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ON SOME TRANSMUTATIONS FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST AND SECOND ORDER IN CLIFFORD ANALYSIS

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Abstract: Using the Clifford algebra we consider a generalization of the Cauchy-Riemann system. These equations are related to some important equations such as the Maxwell equations, the Dirac equation and the Moisil-Theodorescu system.

With the help of first-order and higher-order partial differential operators, transmutations of holomorphic functions, their relatives and of solutions of certain related differential equations to the equations considered here are given. For the first time explicit representations for a class of generalized Clifford holomorphic functions obeying a generalized Bers-Vekua equation are derived and relations between the solutions of the characterizing differential equations with different parameters are presented. Similar representations and relations are given for the solutions of a differential equation of second order also.

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1. Introduction

Let $e_1, e_2, \ldots, e_n$ be the elements of an orthonormal base of $\mathbb{R}^n$ and $Cl_{p,q}$ with

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$p+q=n$ the Clifford algebra with the identity e_0 and the multiplication rules $e_i^2=e_0$ for $i=1,\dots,p, e_i^2=-e_0$ for $i=p+1,\dots,n$, and $e_ie_j+e_je_i=0, i<j$. Identifying each element $x=(x_0,x_1,\dots,x_n)$ of $\mathbb{R}^{n+1}$ with $x=x_0+\sum_{k=1}^n x_ke_k$ the conjugate is given by $\bar{x}=x_0-\sum_{k=1}^n x_ke_k$.

In $\mathbb{R}^{n+1}$ we consider the generalized Cauchy–Riemann operator $\partial=\partial_0+\sum_{i=1}^n e_i\partial_i$ with $\partial_i=\partial/\partial x_i$. The counterpart of ∂ is definded by $\bar{\partial}=\partial_0-\sum_{i=1}^n e_i\partial_i$. Functions which satisfy

$$\partial u = 0 \tag{1}$$

are called left (Clifford) holomorphic (cf. e.g. [5]).

Using Clifford algebras, elliptic, hyperbolic and parabolic differential equations can be considered from a common point of view [8]. Particular cases of (1) in $Cl_{1,2}$ are the Maxwell equations and in $Cl_{1,3}$ the Dirac equation. In $Cl_{0,2}$ the Moisil–Theodorescu system can be written in the form (1).

In the following solutions of (1) are transformed into functions which may be regarded as generalized left (Clifford) holomorphic functions. In [1] corresponding results for the particular case of $Cl_{0,2}$-valued functions were presented which are of interest in treating the Dirac equation or the Schrödinger equation (see e.g. [6] and [7]). In [2] one can find solutions of the iterated differential equation $D^k w=0, k\in\mathbb{N}$, with $Dw=\partial w+(m/x_0)\bar{w}$. As was shown in [3] a differential equation of this type plays an important role for the representation of pseudoanalytic functions in bicomplex variables which appear in the investigation of the complexified Schrödinger equation [9].

In the second part we present solutions for a differential equation of second order by means of differential operators similar to those used in the first part. A differential equation of this type plays an important role in the theory of k-holomorphic Cliffordian automorphic functions, see [4].

2. Transmutations for Generalized Holomorphic Functions

Let us consider a generalization of the differential equation (1) in the form

$$\partial w = \varphi(x_0)\,\bar{w} \tag{2}$$

where φ is an arbitrary suitable function. First we look for a connection between the solutions w of this equation and solutions g of equation (1) which can be expresssed by a differential operator of the form

$$w=\sum_{k=0}^{m} A_k(x_0)\,(g\bar{\partial}^k)+\sum_{k=0}^{n} B_k(x_0)\,(\partial^k\bar{g})\,,\quad m,n\in\mathbb{N} \tag{3}$$

with appropriate coefficients A_k and B_k .

If we insert w according to (3) into equation (2) the relation $n = m-1$ follows immediately and we deduce the following system as a necessary condition on the coefficients:

$$A_m'(x_0) = 0, \tag{4}$$
$$B_{m-1}(x_0) = \varphi(x_0)\, A_m(x_0), \tag{5}$$
$$A_k'(x_0) = \varphi(x_0) B_k(x_0), \qquad k = m-1, \ldots, 0, \tag{6}$$
$$B_{k-1}(x_0) = -B_k'(x_0) + \varphi(x_0) A_k(x_0), \quad k = m-1, \ldots, 1, \tag{7}$$
$$0 = -B_0'(x_0) + \varphi(x_0) A_0(x_0). \tag{8}$$

This system consists of $2m+2$ differential equations for the $2m+1$ functions $A_k, 0 \le k \le m$, and $B_k, 0 \le k \le m-1$, and is overdetermined in general.

Therefore we consider the equations (4)–(7) first and calculate the functions A_k and B_k starting from $k = m$. Using the final condition (8) which is essentially a nonlinear differential equation of order $2m$ for φ one can decide whether this system has a solution.

A suitable factor φ can be chosen in the form $\varphi(x_0) = \alpha/x_0$, $\alpha \in \mathbb{R}$. Then from the preceding system

$$A_m = 1,$$
$$A_{m-k} = \frac{(-1)^k \alpha}{k!\, x_0^k} \prod_{j=1-k}^{k-1} (\alpha + j), \qquad k = 1, \ldots, m,$$
$$B_{m-k} = \frac{(-1)^{k+1}}{(k-1)!\, x_0^k} \prod_{j=1-k}^{k-1} (\alpha + j), \quad k = 1, \ldots, m,$$

follows. Now the final condition (8) yields the relation $\alpha = \pm m$. Thus for $\alpha = m$ the coefficients in (3) can be written in the form $A_k = a_k, B_k = b_k$ with

$$a_k = \frac{(-1)^{m-k}}{(m-k)!} \frac{m\,(2m-k-1)!}{k!\, x_0^{m-k}}, \; b_k = \frac{(-1)^{m-k-1}}{(m-k-1)!} \frac{(2m-k-1)!}{k!\, x_0^{m-k}} \tag{9}$$

and for $\alpha = -m$ we have $A_k = a_k$, $B_k = -b_k$.

With this we can prove the following

Theorem 1. *Let Ω be a suitable domain in $\mathbb{R}^{p+1,q}$ not containing the plane $x_0 = 0$. Then we have*

1. *For each function g left holomorphic in Ω (i.e. $\partial g = 0$) the expression w in (3) with $n = m - 1$ represents a solution of*

$$\partial w = \varepsilon \frac{m}{x_0}\,\bar{w}\,, \quad \varepsilon = \pm 1\,,\ m \in \mathbb{N} \tag{10}$$

   in Ω. For $\varepsilon = 1$ the coefficients A_k and B_k are given by $A_k = a_k, B_k = b_k$, a_k, b_k according to (9), while for $\varepsilon = -1$ these coefficients are given by $A_k = a_k$, $B_k = -b_k$.

2. *The function g in (3) is not determined uniquely by w. Only the expression $(g\bar{\partial}^{2m})$ is given uniquely by*

$$(g\bar{\partial}^{2m}) = 1/(x_0^m)\ \left[(x_0^m\, w)\,\bar{\partial}^m\right].$$

In the following we give some results concerning transmutations between solutions of the differential equation $\partial w = m/x_0\,\bar{w}, m \in \mathbb{R}$, with different parameters m containing differential operators of first order.

Theorem 2. *For $w \in \mathcal{L}_m(\Omega)$ where $\mathcal{L}_m(\Omega)$ denotes the set of solutions of $\partial w = m/x_0\,\bar{w}$, $m \in \mathbb{R}$, defined in Ω we have the following results:*

$$\begin{aligned} u_1 =&(w\bar{\partial}) + \frac{m+1}{x_0}\bar{w} - \frac{2m+1}{x_0}w \in \mathcal{L}_{m+1}(\Omega),\\ u_2 =&(w\bar{\partial}) + \frac{m-1}{x_0}\bar{w} - \frac{2m-1}{x_0}w \in \mathcal{L}_{m-1}(\Omega),\\ u_3 =&(w\bar{\partial}) - \frac{m}{x_0}\bar{w} = (w\bar{\partial}) - (\partial w) \in \mathcal{L}_{-m}(\Omega). \end{aligned}$$

The first of these relations leads to a representation of the solutions of equation (10) with $\varepsilon = 1$ while the second gives a representation of solutions of equation (10) with $\varepsilon = -1$ starting from a solution of (1) in both cases.

3. Transmutations for Solutions of Second Order Differential Equations

Each solution of (1) is also a solution of the differential equation of second order

$$\bar{\partial}\partial\, u = 0 \tag{11}$$

Since

$$\bar{\partial}\partial = \partial\bar{\partial} \quad = \quad \Delta_{(n+1)} \quad \text{in} \quad Cl_{0,n}$$

$$\bar{\partial}\partial = \partial\bar{\partial} \quad = \quad \Delta_{(n)} - \frac{\partial^2}{\partial x_n^2} \quad \text{in} \quad Cl_{1,n-1}$$

where $\Delta_{(k)}$ denotes the Laplacian in $\mathbb{R}^k$, equation (11) contains important cases of applied analysis.

Now we consider a generalization of (11) in the form

$$\bar{\partial}\partial w + \varphi(x_0)\,\partial w + \psi(x_0)\,w = 0$$

and ask under which conditions on the coefficients φ and ψ we can find solutions of this equation in the form

$$w = \sum_{k=0}^{n} \tilde{A}_k(x_0)\,(\bar{\partial}^k g) + \sum_{k=0}^{m} \tilde{B}_k(x_0)\,(\partial^k h)\,, \quad m, n \in \mathbb{N}$$

where g is a left holomorphic function and h is a solution of $\bar{\partial}h = 0$ which may be called left antiholomorphic.

Using the same arguments as in the preceding section we are led to the factors

$$\varphi(x_0) = \frac{n-m}{x_0} \quad \text{and} \quad \psi(x_0) = -\frac{n(m+1)}{x_0^2}\,,\ n, m \in \mathbb{N}$$

The coefficients of the differential operators are now given by

$$\tilde{A}_k = \frac{(-1)^{n-k}(n+m-k)!}{k!(n-k)!x_0^{n-k}}\,, \quad \tilde{B}_k = \frac{(-1)^{m-k}(n+m-k)!}{k!(m-k)!x_0^{n-k}}$$

Thus we can prove the following assertions by direct calculation assuming Ω to have the same properties as in Theorem 1.

Theorem 3. *1. For each function g left holomorphic in Ω (i.e. $\partial g = 0$) and h left antiholomorphic in Ω (i.e. $\bar{\partial}h = 0$)*

$$w = \sum_{k=0}^{n} \frac{(-1)^{n-k}(n+m-k)!}{k!(n-k)!x_0^{n-k}}\,\bar{\partial}^k g + \sum_{k=0}^{m} \frac{(-1)^{m-k}(n+m-k)!}{k!(m-k)!x_0^{n-k}}\,\partial^k h \quad (12)$$

represents a solution of

$$\partial\bar{\partial}\,w + \frac{n-m}{x_0}\,\partial\,w - \frac{n(m+1)}{x_0^2}\,w = 0\,,\ n, m \in \mathbb{N} \quad (13)$$

in Ω.

2. *For a given solution* w *of* (13) *in the representation* (12) *the functions* $\bar{\partial}^{n+m+1}g$ *and* $\partial^{n+m+1}h$ *are determined uniquely by*

$$\bar{\partial}^{n+m+1}g = \frac{n!}{m!}\frac{1}{x_0^{n+m+2}}\,\tilde{P}^{m+1}(x_0^{n-m}\,w) \quad \textit{with} \quad \tilde{P} = x_0^2\,\bar{\partial}$$

$$\partial^{n+m+1}h = \frac{m!}{n!}\frac{1}{x_0^{n+m+2}}\,\tilde{Q}^{n+1}(w) \quad \textit{with} \quad \tilde{Q} = x_0^2\,\partial$$

3. *For a solution of* (13) *which can be represented by only one generator* g *or* h, *i.e.*

$$w_1 = \sum_{k=0}^{n} \frac{(-1)^{n-k}(n+m-k)!}{k!(n-k)!x_0^{n-k}}\,\bar{\partial}^k g$$

or

$$w_2 = \sum_{k=0}^{m} \frac{(-1)^{m-k}(n+m-k)!}{k!(m-k)!x_0^{n-k}}\,\partial^k h$$

the functions g *and* h *are determined uniquely by*

$$g = \frac{1}{(m+n)!}\,\tilde{Q}^n\,w_1 \quad \text{and} \quad h = \frac{1}{(m+n)!}\,\tilde{P}^m\,(x_0^{n-m}\,w_2) \quad \textit{resp.}$$

Now we shall investigate some transmutations between the solutions of a differential equation of type (13) containing different parameters n and m. Let $\mathcal{L}_{n,m}(\Omega)$ denote the set of solutions of (13) defined in Ω. By direct calculation we can prove that with the operators

$$R_{n,m} = \frac{x_0^2}{n(m+1)}\,\partial \quad \text{and} \quad S_{n,m} = \bar{\partial} + \frac{n-m}{x_0}\,id_{\mathsf{L}}$$

where id_{L} denotes the identity mapping on the corresponding set $\mathcal{L}$ the following results hold.

Theorem 4. 1. *For* $w \in \mathcal{L}_{n,m}(\Omega)$ *we have*

$$\begin{aligned} v = R_{n,m}w &= \frac{x_0^2}{n(m+1)}\,\partial\,w \;\in \mathcal{L}_{n-1,m+1}(\Omega) \\ u = S_{n,m}w &= \bar{\partial}\,w + \frac{n-m}{x_0}\,w \;\in \mathcal{L}_{n+1,m-1}(\Omega) \end{aligned}$$

2. *For the composition of these operators the following relations hold*

$$\begin{aligned} R_{n+1,m-1} \circ S_{n,m} &= id_{\mathsf{L}_{n,m}(\Omega)} \\ S_{n-1,m+1} \circ R_{n,m} &= id_{\mathsf{L}_{n,m}(\Omega)} \end{aligned}$$

3. *Starting from a solution w_n of* (13) *the function*

$$w_{n-k} = (R_{n-k+1,m+k-1} \circ \dots R_{n-1,m+1} \circ R_{n,m})\, w_n\,, \quad k = 1, \dots, n$$

is a solution of

$$\partial\bar{\partial} w_{n-k} + \frac{n-m-2k}{x_0}\partial w_{n-k} - \frac{(n-k)(m+1+k)}{x_0^2} w_{n-k} = 0$$

For $k = n$ we get

$$\partial\bar{\partial} w_0 - \frac{n+m}{x_0}\partial w_0 = 0 \tag{14}$$

4. *On the other hand starting from equation* (14) *the functions*

$$w_k = (S_{k-1,m+n-k+1} \circ \dots \circ S_{1,m+n-1} \circ S_{0,m+n})\, w_0$$

for $k = 1, \dots, n$, are solutions of

$$\partial\bar{\partial} w_k + \frac{2k-n-m}{x_0}\partial w_k - \frac{k(m+n+1-k)}{x_0^2} w_k = 0$$

Using here $k = n$ we are led to equation (13).

Thus the operators $R_{k,l}$ and $S_{k,l}$ allow us to build a chain of differential equations with ascending and descending parameters m and n. In view of property 2 of Theorem 4 we can state that if one has a representation for all the solutions of the more simple differential equation (14) one can give all the solutions of (13) also.

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