).

Theorem 2. *Fix an integer $m \ge 4$. Let $S \subset \mathbb{P}^r$, $r \ge 3$, be a finite subset such that $\sharp(S) \le 4m + r - 4$, $\sharp(S \cap A) \le 4m - 5$ for every plane $A \subseteq \mathbb{P}^r$ and S spans $\mathbb{P}^r$. We have $h^1(\mathcal{I}_S(m)) > 0$ if and only if there is $W \subseteq S$ as in one of the following cases:*

(a) $\sharp(W) = m+2$ and W is contained in a line;

(b) $\sharp(W) = 2m+2$ and W is contained in a plane conic;

(c) $\sharp(W) = 3m$ and W is the complete intersection of a degree 3 plane curve and a degree m surface;

(d) $\sharp(W) \ge 3m+1$ and W is contained in a degree 3 plane curve;

(e) $\sharp(W) = 3m+2$ and W is contained in a reduced and connected degree 3 curve spanning $\mathbb{P}^3$.

We work over an algebraically closed base field $\mathbb{K}$

2. The proofs

Lemma 1. *Fix finite sets $A \subset B \subset \mathbb{P}^r$, a linear subspace $M \subset \mathbb{P}^r$ and an integer $t \ge 0$.*

(i) If $h^1(\mathcal{I}_A(t)) > 0$, then $h^1(\mathcal{I}_B(t)) > 0$.

(ii) If $h^1(M, \mathcal{I}_{B\cap M}(t)) > 0$, then $h^1(\mathcal{I}_B(t)) > 0$.

Proof. Part (i) follows from the surjectivity of the restriction map

$$H^0(B, \mathcal{O}_B(t)) \to H^0(A, \mathcal{O}_A(t)).$$

Since the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(M, \mathcal{O}_M(t))$ is surjective, we have $h^1(\mathcal{I}_{B\cap M}(t)) = h^1(M, \mathcal{I}_{B\cap M}(t))$. Hence part (ii) follows from part (a) applied to the set $A := B \cap V$. □

Proof of Theorem 1. The "if" part of (i) follows from part (ii) of Lemma 1. Assume $h^1(\mathcal{I}_S(k)) > 0$. Set $S_0 := S$. Let $H_1 \subset \mathbb{P}^r$ be any hyperplane such that $a_1 := \sharp(H_1 \cap S)$ is maximal. Set $S_1 := S_0 \setminus S_0 \cap H_1$. For each integer $i \geq 2$ define inductively the hyperplane H_i, the integer $a_i \geq 0$ and the set S_i in the following way. Let H_i be any hyperplane such that $a_i := S_{i-1} \cap H_i$ is maximal and set $S_i := S_{i-1} \setminus S_{i-1} \cap H_i$. We have $a_1 \leq r$ if and only if S is in linearly general position. If S is in linearly general position, then $h^1(\mathcal{I}_S(k)) = 0$ (see [3], Theorem 3.2). Hence from now on we assume $a_1 \geq r+1$. The sequence $\{a_i\}_{i \geq i}$ is non-decreasing. Since any r points of $\mathbb{P}^r$ are contained in a hyperplane, if $a_i \leq r-1$, then $a_{i+1} = 0$. Since $a_r \geq r+1$, we get $a_{k+1} = 0$. Notice that $d = a_1 + \cdots + a_k$. For each integer $i \geq 1$ we have an exact sequence

$$0 \to \mathcal{I}_{S_i}(k-i) \to \mathcal{I}_{S_{i-1}}(k+1-i) \to \mathcal{I}_{S_{i-1} \cap H_i, H_i}(k+1-i) \to 0 \tag{1}$$

Since $h^1(\mathcal{I}_S(k)) > 0$, from (1) for $i = 1, \dots, k$ we get the existence of an integer $x \in \{1, \dots, k\}$ such that $h^1(H_x, \mathcal{I}_{S_{t-1} \cap H_x}(k+1-x)) > 0$. Let e be the minimal such an integer x.

(a) Assume $e \geq 2$. Since $a_1 + \cdots + a_e \leq d$, we have $a_e \leq (rk+1)/e$. For any $t \in \mathbb{R}$ set $\psi(t) := t(k+1-t)$. The function $\psi : \mathbb{R} \to \mathbb{R}$ is increasing if $0 \leq t \leq (k+1)/2$ and decreasing if $(k+1)/2 \leq t \leq k+1$. Since $H_e \cap S$ is in linearly general position in H_e, [3], Theorem 3.2, gives $a_e \geq (r-1)(k+1-e)+2$. If $e = k$, we get $a_k \geq r+2$ and hence $d \geq k(r+2)$, a contradiction. If $(k+1)/2 \leq e \leq k-1$, then we get $a_e \geq 2(r-1)+2$ and hence $d \geq (k+1)(r-1)+k-1$, a contradiction. If $2 \leq e \leq (k+1)/2$, then we get $d \geq 2(r-1)(k-1)+2$, a contradiction.

(b) Now assume $e = 1$. To prove part (i) of Theorem 1 it is sufficient to prove $h^1(H, \mathcal{I}_{S \cap H}(k)) = 0$ for each hyperplane $H \neq H_1$. By the definition of e_2 we have $\sharp(H \cap S) \leq e_2$. We saw in step (a) that $e_2 \leq (rk+1)/2 \leq (r-1)k+1$. Hence $h^1(H, \mathcal{I}_{S \cap H}(k)) = 0$ by [3], Theorem 3.2.

(c) For each hyperplane $H \subset \mathbb{P}^r$ the set $S \cap H$ is in linearly general position. Hence part (ii) follows from the first assertion part (i) and [3], Theorem 3.2. □

Proposition 1. *Fix integers $r \geq 3$ and $k > t > 0$. Let $S \subset \mathbb{P}^r$ be a finite set such that $h^0(\mathcal{I}_{S'}(t)) = \max\{0, \binom{r+k}{k} - \sharp(S')\}$ for every $S' \subseteq S$. Set $y := k - \lfloor k/t \rfloor$. If $y > 0$, then assume $h^0(\mathcal{I}_{S'}(k)) = \max\{0, \binom{r+y}{ky} - \sharp(S')\}$ for every $S' \subseteq S$. If $\sharp(S) \leq (\binom{r+t}{r} - 1)\lfloor k/t \rfloor + \binom{r+y}{r}$, then $h^1(\mathcal{I}_S(k)) = 0$.*

Proof. Adding $(\binom{r+t}{r} - 1)\lfloor k/t \rfloor + \binom{r+y}{r} - \sharp(S)$ general points of $\mathbb{P}^r$ we reduce to the case $\sharp(S) = (\binom{r+t}{r} - 1)\lfloor k/t \rfloor + \binom{r+y}{r}$ (Lemma 1). First assume

$y > 0$. Set $S_0 := S$. Fix any degree t hypersurface Q_1 of $\mathbb{P}^r$ containing at least $\binom{r+t}{r} - 1$ points of S_0. Since $h^0(\mathcal{I}_{S'}(t)) = \max\{0, \binom{r+k}{k} - \sharp(S')\}$ for every $S' \subseteq S_0$, we have $\sharp(S_0 \cap Q_1) = \binom{r+t}{r} - 1$. Set $S_1 := S_0 \setminus S_0 \cap Q_1$. For each integer $i \in \{2, \dots, \lfloor k/t \rfloor\}$ (if any) define inductively the degree t hypersurface Q_i and the set S_i in the following way. Let Q_i be any hypersurface containing at least $\binom{r+t}{r} - 1$ points of S_{i-1}. Since $h^0(\mathcal{I}_{S'}(t)) = \max\{0, \binom{r+k}{k} - \sharp(S')\}$ for every $S' \subseteq S_0$, we have $\sharp(S_{i-1} \cap Q_i) = \binom{r+t}{r} - 1$. Set $S_i := S_{i-1} \setminus S_{i-1} \cap Q_1$. Notice that $\sharp(S_i) = (\binom{r+t}{r} - 1)(\lfloor k/t \rfloor - i) + \binom{r+y}{r}$ and in particular $\sharp(S_{\lfloor k/t \rfloor}) = \binom{r+y}{y}$. Let $T \subset \mathbb{P}^r$ be any degree y hypersurface containing at least $\binom{r+y}{y} - 1$ points of $S_{\lfloor k/t \rfloor}$. By assumption $B := S_{\lfloor k/t \rfloor} \setminus A \cap S_{\lfloor k/t \rfloor}$ is formed by a unique point. We have the exact sequence

$$0 \to \mathcal{I}_B \to \mathcal{I}_{S_{\lfloor k/t \rfloor}}(y) \to \mathcal{I}_{A \cap S_{\lfloor k/t \rfloor}}(y) \to 0 \tag{2}$$

Since $\sharp(B) = 1$, we have $h^1(\mathcal{I}_B) = 0$. Since $h^0(\mathcal{I}_{A \cap S_{\lfloor k/t \rfloor}}(y)) = 1 = \binom{r+y}{r} - \sharp(A \cap S_{\lfloor k/t \rfloor})$, the set $A \cap S_{\lfloor k/t \rfloor}$ gives independent conditions to $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(y))$. Hence it gives independent conditions to $H^0(A, \mathcal{O}_A(y))$. Since A is a hypersurface and $r \geq 3$, we have $h^1(A, \mathcal{O}_A(y)) = 0$. Hence $h^1(A, \mathcal{I}_{A \cap S_{\lfloor k/t \rfloor}}(y)) = 0$. Hence (2) gives $h^1(\mathcal{I}_{S_{\lfloor k/t \rfloor}}(y)) = 0$. For each $i = 1, \dots, \lfloor k/t \rfloor$ we have the exact sequences

$$0 \to \mathcal{I}_{S_i}(k - it) \to \mathcal{I}_{S_{i-1}}(k - it + t) \to \mathcal{I}_{Q_i \cap S_{i-1}, Q_i}(k - it + t) \to 0 \tag{3}$$

Since the set $Q_i \cap S_{i-1}$ gives independent conditions to $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t))$, it gives independent conditions to $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k - it + t))$. Hence it gives independent conditions to $H^0(Q_i, \mathcal{O}_{Q_i}(k - it + t))$. Since Q_i is a hypersurface and $r \geq 3$, we have $h^1(Q_i, \mathcal{O}_{Q_i}(k - it + t)) = 0$. Hence $h^1(Q_i, \mathcal{I}_{Q_i \cap S_{i-1}, Q_i}(t - it + t)) = 0$. From (2) applied for $i = \lfloor k/t \rfloor, \dots, 1$ we get $h^1(\mathcal{I}_S(k)) = 0$.

If $y = 0$, then we make the same proof, just stopping with $S_{\lfloor k/t \rfloor}$. □

Let $S \subset \mathbb{P}^r$ be a finite set. Fix an integer $t > 0$. We say that S has the uniform position property in degree t if $h^0(\mathcal{I}_A(x)) = h^0(\mathcal{I}_B(x))$ for all subsets A, B of S such that $\sharp(A) = \sharp(B)$. For instance, S has the uniform position property in degree 1 if and only if it is in linearly general position. In characteristic zero a general hyperplane section of an integral and non-degenerate curve $C \subset \mathbb{P}^{r+1}$ has the uniform position for all degrees. In positive characteristic this is often true (see [5]).

Proposition 2. *Fix integers $r \geq 3$ and $k > t > 0$. Let $S \subset \mathbb{P}^r$ be a finite set with the uniform position property in degree t. Set $y := k - \lfloor k/t \rfloor$. If $y > 0$, then assume that S has the uniform position property in degree y. Assume $\sharp(S) \leq (\binom{r+t}{r} - 1)\lfloor k/t \rfloor + \binom{r+y}{r}$. Then either $h^1(\mathcal{I}_S(k)) = 0$ or $h^0(\mathcal{I}_S(t)) > 0$.*

Proof. Assume $h^0(\mathcal{I}_S(t)) = 0$ (notice that this assumption implies

$$h^0(\mathcal{I}_S(y)) = 0.$$

Fix a set $A \subset S$ such that $\sharp(A) < \binom{r+t}{r}$. If $h^1(\mathcal{I}_A(t)) > 0$, then the uniform position property of S in degree t implies $h^0(\mathcal{I}_S(t)) = h^0(\mathcal{I}_A(t)) > 0$, a contradiction. Hence Proposition 2 follows from Proposition 1. □

Proposition 3. *Fix integers $r \geq 3$, $k \geq 3$ and $w > 0$ such that $w \leq \binom{r+2}{2} - 2r - 2$. Let $S \subset \mathbb{P}^r$ be a finite subset in linearly general position such that $h^1(\mathcal{I}_A(2)) = 0$ for each $A \subseteq S$ with $\sharp(A) \leq 2r + w + 1$. Assume $\sharp(S) \leq \lfloor k/2 \rfloor (2r + w) + \epsilon r + 1$, with $\epsilon = 0$ if k is even and $\epsilon = 1$ if k is odd. Then $h^1(\mathcal{I}_S(k)) = 0$.*

Proof. First assume k odd. Adding general points of $\mathbb{P}^r$ we reduce to the case $\sharp(S) = (k-1)(2r+w)/2$. Notice that for each $B \subset S$ with $\sharp(B) \leq 2r + w$ the base locus of $|\mathcal{I}_B(2)|$ intersects S only in B. Set $S_0 := S$. Fix any degree 2 hypersurface $Q_1 \subset \mathbb{P}^r$ containing exactly $2r + w$ points of S_0 (we just saw that Q_1 exists). Set $S_1 := S_0 \setminus S_0 \cap Q_1$. For each integer $i \in \{2, \dots, \lfloor k/t \rfloor\}$ (if any) define inductively the degree 2 hypersurface Q_i and the set S_i in the following way. Let Q_i be any degree 2 hypersurface containing exactly $2r + w$ points of S_{i-1}. Set $S_i := S_{i-1} \cap Q_i \cap S_{i-1}$. As in the proof of Proposition 1 we have $h^1(Q_i, \mathcal{I}_{S_{i-1} \cap Q_i, Q_i}(2)) = 0$ and hence $h^1(Q_i, \mathcal{I}_{S_{i-1} \cap Q_i, Q_i}(k - 2i + 2)) = 0$. From (3) we get $h^1(\mathcal{I}_S(k)) = h^1(\mathcal{I}_{S_{(k-1)/2}}(1))$. Since $\sharp(S_{(k-1)/2}) = r + 1$ and S is in linearly general position, we have $h^1(\mathcal{I}_{S_{(k-1)/2}}(1)) = 0$.

The case in which k is even is similar. □

We may often apply Proposition 3 to sets with the uniform position principle in degree 1 and 2. We now look at a case in which the assumptions of Proposition 3 fail in a controlled way.

Proposition 4. *Fix integers $r \geq 3$, $k \geq 3$ and $w > 0$ such that $w \leq \binom{r+2}{2} - 2r - 2$. Let $S \subset \mathbb{P}^r$ be a finite subset in linearly general position such that $h^1(\mathcal{I}_A(2)) = 0$ for each $A \subseteq S$ with $\sharp(A) \leq 2r + w$. Let $\mathcal{B}$ denote the set of all $B \subseteq S$ such that $\sharp(S) = 2r + w$ and $h^1(\mathcal{I}_B(2)) > 0$. Set $\mathbb{B} := \cup_{B \in \mathcal{B}} B$. Assume $\sharp(\mathbb{B}) \leq 4r + 2w - 1$ and $\sharp(S) \leq \lfloor k/2 \rfloor (2r + w) + \epsilon r + 1$, with $\epsilon = 0$ if k is even and $\epsilon = 1$ if k is odd. Then $h^1(\mathcal{I}_S(k)) = 0$.*

Proof. Obviously we may assume $\mathcal{B} \neq \emptyset$ and hence $\sharp(\mathbb{B}) \geq 2r+w+1$. Look at the proof of Proposition 1. We start with $B \subset S$ such that $\sharp(B) = 2r+w$ and $\sharp(B \cap \mathbb{B}) = 2r+w-1$. Notice that B cannot be contained in any $B' \in \mathcal{B}$. Hence $h^1(\mathcal{I}_{B_1}(2)) = 0$ for every $B_1 \subset S$ such that $B_1 \supset B$ and $\sharp(B_1) = 2r + w + 1$.

Hence the base locus of $|\mathcal{I}_B(2)|$ contains no point of $S\backslash B$. Since $\sharp(\mathbb{B} \setminus \mathbb{B} \cap B) \le 2r + w$, we have $B_2 \notin \mathcal{B}$ for any $B_2 \subset S \setminus B$ with $\sharp(B_2) = 2r + w + 1$. Hence from the second step on we may repeat the proof of Proposition 3. □

3. Proof of Theorem 2

We need to use the notation in [1] and only point out a few small differences. To see how to use the case in which many points of S are contained in a plane we prove the following result.

Lemma 2. *Let $H, M \subset \mathbb{P}^3$ be distinct hyperplanes. Fix an integer $m \ge 2$ and a set $S \subset H \cup M$ such that $h^1(\mathcal{I}_S(m)) > 0$ and $\sharp(S) \le 4m - 1$. Then one of the following cases occurs:*

- *(i) there is a line D contained in one of the planes H, M such that $\sharp(D \cap S) \ge m + 2$;*
- *(ii) there is a conic $D' \subset H \cup M$ such that $\sharp(D' \cap S) \ge 2m + 2$;*
- *(iii) there is a cubic curve D'' contained in one of the planes H, M (say H) such that $S \cap D''$ is the complete intersection of D'' and a degree m subcurve of H;*
- *(iv) there is a cubic curve D_2 contained in one of the planes and $\sharp(S \cap D_2) \ge 3m + 1$;*
- *(v) there is a quartic curve T contained in one of the planes H, M, such that $\sharp(D_2 \cap S) \ge 4m - 4$ and $h^1(T, \mathcal{I}_{T\cap S,T}(4)) > 0$.*

Proof. Set $L := H \cap M$. First assume $h^1(H, \mathcal{I}_{S\cap H}(m) > 0$. We get that we are in one of the 5 listed cases using [4], Corollaire 2. In the same way we conclude if $h^1(M, \mathcal{I}_{M\cap H}(m)) > 0$. Without losing generality we may assume $\sharp(S \cap H) \ge \sharp(S \cap M)$ and hence $\sharp(S \cap M) \le 2m - 1$. Since $h^1(H, \mathcal{I}_{S\cap H}(m)) = 0$, a residual exact sequence gives $h^1(M, \mathcal{I}_{S\backslash S\cap H}(m - 1)) > 0$. Since $\sharp(S \cap M) \le 2m - 1$, we get the existence of a line $D \subset M$ such that $\sharp(D \cap (S\backslash S \cap H)) \ge m + 1$. In particular $D \ne L$. Since $h^1(M, \mathcal{I}_{M\cap S}(m)) = 0$, we get $L \cap D \notin S$. Set $\{P\} := L \cap D$.

(a) In this step we assume $h^1(H, \mathcal{I}_{\{P\}\cup(S\cap H)}(m)) > 0$. Since $\sharp(\{P\} \cup (S \cap H)) \le z + 1 - m - 1 < 3m$, either there is a line $L_1 \subset H$ such that $\sharp(L_1 \cap (\{P\} \cup (S \cap H))) \ge m + 2$ or there is a conic $L_2 \subset H$ such that $\sharp(L_1 \cap (\{P\} \cup$

$(S \cap H))) \geq m+2$. First assume the existence of a line $L_1 \subset H$ such that $\sharp(L_1 \cap (\{P\} \cup (S \cap H))) \geq m+2$. If $P \notin L_1$, then we are in case (i) with respect to the line L_1. If $P \in L_1$, then the curve $L_1 \cup D$ is a conic and $\sharp(S \cap (L_1 \cup D)) \geq 2m+2$. Hence we are in case (ii). Now assume the existence of a conic L_2 such that $\sharp(L_2 \cap (\{P\} \cup (S \cap H))) \geq m+2$. If $P \notin L_2$, then we are in case (ii) with respect to the conic L_2. If $P \in L_2$, then we are in case (v) (even with one more point) with $L_1 \cup L_2$ as the connected, non-degenerate degree 3 curve with arithmetic genus 0.

(b) From now on we assume $h^1(H, \mathcal{I}_{\{P\} \cup (S \cap H)}(m)) = 0$. Let $M' \subset \mathbb{P}^3$ be a general plane containing D. Since S is finite, we have $S \cap M' = S \cap D$.

Claim. $h^1(\mathcal{I}_{S \cap (H \cup M')}(m)) = 0$.

Proof of the Claim. The claim is equivalent to $h^0(\mathcal{I}_{S \cap (H \cup M')}(m)) = \binom{m+3}{3} - m - 1 - \sharp(S \cap H)$. Fix $P' \in S \cap D$ and write $S' := (S \cup H) \cup \{P\} \cup ((S \cap D) \setminus \{P'\})$. Notice that $H^0(\mathcal{I}_{S \cap (H \cup M'}(m)) = H^0(\mathcal{I}_{S'}(m)$. Hence to prove the Claim it is sufficient to prove $h^1(\mathcal{I}_{S'}(m)) = 0$. Since $(S \cup H) \cup \{P\} = S' \cap H$, we have $h^1(H, \mathcal{I}_{S' \cap H}(m)) = 0$. Since $\sharp(S' \setminus S' \cap H) = m$, we have $h^1(\mathcal{I}_{S' \setminus S' \cap H}(m)) = 0$. Apply [1], Remark 1, i.e. the so-called Horace lemma or Castelnuovo's lemma.

Since $\sharp(S \setminus S \cap (H \cup M')) \leq 2m - 1 - m - 1$, we have $h^1(\mathcal{I}_{S \setminus S \cap (H \cup M)}(m-2)) = 0$. Since $\deg(H \cup M') = 2$, the Claim and [1], Remark 1, give $h^1(\mathcal{I}_S(k)) = 0$. □

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Let $T \subset \mathbb{P}^3$ be an integral quadric cone. Let O denote the vertex of T. Let $\pi : Y \to T$ be a minimal resolution of T. The surface Y is isomorphic to the Hirzebruch surface F_2 and π is described in the following way. Set $h := \pi^{-1}(O)$. Let $u : Y \to \mathbb{P}^1$ denote the ruling of Y. Call f any fiber of u. We have $f \cong \mathbb{P}^1$ and $\pi(f)$ is a line.

Just to be explicit, we reproduce the proof of [1], Lemma 11, with the minimal modifications needed when $\sharp(S) = 4m - 1$.

Lemma 3. *Fix an integer $m \geq 1$ and a finite set $S \subset Q$ such that $z := \sharp(S) \leq 4m - 1$. We have $h^1(\mathcal{I}_S(m)) > 0$ if and only if there is $W \subseteq S$ as in one of the following cases:*

(a) $\sharp(W) = m + 2$ and W is contained in a line;

(b) $\sharp(W) = 2m + 2$ and W is contained in a plane conic;

(c) $\sharp(W) = 3m + 2$, $m \geq 2$, and W is contained in a reduced element of $|\mathcal{O}_Q(2,1)|$ or $|\mathcal{O}_Q(1,2)|$.

Proof. The curves in case (c) are the degree 3 connected curves of Q and each of them has arithmetic genus 0. Hence the " if " part is obvious. Assume $h^1(\mathcal{I}_S(m)) > 0$. By [2], Theorem 3.8, we may assume $\sharp(S) \ge 3m+1$. Hence from now on we assume $m \ge 4$. We often silently assume $\sharp(S) \ge 3m+1$ (e.g., to say $s_1 \ge 8$ below we need $\sharp(S) \ge 8$).

Fix $E_1 \in |\mathcal{O}_Q(2,2)|$ such that $\sharp(S \cap E_1)$ is maximal. Set $S_0 := S$, $s_1 := \sharp(S \cap E_1)$ and $S_2 := \mathrm{Res}_{E_1}(S)$. Define recursively the curve $E_i \in |\mathcal{O}_Q(2,2)|$, the non-negative integer s_i and the set $S_i \subseteq S_{i-1}$ in the following way. Let E_i be any element of $|\mathcal{O}_Q(2,2)|$ such that $s_i := \sharp(E_i \cap S_{i-1})$ is maximal. Set $S_i := \mathrm{Res}_{E_i}(S_{i-1})$. The sequence $\{s_i\}_{i\ge1}$ is non-decreasing. Since S is a reduced scheme, without loosing generality we may find a sequence $\{E_i\}_{i\ge1}$ with each E_i reduced. Since $h^0(Q, \mathcal{O}_Q(2,2)) = 9$, either $s_i \ge 8$ or $s_{i+1} = 0$. The so-called Horace lemma or Castelnuovo's lemma (see [1], Remark 1) shows the existence of an integer $t > 0$ such that $h^1(E_t, \mathcal{I}_{E_t \cap S_{t-1}}(m+2-2t)) > 0$. Let e be the minimal integer $t > 0$ such $h^1(E_t, \mathcal{I}_{E_t \cap S_{t-1}}(m+2-2t)) > 0$, if either m is even or there is such an integer $t \le \lfloor (m+2)/2 \rfloor$. We will handle the case in which m is odd and there is no such integer $\le (m+1)/2$ in step (iv). If $e = 1$, then we may apply [1], Lemma 9. Hence we may assume $e \ne 1$. We use induction on m, the case $m = 1$, being obvious. In steps (iii), ..., (vii) we assume $e \ge 3$ and find a contradiction (in each step for a different numerical reason).

In steps (i) and (ii) we assume $e = 2$. In these steps we need either to find a contradiction or a subset of $S \backslash S \cap E_1$ which together with a subset of $S \cap E_1$ gives a set W as (a), (b) or (c). Assume $e = 2$. Since $s_1 \ge s_2$ and $s_1 + s_2 \le 4m-1$, we have $s_2 \le 2m-1$. Hence either there is $D \in (|\mathcal{O}_Q(1,0)| \cup |\mathcal{O}_Q(0,1)|)$ such that $\sharp(\mathrm{Res}_{E_1}(S) \cap D) \ge m$ or there is $F \in |\mathcal{O}_Q(1,1)|$ such that $\sharp(\mathrm{Res}_{E_1}(S) \cap F) \ge 2m-2$ (e.g. by [1], Lemma 9, applied to the integer $m-2$).

(i) Assume the existence of F as above. Hence $s_2 \ge 2m-2$. Take $U_2 \in |\mathcal{O}_Q(2,2)|$ containing F and such that $\sharp(U_2 \cap S)$ is maximal.

Since $h^0(Q, \mathcal{O}_Q(2,2)(-F)) = 4$, we have $\sharp(U_2 \cap S) \ge 2m+1$. We get $z = 4m-1$, $s_1 = 2m+1$, $s_2 = 2m-2$ and $S \subset E_1 \cup F$. Fix any $E' \in |\mathcal{O}_Q(2,2))|$ containing F and at least 3 points, say P_1, P_2, P_3, of $S \setminus S \cap F$. We get $\sharp(E' \cap S) = 2m+1$. Since the case $e = 1$ is done for each choice of the curve $E_1 \in |\mathcal{O}_Q(2,2)|$, we get $h^1(Q, \mathcal{I}_{S \backslash S \cap E'}(m-2)) > 0$. Hence either there is a line L' with $\sharp(L' \cap (S \setminus S \cap E')) \ge m$ or there is $F' \in |\mathcal{O}_Q(1,1)|$ with $\sharp(F' \cap (S \setminus S \cap E')) \ge 2m-2$. Assume the existence of L'. We change the choice of one of the points P_1, P_2, P_3 taking instead of P_1 a points $P_1' \in L' \cap (S \setminus S \cap E')$ (call $E'' \in |\mathcal{O}_Q(2,2)|$ the curve containing F and P_1', P_2, P_3). Since $s_2 \le 2m-2$, the same line must work also for the set $S \backslash S \cap E''$. Taking a different $P_2' \in L \cap S$ and then P_1', P_2', P_3 (and then taking $P_3' \in L \cap S$), we get $\sharp(L' \cap S) \ge m+3$. If there is F', then again we get $\sharp(S \cap F') \geq 2m+1$. Since $h^0(Q, \mathcal{I}_{F'}(2,2)) = 4$, we get $s_1 \geq 2m+1+3$, a contradiction.

(ii) From now on we assume the non-existence of F. Hence there is $D \in (|\mathcal{O}_Q(1,0)| \cup |\mathcal{O}_Q(0,1)|)$ such that $\sharp(\mathrm{Res}_{E_1}(S) \cap D) \geq m+1$. Fix $V_1 \in |\mathcal{O}_Q(2,2)|$ containing D and such that $\sharp(V_1 \cap S)$ is maximal. Since $h^0(Q, \mathcal{O}_Q(2,2)(-D)) = 6$, we have $z_1 := \sharp(U_1 \cap S) \geq m+5$. Set $Z_1 := \mathrm{Res}_{V_1}(S)$. Define recursively the curve $V_i \in |\mathcal{O}_Q(2,2)|$, the non-negative integer z_i and the set $Z_i \subseteq Z_{i-1}$, $i \geq 2$, in the following way. Let V_i be any element of $|\mathcal{O}_Q(2,2)|$ such that $z_i := \sharp(V_i \cap Z_{i-1})$ is maximal. Set $Z_i := \mathrm{Res}_{V_i}(Z_{i-1})$. The sequence $\{z_i\}_{i\geq 1}$ is non-decreasing. Since S is a reduced scheme, without loosing generality we may find a sequence $\{V_i\}_{i\geq 1}$ with each V_i reduced. Since $h^0(Q, \mathcal{O}_Q(2,2)) = 9$, either $z_i \geq 8$ or $z_{i+1} = 0$. By [1], Remark 1, there is an integer $t > 0$ such that $h^1(V_t, \mathcal{I}_{V_t \cap Z_{t-1}}(m+2-2t)) > 0$. Let c be the minimal integer t such $h^1(V_t, \mathcal{I}_{V_t \cap Z_{t-1}}(m+2-2t)) > 0$. If $c = 1$, then we may apply [1], Lemma 9. For the case $c \geq 3$, see from step (iii) on.

(ii.1) Here we assume $c = 2$. Since $z_1 \geq z_2$ and $z_1 + z_2 \leq 4m-1$, as in step (i) we get the existence of a line $D' \subset V_2$ such that $\sharp(D' \cap \mathrm{Res}_{V_1}(S)) \geq m$. Since $\sharp(S \cap D' \setminus (D \cap D')) \geq m$, we have $\sharp(S \cap (D \cup D')) \geq 2m$. Take $E(1) \in |\mathcal{O}_Q(2,2)|$ containing $D \cup D'$ and with maximal $s(1) := \sharp(E(1) \cap S)$. Set $S(0) := S$ and $S(2) := \mathrm{Res}_{E(1)}(S)$. Define recursively the curve $E(i) \in |\mathcal{O}_Q(2,2)|$, the non-negative integer $s(i)$ and the set $S(i) \subseteq S(i-1)$ in the following way. Let $E(i)$ be any element of $|\mathcal{O}_Q(2,2)|$ such that $s(i) := \sharp(E(i) \cap S(i-1))$ is maximal. Set $S(i) := \mathrm{Res}_{E(i)}(S(i-1))$. Since S is reduced, we may find a sequence $\{E(i)\}$ with each $E(i)$ reduced. The sequence $\{s(i)\}$ is non-decreasing. By [1], Remark 1, there is an integer $t \geq 1$ such that $h^1(E(t), \mathcal{I}_{E(t)\cap S(t-1)}(m+2-2t)) > 0$. Let $e(1)$ be the minimal integer t such that $h^1(E(t), \mathcal{I}_{E(t)\cap S(t-1)}(m+2-2t)) > 0$. If $e(1) = 1$, then we may apply [1], Lemma 9. See step (v) below for the case $e(1) \geq 3$. Now assume $e(1) = 2$. Since $s(1) \geq s(2)$ and $s(1) + s(2) \leq 4m-1$, as in step (i) we get the existence of a line $D_1 \subset E(2)$ such that $\sharp(D_1 \cap S(2)) \geq m$. Since $D_1 \cap S_2 \neq \emptyset$, we have $D_1 \neq D$ and $D_1 \neq D'$. If the lines D, D' and D_1 are not disjoint, then their union is contained in an element of $|\mathcal{O}_Q(2,2)|$. Hence $s_1 \geq 3m$. Hence $s_2 \leq m-1$. Thus $h^1(\mathcal{I}_{\mathrm{Res}_{E_1}(S)}(m-2)) = 0$, a contradiction. Hence D, D' and D'' are lines in the same system of lines, say $|\mathcal{O}_Q(1,0)|$. Since each of these lines contains at most $m+1$ points of S, we have $h^1(D \cup D' \cup D_1, \mathcal{I}_{S\cap(D\cup D'\cup D_1)}(m)) = 0$. Hence $h^1(Q, \mathcal{I}_{\mathrm{Res}_{D\cup D'\cup D_1}(S)}(m-3, m)) > 0$ (see [1], Remark 1). Since $\sharp(\mathrm{Res}_{D\cup D'\cup D_1}(S)) \leq m-1$, we get $\sharp(S \cap T) = m$ for each $T \in \{D, D', D'\}$, $z = 4m-1$ and the existence of a line $L' \in |\mathcal{O}_Q(0,1)|$ such that $S \subset L' \cup D \cup D' \cup D''$ and $\sharp(L' \cap (S \setminus S \cap (D \cup D' \cup D_1)) = m-1$. We have

$h^1(D \cup D' \cup L', \mathcal{I}_{S\cap(D\cup D'\cup L')}(m,m)) = 0$ and $h^1(Q, \mathcal{I}_{D_1\cap S}(m-2,m-1)) = 0$. Hence $h^1(Q, \mathcal{I}_S(m)) = 0$ (see [1], Remark 1), a contradiction.

(iii) In this step we handle all cases with $m/2 - 2 \le e \le m/2$, $e \ge 3$ and $m \ge 6$. First assume $e \ge 4$. By [1], lemma 9, either $\sharp(S_{e-1} \cap E_e) \ge 4m+16-8e$ or $S_{e-1} \cap E_e$ contains at least $m+4-2e$ collinear points or $S_{e-1} \cap E_e$ contains at least $2m+6-4e$ points on a conic or $\sharp(S_{e-1} \cap E_e) \ge 3(m+4-2e)$. In the first case (resp. second) case we get $s_{e-1} \ge 5 + (m+4-2e)$ (resp. $s_{e-1} \ge 2m+9-4e$. Hence $\sharp(S) \ge 5(e-1) + e(m+4-2e) \ge 4m-1$ with strict inequality if $e \ge 5$, because if $e \ge 5$, then $m \ge 10$. Now assume $e = 4$. Hence $8 \le m \le 10$. In this case we may assume $s_5 = 0$, $s_4 = m-4$, $S_3 \cap E_4$ formed by $m-4$ collinear points (say on a line $\widetilde{L}$) and $s_1 = s_2 = s_3 = m+1$. We take $\widetilde{E} \in |\mathcal{O}_Q(2,2)|$ containing $\widetilde{L}$ and (among these curves) one with largest $\sharp(\widetilde{E} \cap S)$. Since $\sharp(\widetilde{E} \cap S) \ge m-4+5 = s_1$, we may take $\widetilde{E}$ as E_1. Then we continuous and at a certain point (at e') we get a set $S_{e'-1} \cap E_{e'}$ formed by $m+4-2e'$ collinear points. Any two lines of Q are contained in a form of type $(2,2)$. Hence $s_1 \ge 3 + (m-4) + (m+4-2e')$. Since $e' \in \{3,4\}$, we get $s_1 \ge m+2$, a contradiction. Now assume $e = 3$ and hence $m = 6$. Since $s_1 \ge s_2 \ge s_3$, we have $s_3 \le 8$. If $S_2 \cap E_3$ contains 6 points on a conic, then $s_2 \ge 9$ and hence $\sharp(S) \ge 24$, a contradiction. Now assume that $S_2 \cap E_3$ contains $m-2 = 4$ collinear points. Repeating twice the trick with $\widetilde{L}$ we get $s_1 \ge 11$, $s_2 \ge 9$ and hence $\sharp(S) \ge 24$, a contradiction.

(iv) Here we assume m even and $e = (m+2)/2$. Hence $m+2-2e = 0$. Since $h^1(\mathcal{I}_{S_{m/2}\cap E_{(m+2)/2}}(m+2-2e)) > 0$, we get $\sharp(S_{m/2} \cap E_{(m+2)/2}) \ge 2$. Since $s_i \ge 8$ if $s_{i+1} > 0$, we get $\sharp(S) \ge 4m+2$, a contradiction. In the same way we handle the cases $c = (m+2)/2$ and $e(1) = (m+2)/2$.

(v) Here we assume m odd and $h^1(E_t, \mathcal{I}_{E_t\cap S_{t-1}}(m+2-2t)) = 0$ for all $t \le (m+1)/2$. Applying $(m-1)/2$ times [1], Remark 1, we get $h^1(\mathcal{I}_{S_{(m-1)/2}}(1)) = 0$. Hence $\sharp(S_{(m-1)/2}) \ge 3$ and either $\sharp(S_{(m-1)/2}) \ge 4$ or $S_{(m-1)/2}$ contains 3 collinear points (say on a line R). In the former case we get $\sharp(S) \ge 8(m-1)/2 + 4 > 4m-1$, absurd. In the latter case we get $s_i = 8$ for all $i \le (m-1)/2$. Since $s_i = 8$ for $i \le (m-1)/2$ we may take as $E_1, \dots, E_{(m-1)/2}$ any $E_i \in |\mathcal{O}_Q(2,2)|$ containing 8 points of $S \setminus (S \cap E_1 \cup \dots \cup E_{i-1})$. In this way we may find $E_1, \dots, E_{m-1}$ such that $S_{(m-1)/2}$ is any prescribed set of 3 points of S and in particular it is not contained in a line, a contradiction.

(vi) Assume $m = 5$ and $e = 3$. As in step (v) we get that $E_3 \cap S_2$ contains 3 collinear points. As in step (v) we get $s_1 = s_2 = 8$, $s_3 = 3$. We conclude as in step (v).

(vii) Here we assume $e \geq 3$ and $e < m/2 - 2$. The last inequality implies $m+2-2e > 0$. Q contains no plane curve of degree ≥ 3. First assume the non-existence of a plane curve F of degree ≤ 2 such that $h^1(F, \mathcal{I}_{S_{e-1}\cap F}(m+2-2e)) > 0$. By [2], Theorem 3.8, we get $\sharp(F \cap S_{e-1}) \geq 3(m+2-2e)+1$ and hence $s_e \geq 3(m+2-2e)+1$. Therefore $\sharp(S) \geq e(3m+7-6e)$. Since $6e < 3m - 12$ and $e \geq 3$, we get $\sharp(S) \geq 3(3m-11)$. Hence $9m - 33 \leq 4m - 1$, i.e. $m \leq 6$, contradicting the assumption $e < m/2 - 2$. Now assume that such a curve F exists, but it is not a line. We get $s_e \geq 2m+6-4e$; since $h^0(Q, \mathcal{O}_Q(1,1)) = 4$, there is a curve $A \in |\mathcal{O}_Q(2,2)|$ containing F and at least 3 further points of S_{e-1} (unless $\sharp(S_{e-1} \cap F) \geq \sharp(S_{e-1}) + 2$, but this inequality and $h^0(Q, \mathcal{O}_Q(1,1)) = 4$ would imply $s_e = 0$, absurd). We get $s_{e-1} \geq 3 + 2m + 6 - 4e$. Hence $\sharp(S) \geq 3(e-1) + e(2m+6-4e)$; since $2m+6-4e \geq 3$, we get $\sharp(S) \geq 6m - 12$; hence $m \leq 5$, absurd. From now on in this step we assume that such a curve F has degree 1, i.e. we assume the existence of $D_0 \in (|\mathcal{O}_Q(1,0)| \cup |\mathcal{O}_Q(0,1)|)$ such that $\sharp(\mathrm{Res}_{E_e}(S_{e-1}) \cap D_0) \geq m+4-2e$. Fix $E[1] \in |\mathcal{O}_Q(2,2)|$ containing D_0 and with maximal $s[1] := \sharp(E[1] \cap S)$. Define recursively the curve $E[i] \in |\mathcal{O}_Q(2,2)|$, the non-negative integer $s[i]$ and the set $S[i] \subseteq S[i-1]$ in the following way. Let $E[i]$ be any element of $|\mathcal{O}_Q(2,2)|$ such that $s[i] := \sharp(E[i] \cap S[i-1])$ is maximal. Set $S[i] := \mathrm{Res}_{E[i]}(S[i-1])$. Since S is reduced, we may find a sequence $\{E[i]\}$ with each $E[i]$ reduced. The sequence $\{s[i]\}$ is non-decreasing. By [1], Remark 1, there is an integer $t \geq 1$ such that $h^1(E[t], \mathcal{I}_{E[t]\cap S[t-1]}(m+2-2t)) > 0$. Let $e[1]$ be the minimal integer t such that $h^1(E[t], \mathcal{I}_{E[t]\cap S[t-1])}(m+2-2t)) > 0$ (with again the need to look at step (iv) if m is odd). If $e[1] = 1$, then we apply [1], Lemma 9. Now assume $e[1] = 2$. We get a line $D[1] \subset E[2]$ such that $\sharp(D[1] \cap S[2]) \geq m$. Since $D[1] \cap S[2] \neq \emptyset$, we have $D[1] \neq D_0$. Since $\sharp(S \cap (D_0 \cup D[1])) \geq 2m$ and there is an element of $|\mathcal{O}_Q(2,2)|$ containing $D_0 \cup D[1]$ and at least two other points of S (or 3 if $D_0 \cap D[1] \neq \emptyset$ (unless $S \subset D_0 \cup D[1]$, an inclusion which implies that we are either in case (a) or in case (b) of the lemma), we get $s[1] \geq 2m+2$ with strict inequality if $D_0 \cap D[1] \neq \emptyset$. Hence we may assume $s_1 \geq 2m+2$. Since $s_i \geq 5 + (m+4-2e)$ for $i < e$, we get $\sharp(S) \geq 2m + 2 + 5(e-2) + (e-1)(m+4-2e)$, absurd. □

Remark 1. We may drastically cut the proofs in steps (iii),…,(v), quoting [1], Theorem 1, instead of [2], Theorem 3.8 (e.g. in step (vii) we may use $\sharp(F \cap S_{e-1}) \geq 4(m+2-2e) - 2$).

We may apply the proof of [1], Lemma 12, quoting the proof of Lemma 3 instead of the one of [1], Lemma 11, and get the following result.

Lemma 4. *Fix an integer $m > 0$ and a set $S \subset Y \setminus h$ such that $\sharp(S) \leq 4m-1$. We have $h^1(Y, \mathcal{I}_S(mh+(2m)f)) = h^1(\mathbb{P}^3, \mathcal{I}_{\pi(S)}(m))$. We have $h^1(Y, \mathcal{I}_S(mh+$*

$(2m)f)) > 0$ *if and only if there is* $W \subseteq S$ *as in one of the following cases:*

(a) $\sharp(W) = m+2$ *and* W *is contained in an element of* $|\mathcal{O}_Y(f)|$ *(equivalently,* $\pi(W)$ *is contained in a line through* O*);*

(b) $\sharp(W) = 2m+2$ *and* W *is contained in an integral element of* $|\mathcal{O}_Y(h+2f)|$ *(equivalently,* $\pi(W)$ *is contained in a smooth plane section of* T*);*

(c) there are $F_1, F_2 \in |\mathcal{O}_Y(f)|$*, such that* $S \subset (F_1 \cup F_2)$*,* $F_1 \neq F_2$ *and* $\sharp(W \cap F_1) = \sharp(W \cap F_2) = m+1$ *(equivalently,* $O \notin \pi(W)$*,* $m+1$ *of its points are on a line through* O *and the other* $m+1$ *ones are on another line though* O*);*

(d) $\sharp(W) = 3m+2$ *and* W *is contained in an element of* $|\mathcal{O}_Y(h+3f)|$ *(equivalently,* $\pi(W)$ *is contained in a reduced degree 3 subcurve of* T*).*

Lemma 5. *Fix a finite set* $A \subset T$ *such that* $O \in A$*. Take* $S' \subset Y \setminus h$ *such that* $\pi(S') = A \setminus \{O\}$*. We have* $h^1(\mathcal{I}_A(m)) > 0$ *and* $h^1(\mathcal{I}_{A\setminus\{O\}}(m)) = 0$ *if and only if* $h^0(Y, \mathcal{I}_{S'}(mh+2mf)) = h^0(Y, \mathcal{I}_{S'}((m-1)h+2mf))$ *and* $h^1(Y, \mathcal{I}_{S'}(mh+2mf)) = 0$*.*

Proof. We have $h^1(\mathcal{I}_{A\setminus\{O\}}(t)) = h^1(Y, \mathcal{I}_{S'}(mh+2mf)) = 0$. We have $h^1(\mathcal{I}_A(m)) > 0$ and $h^1(\mathcal{I}_{A\setminus\{O\}}(t)) = 0$ if and only if $h^1(\mathcal{I}_{A\setminus\{O\}}(t)) = 0$ and O is in the base locus of $|\mathcal{I}_{A\setminus\{O\}}|$, i.e. if and only if h is in the base locus of $|\mathcal{I}_{S'}(mh+2mf)|$. □

Lemma 6. *Fix a finite set* $A \subset T$ *such that* $O \in A$ *and* $\sharp(A) \leq 4m-1$*. Take* $S' \subset Y \setminus h$ *such that* $\pi(S') = A \setminus \{O\}$*. We have* $h^1(Y, \mathcal{I}_{S'}(mh+2mf)) = 0$ *and* h *is in the base locus of* $|\mathcal{I}_{S'}(mh+2mf)|$ *if and only if there is* $W \subseteq A$ *as in one of the following cases:*

(i) there is a line $L \subset T$ *such that* $\sharp(T \cap W) = m+2$*;*

(ii) there is a smooth conic $B \subset T$ *such that* $O \in W \subset B$ *and* $\sharp(W) = 2m+2$*;*

(iii) there is a rational normal curve $C \subset T$ *such that* $O \in W \subset C$ *and* $\sharp(W) = 2m+2$*.*

Proof. Since the “ if ” part is obvious, we only prove the “ only if ” part. Fix any $P \in h$ and set $S := S' \cup \{P\}$. We repeat the proof of Lemma 4, i.e. of [1], Lemma 12, stopping at a step in which we met a curve $E_i \in |\mathcal{O}_Y(2h+4f)|$ containing O. Such a curve has h as an irreducible component. There is no essential problem, because $h^0(Y, \mathcal{O}_Y(h+4f)) = 8 = h^0(Y, \mathcal{O}_Y(2h+4f)) - 1$ and then we may use $|\mathcal{O}_Y(h+4f)|$ for S'. □

Proof of Theorem 2. (i) Assume $r = 3$. All cases in which $h^0(\mathcal{I}_S(2)) \geq 2$ are covered by [1], Lemmas 9, 10. For the cases in which a large subset of S is contained in a reducible (resp. irreducible, resp. smooth) quadric see Lemma 2 (resp. Lemmas 3, resp. Lemmas 4 and 6). We need to extend [1], Lemma 18, to the case $\sharp(S) = 4m-1$. Look at the proof of [1], Lemma 18. Take the set-up of step (a). Since $s_1 \geq s_2$ and $s_1 + s_2 \leq 4m-1$, we still have $s_2 \leq 2m-1$. First assume the existence of a conic F such that $\sharp(\mathrm{Res}_{E_1}(S) \cap F) \geq 2m-2$. Since $h^0(\mathcal{I}_F(2)) = 5$, there is a quadric surface containing at least $2m+2$ points of S. Hence $s_1 \geq 2m+2$. Hence $s_2 \leq 2m+3$, a contradiction. In the same way we exclude the existence of a plane conic $F_2 \subset V_2$ such that $\sharp(\mathrm{Res}_{V_1}(S) \cap F) \geq 2m-2$.

In step (b) we met an inequality $\sharp(S) \geq 3(3m-11)$, which implies $m \leq 7$ (as in [1]) and an inequality $\sharp(S) \geq 6m-12$, which implies $m \leq 5$ (as in [1]). At the end we get $\sharp(S) \geq 4m+7$, a contradiction.

(ii) Now assume $r \geq 4$. In this case the proofs in [1] works verbatim, because they only used weaker numerical assumption on $\sharp(S)$. We only point out two of these computations.

At the end of step (a2) (case $r = 4$), there are 4 lines $L_1, L_2, L_3, R \subset \mathbb{P}^r$ such that $L_i \cap L_j = \emptyset$, $\sharp(S \cap L_i) = m+1$ for all i and a quadric hypersurface $A'' \subset \mathbb{P}^4$ such that $A'' \supset R \cup L_1 \cup L_2 \cup L_3$ and $A'' \cap S = (R \cup L_1 \cup L_2 \cup L_3) \cap S$. Since (case $r = 4$) we have $\sharp(S) \leq 4m$, we have $\sharp((S \cap (R \setminus L_1 \cup L_2 \cup L_3)) \leq 4m - 3m - 3$. In particular we have $\sharp((S \cap (R \setminus L_1 \cup L_2 \cup L_3)) \leq m-2$. Hence $h^1(A'', \mathcal{I}_{S \cap A''}(m)) = 0$ (see [1], Lemma 21). Since $\sharp(S \backslash S \cap A'') \leq 4m - 3m - 3$, we have $h^1(\mathcal{I}_{S \backslash S \cap A''}(m-2)) = 0$. Hence $h^1(\mathcal{I}_S(m)) = 0$. Look at step (b), subcase $e = m-1$. First assume the existence of a line R_1 such that $\sharp(S_{m-2} \cap R_1)) \geq 4$. Since $h^0(\mathbb{P}^r, \mathcal{I}_{R_1}(1)) = r-1$, we get $s_i \geq r+2$ for all $i \leq m-2$. By assumption we have $s_{m-1} \geq 4$. Hence $\sharp(S) \geq (m-2)(r+2)+4$. Since $r \geq 4$, we get $\sharp(S) \geq 6m-8 > 4m$. Now assume the existence of a plane $T \subset \mathbb{P}^r$ such that $\sharp(T \cap S_{m-2}) \geq 6$. Since $h^0(\mathbb{P}^r, \mathcal{I}_T(1)) = r-2$, we get $s_i \geq 3+r$ for all $i \leq m-2$. Since $s_{m-1} \geq 6$, we get $\sharp(S) \geq (m-2)(r+3)+6 > 4m$, a contradiction.

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References

[1] E. Ballico, Finite subsets of projective spaces with bad postulation in a fixed degree, *Beitr. Algebra Geom.*, DOI 10.1007/s13366-012-0104-8

[2] A. Couvreur, The dual minimum distance of arbitrary dimensional algebraic-geometric codes, *J. Algebra*, **350**, No. 1 (2012), 84-107.

[3] D. Eisenbud, J. Harris, Finite projective schemes in linearly general position, *J. Algebraic Geom.*, **1**, No. 1 (1992), 15-30.

[4] Ph. Ellia, Ch. Peskine, Groupes de points de $\mathbf{P}^2$: Caractère et position uniforme, *Algebraic Geometry*, L'Aquila (1988), 111-116; *Lecture Notes in Math.*, **1417**, Springer, Berlin (1990).

[5] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276**, No. 4 (1987), 565-579.