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SOME FIXED POINT THEOREMS OF A-TYPE CONTRACTIONS IN G-METRIC SPACE

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Abstract: In this paper, we extended the idea of A-contractions for G-metric space and proved some fixed point theorems for self mapping using A-type contraction in G-metric space. Also, we showed that our results generalize and improve the corresponding results of Z. Mustafa and others.

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1. Introduction

The study of Banach contraction principle have led to number of generalizations and modification of the principle. It concerns certain mappings of a complete metric space into itself. It states sufficient condition for the existences and uniqueness of fixed points. The theorem also gives an iterative process by which we can obtain the approximation to the fixed points. Many authers have generalized the well known Banach contraction principle in several different forms, we may see for example [6, 8, 9, 15, 13, 17].

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In [5] Dhage introduced D-metric space as a generalization of metric space and proved many results in this setting. But in 2005, Z. Mustafa and B. Sims [11] proved that these results are not true in topological structure and hence they introduced G-metric space as a generalized form of metric space. Since then many authors have been studying fixed point results in G-metric spaces. In [1] M. Akram, A. A. Siddiqui and A. A. Zafar introduced a class of contractions, called A-contractions and proved some fixed theorems for self maps using A-contractions. This general class of contractions properly contains some of the contractions studied by R. Kannan [7], Bianchini [4], M. S. Khan [8] and Reich [14] for details see [1]. Further, M Akram, A. A. Siddiqui and A. A. Zafar have studied some fixed point theorems using A-contraction in generalized metric spaces (gms), for detail see [2] and [3]. In this paper, we prove some fixed point theorems for self mapping using A-type contraction in G-metric space.

Throughout the article X will denote a complete G-metric space and $\mathbb{R}_+$ will denote the set of non-negative real numbers.

2. Preliminaries

In this section, we give some basic definitions and results on G-metric space from [11], which we require in the sequel.

Definition 2.1. Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties,

(i) $G(x, y, z) = 0$ if $x = y = z$,

(ii) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$(symmetry in all three variables),

(v) $G(x, y, z) \leq$ G$(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangular inequality).

Then the function G is called a generalized metric or more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 2.2. A G-metric space (X, G) is called symmetric G-metric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.3. Let (X,G) be a G-metric space, and (x_n) be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence (x_n) if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, and one can says that the sequence (x_n) is G-convergent to x.

Thus, if $x_n \to x$ in a G-metric space (X,G), then for any $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that G$(x,x_n,x_m)<\epsilon$, for all $n,m \geq N$.

Proposition 2.4. *Let (X,G) be a G-metric space, then the following are equivalent,*

(i) (x_n) *is G-convergent to* x,

(ii) $G(x_n,x_n,x) \to 0$, *as* $n \to \infty$,

(iii) $G(x_n,x,x) \to 0$, *as* $n \to \infty$,

(iv) $G(x_m,x_n,x) \to 0$, *as* $m,n \to \infty$.

Definition 2.5. Let (X,G) be a G-metric space, a sequence (x_n) is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$, for $n,m,l \geq N$; that is, if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 2.6. *Let (X,G) be a G-metric space, then the following are equivalent,*

(i) (x_n) *is G-Cauchy,*

(ii) for $\epsilon > 0$, *there exist* $N \in \mathbb{N}$ *such that* $G(x_n,x_m,x_m) < \epsilon$, *for all* $n,m \geq N$.

Definition 2.7. A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in X.

Definition 2.8. Let (X,G) and $\left(X',G'\right)$ be G-metric spaces and let $f:(X,G) \to (X',G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x,y \in X$; and $G(a,x,y) < \delta$ implies $G'(f(a),f(x),f(y)) < \epsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

3. A-Type Contraction in G-Metric Space

Definition 3.1. (see [1]) Let A stands for the set of all functions $\alpha{:}\mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying,

(i) α is G-continuous on the set $\mathbb{R}_+^3$ of all triplets of nonnegative reals (with respect to the Euclidean G-metric on $\mathbb{R}_+^3$).

(ii) $a \leq kb$, for some $k \in [0,1)$, whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$, for all $a, b \in \mathbb{R}_+$.

Definition 3.2. **(A-type Contraction)** A self map T on a G-metric X is said to be A-type contraction of X if there exists $\alpha \in A$ such that,

$$G(Tx,Ty,Ty) \leq \alpha(G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty)),$$

for all x, y in X.

Theorem 3.3. *The self map $T : X \to X$ satisfying the condition,*

$$G(Tx,Ty,Ty) \leq \beta max\{G(x,Tx,Tx) + G(y,Ty,Ty), G(y,Ty,Ty) + G(x,y,y), G(x,Tx,Tx) + G(x,y,y)\},$$

where $\beta \in [0, \frac{1}{2})$, is an A*-type contraction.*

Proof. Defining $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ by $\alpha(w,u,v) = \beta max\{u+v, v+w, u+w\}$ for all $u, v, w \in \mathbb{R}_+$. It is easy to show that $\alpha \in A$. Taking $u = G(x,Tx,Tx)$, $v = G(y,Ty,Ty)$ and $w = G(x,y,y))$, we get

$$\begin{aligned} G(Tx,Ty,Ty) \leq& \beta max\{G(x,Tx,Tx) + G(y,Ty,Ty), G(y,Ty,Ty) \\ &+ G(x,y,y), G(x,Tx,Tx) + G(x,y,y)\} \\ =& \alpha(G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty)). \end{aligned}$$

Which implies that T is an A-type contraction. Now, we, present an example of A-type contraction in G-metric space.

Example 3.4. Consider $X = \{0,1,2,3,4\}$ with usual G-metric and T be a self map on X given by,
$T0 = 2,\ T1 = 1,\ T2 = 1,\ T3 = 1,\ T4 = 1.$
By above result T is an A-type contraction if the following inequality holds.

$$G(Tx,Ty,Ty) \leq \beta max\{G(x,Tx,Tx) + G(y,Ty,Ty), G(y,Ty,Ty) + G(x,y,y), G(x,Tx,Tx) + G(x,y,y)\}.$$

Here by definition of usual G-metric,

$$G\,(Tx,Ty,Ty) = 2|Tx - Ty|,\ G\,(x,Tx,Tx) = 2|x - Tx|,$$

$$G\,(y,Ty,Ty) = 2|y - Ty|,\ G\,(x,y,y) = 2|x - y|.$$

For $x = 0,\ y = 1$, $G(T0, T1, T1) = 2(|T0 - T1|) = 2|2 - 1| = 2$.

Similarly, calculating other values and than putting in above inequality, we get

$$\begin{aligned} 2 &\leq \beta max\{4+0, 0+2, 4+2\}, \\ &\leq \beta max\{4, 2, 6\}, \\ &\leq \beta 6. \end{aligned}$$

Which gives $1 \leq 3\beta$, is true.

Similarly for all values of x and y the above inequality is satisfied.

Thus T is an A-type contraction.

Also note that 1 is the fixed point of T.

4. Comparison of Some Contractions with A-Type Contraction

All the contractions defined below can be found in [16]. These contractions are A-type contractions (see [18]) and hence class of all these contractions contained in the class of A-type contractions.

K-Type Contraction: There exist a number $a \in [0, \frac{1}{2})$ such that for all x, y in X,

$$G(Tx, Ty, Ty) \leq a(G(x, Tx, Tx) + G(y, Ty, Ty)).$$

Khan-Type Contraction. There exist a number $h \in [0, 1)$ such that for all x, y in X,

$$G(Tx, Ty, Ty) \leq h\sqrt{G(x, Tx, Tx)G(y, Ty, Ty)}.$$

B-Type Contraction. There exist a number $h \in [0, 1)$ such that for all x, y in X,

$$G(Tx, Ty, Ty) \leq hmax\{G(x, Tx, Tx), G(y, Ty, Ty)\}.$$

R-Type Contraction. There exist numbers $a, b, c \in [0, 1)$ such that $a + b + c <1$ and for all x, y in X,

$$G(Tx, Ty, Ty) \leq aG(x, y, y) + bG(x, Tx, Tx) + cG(y, Ty, Ty).$$

For details see [12].

Theorem 4.1. *Let $T : X \to X$ be a contraction defined for all x, y in X and $h \in [0, 1)$ as,*

$$G(Tx, Ty, Tz) \le h max\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}.$$

Then T is an A*-type contraction.*

Proof. Proof follows from B-type contraction by taking $y = z$ in the given contraction mapping.

Theorem 4.2. *Let $T : X \to X$ be a contraction defined as,*

$$G(Tx, Ty, Tz) \le aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz),$$

for all x, y in X and $a, b, c, d \in [0, 1)$, such that $a + b + c + d < 1$.

Then T is an A*-type contraction.*

Proof. Proof follows from R-type contraction by taking $y = z$ in the given contraction mapping.

Now, with the help of an example, we prove that A-type contraction is the proper super class of K-type contractions.

We know that the self map $T : X \to X$ satisfying the condition,

$$G(Tx, Ty, Ty) \le \beta max\{G(x, Tx, Tx) + G(y, Ty, Ty), G(y, Ty, Ty) + G(x, y, y), G(x, Tx, Tx) + G(x, y, y)\},$$

where $\beta \in [0, \frac{1}{2})$, is an A-type contraction.

Also by using this result, we have proved in Example 3.5 that the mapping $T : X \to X$ defined as, $T0 = 2,\ T1 = 1,\ T2 = 1,\ T3 = 1,\ T4 = 1$, on the set $X = \{0, 1, 2, 3, 4\}$ with usual G-metric is an A-type contraction.

But we observe that at $x = 0,\ y = 1$, the conditions of K-type contraction is not satisfied.

As $G(T0, T1, T1) = 2,\ G(0, T0, T0) = 4,\ G(1, T1, T1) = 0.$

So,

$$G(Tx, Ty, Ty) \le a(G(x, Tx, Tx) + G(y, Ty, Ty)),$$

implies that, $2 \le a(4 + 0)$.

Which gives, $1 \le 2a < 1$, as $a \in [0, \frac{1}{2})$, a contradiction.

5. Some Fixed Point Theorems

The following theorem extended the Theorem 5 of [1] from metric space setup to the G-metric space setup. Through out the sequel X denotes the complete G-metric space unless it is stated otherwise.

Theorem 5.1. *Let $T : X \to X$ be an* A*-type contraction. Then T has a unique fixed point(say $x^{'}$) in X and for each $x_0 \in X$ the sequence of iterates $\{T^n x_0\}$ converges to this fixed point, also T is G-continuous at $x^{'}$.*

Proof. Select $x_0 \in X$, define the iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ (equivalently $x_n = T^n x_0$) and applying the definition of A-type contraction in G-metric, we can write

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \alpha(G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n)) \\ &= \alpha(G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})) \\ &\leq kG(x_{n-1}, x_n, x_n), \text{ for some k} \in [0, 1) \\ &\leq k(kG(x_{n-2}, x_{n-1}, x_{n-1})) \\ &= k^2 G(x_{n-2}, x_{n-1}, x_{n-1}). \end{aligned}$$

Proceeding in the same way, we get $G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1)$.

As k <1, so when $n \to \infty$, then $G(x_n, x_{n+1}, x_{n+1}) \to 0$.

Now, by repeated use of the rectangular inequality of G-metric, for every integer $P > 0$, we can write

$$\begin{aligned} G(x_n, x_{n+p}, x_{n+p}) \leq\ & G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+ G(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G(x_{n+p-1}, x_{n+p}, x_{n+p}). \end{aligned}$$

This gives $\lim_{n\to\infty} G(x_n, x_{n+p}, x_{n+p}) = 0$, which implies $\{x_n\}$ is a Cauchy sequence and since X is complete there exist $x^{'} \in X$ such that $x_n \to x^{'}$ as $n \to \infty$.

Again, we have,

$$\begin{aligned} G(Tx_n, Tx^{'}, Tx^{'}) &\leq \alpha(G(x_n, x^{'}, x^{'}), G(x_n, Tx_n, Tx_n), G(x^{'}, Tx^{'}, Tx^{'})) \\ &\leq \alpha(G(x_n, x^{'}, x^{'}), G(x_n, x_{n+1}, x_{n+1}), G(x^{'}, Tx^{'}, Tx^{'})). \end{aligned}$$

Taking limit as $n \to \infty$ and by the continuity of α, we have

$$\begin{aligned} G(x^{'}, Tx^{'}, Tx^{'}) &\le \alpha(G(x^{'}, x^{'}, x^{'}), G(x^{'}, x^{'}, x^{'}), G(x^{'}, Tx^{'}, Tx^{'})) \\ &\le k(G(x^{'}, x^{'}, x^{'})) \\ &= k0 \\ &= 0. \end{aligned}$$

Which gives $Tx^{'}{=}x^{'}$.

Now if $w \in X$ is another fixed point of T in X, that is, $Tw = w$, then we have

$$\begin{aligned} G(w, x^{'}, x^{'}) &= G(Tw, Tx^{'}, Tx^{'}) \\ &\le \alpha(G(w, x^{'}, x^{'}), G(w, Tw, Tw), G(x^{'}, Tx^{'}, Tx^{'})) \\ &\le \alpha(G(w, x^{'}, x^{'}), G(w, w, w), G(x^{'}, x^{'}, x^{'})) \\ &\le \alpha(G(w, x^{'}, x^{'}), 0, 0) \\ &\le k0 \\ &= 0. \end{aligned}$$

Thus we get $w = x^{'}$.

Now, we show that T is G-continuous at $x^{'}$, for this, let (y_n) be a sequence such that $\lim_{n\to\infty}(y_n) = x^{'}$.

Now, by using definition of α, we have

$$\begin{aligned} G(x^{'}, T(y_n), T(y_n)) &= G(Tx^{'}, T(y_n), T(y_n)) \\ &\le \alpha(G(x^{'}, (y_n), (y_n)), G(x^{'}, Tx^{'}, Tx^{'}), G((y_n), T(y_n), T(y_n))). \end{aligned}$$

Taking limit $n \to \infty$, we get

$$\begin{aligned} G(x^{'}, \lim T(y_n), \lim T(y_n)) &\le \alpha(G(x^{'}, x^{'}, x^{'}), \\ &\qquad G(x^{'}, x^{'}, x^{'}), G(x^{'}, \lim T(y_n), \lim T(y_n))) \\ &\le \alpha(0, 0, G(x^{'}, \lim T(y_n), \lim T(y_n))) \\ &= k0 \\ &= 0. \end{aligned}$$

Which gives, $T(y_n) \to x^{'} = Tx^{'}$ as $n \to \infty$. Hence T is G-continuous at $x^{'}$.

This completes the proof.

Corollary 5.2. *Let $T : X \to X$ be a mapping satisfying any one of the following contractive condition,* *(i)* *There exist a number* $a \in [0, \frac{1}{2})$ *such that for all* x, y *in* X,

$$G\,(Tx, Ty, Ty) \leq a(G(x, Tx, Tx) + G(y, Ty, Ty)).$$

(ii) *There exist a number* $h \in [0, 1)$ *such that for all* x, y *in* X,

$$G\,(Tx, Ty, Ty) \leq h\sqrt{G(x, Tx, Tx)G(y, Ty, Ty)}.$$

(iii) *There exist a number* $h \in [0, 1)$ *such that for all* x, y *in* X,

$$G\,(Tx, Ty, Ty) \leq hmax\{G(x, Tx, Tx), G(y, Ty, Ty)\}.$$

(iv) *There exist numbers a,b,c* $\in [0, 1)$ *such that a+b+c<1 and for all* x, y *in* X,

$$G\,(Tx, Ty, Ty) \leq aG(x, y, y) + bG(x, Tx, Tx) + cG(y, Ty, Ty).$$

Then T *has a unique fixed point(say* x*) in* X *and for each* $x_0 \in X$ *the sequence of iterates* $\{T^n x_0\}$ *converges to this fixed point, also* T *is G-continuous at* x.

Proof. Since we have proved that above contractions are A-type contraction, also we have proved that A-type contraction has a unique fixed point in complete G-metric space and T is G-continuous at x. Hence the result follows from Theorem 5.1.

Now, we prove that the results 2.1, 2.2, 2.3 and 2.4 given in [10] becomes the Corollary 5.3,5.4, 5.5 and 5.6 respectively of Theorem 5.1 as follows.

Corollary 5.3. *Let* (X, G) *be a complete symmetric G-metric space and let* $T : X \to X$ *be a mapping satisfying the following condition,*

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz),$$

for all $x, y, z \in X$, *where* $0 \leq a + b + c + d < 1$, *then* T *has a unique fixed point(say* x*), and* T *is G-continuous at* x.

Proof. Since we have proved that above contraction is A-type contraction, also we have proved that A-type contraction has a unique fixed point in complete G-metric space and is G-continuous at the fixed point, hence the result follows for above contraction mapping.

Corollary 5.4. *Let* (X, G) *be a complete symmetric G-metric space and let* $T : X \to X$ *be a mapping satisfying the following condition,*

$$G(T^m x, T^m y, T^m z) \leq aG(x, y, z) + bG(x, T^m x, T^m x) + cG(y, T^m y, T^m y) + dG(z, T^m z, T^m z),$$

for all $x, y, z \in X$, where $0 \leq a + b + c + d < 1$, then T has a unique fixed point(say x), and T^m is G-continuous at x.

Proof. By above result T^m has a unique fixed point(say x), that is $T^m(x) = x$ and T^m is G-continuous at x.

But $T(x) = T(T^m(x)) = T^{m+1}(x) = T^m(T(x))$.

Which implies that $T(x)$ is another fixed point of T^m. Now by the uniqueness of fixed point, we can get $Tx = x$.

Corollary 5.5. *Let (X, G) be a complete symmetric G-metric space and let $T : X \to X$ be a mapping satisfying the following condition,*

$$G(Tx, Ty, Tz) \leq kmax\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\},$$

for all $x, y, z \in X$ and for some $k \in [0, 1)$, then T has a unique fixed point(say x), and T is G-continuous at x.

Corollary 5.6. *Let (X, G) be a complete symmetric G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for some $m \in \mathbb{N}$,*

$$G(T^m x, T^m y, T^m z) \leq kmax\{G(x, T^m x, T^m x), G(y, T^m y, T^m y), G(z, T^m z, T^m z)\},$$

for all $x, y, z \in X$ and for some $k \in [0, 1)$, then T has a unique fixed point(say x), and T^m is G-continuous at x.

6. Fixed Point Theorems for Pair of Self Maps

Next theorem describes common fixed point of two self-maps on X having two related G-metrics. This result generalizes Theorem 7 of [1] and Theorem 4 of [13].

Theorem 6.1. *Let X be a set with two G-metrics G_1 and G_2 satisfying the following conditions*

(i) $G_1(x, y, y) \leq G_2(x, y, y)$, *for all $x, y \in X$.*

(ii) *X is complete with respect to G_1.*

(iii) *$T, S : X \to X$ are* A*-type contractions on X such that T is G-continuous with respect to G_1 and*

$$G_2(Tx, Sy, Sy) \leq \alpha(G_2(x, y, y), G_2(x, Tx, Tx), G_2(y, Sy, Sy)),$$

for all x, y in X and for some $\alpha \in A$. Then T and S have a unique common fixed point.

Proof. Define a sequence $\{x_n\}$ in X as $Tx_{2n} = x_{2n+1}$ and $Sx_{2n-1} = x_{2n}$. Consider, $G_2(x_1, x_2, x_2) = G_2(Tx_0, Sx_1, Sx_1)$ and applying (iii), we get

$$\begin{aligned} G_2(x_1, x_2, x_2) &\leq \alpha(G_2(x_0, x_1, x_1), G_2(x_0, Tx_0, Tx_0), G_2(x_1, Sx_1, Sx_1)) \\ &= \alpha(G_2(x_0, x_1, x_1), G_2(x_0, x_1, x_1), G_2(x_1, x_2, x_2)) \\ &\leq \ kG_2(x_0, x_1, x_1), \text{ for some k} \in [0, 1). \end{aligned}$$

Similarly, we get $G_2(x_2, x_3, x_3) \leq \ k^2 G_2(x_0, x_1, x_1)$, for some $k \in [0, 1)$.

Continuing in this way, we have $G_2(x_n, x_{n+1}, x_{n+1}) \leq \ k^n G_2(x_0, x_1, x_1)$, for some $k \in [0, 1)$.

Also, by (i)

$$\begin{aligned} G_1(x_n, x_{n+1}, x_{n+1}) &\leq G_2(x_n, x_{n+1}, x_{n+1}) \\ &\leq k^n G_2(x_0, x_1, x_1). \end{aligned}$$

As $k < 1$, $G_1(x_n, x_{n+1}, x_{n+1}) \to 0$, as $n \to \infty$.

Now, by repeated use of the rectangular inequality of G-metric, for every integer $p > 0$, we can write

$$\begin{aligned} G_1(x_n, x_{n+p}, x_{n+p}) \leq G_1(x_n, x_{n+1}, x_{n+1}) &+ G_1(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+ G_1(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G_1(x_{n+p-1}, x_{n+p}, x_{n+p}). \end{aligned}$$

This implies $\lim_{n\to\infty} G_1(x_n, x_{n+p}, x_{n+p}) = 0$, Which gives $\{x_n\}$ is a G-Cauchy sequence and since X is complete with respect to G_1 there exist $x \in X$ such that $x_n \to x$ as $n \to \infty$. Since T is G-continuous and $x = \lim_{n\to\infty} x_{2n+1}$, we can get $x = \lim_{n\to\infty} Tx_{2n} = T(\lim_{n\to\infty} x_{2n})$. That is $x = Tx$.

Now,

$$\begin{aligned} G_2(x, Sx, Sx) &= G_2(Tx, Sx, Sx) \\ &\leq \alpha(G_2(x, x, x), G_2(x, Tx, Tx), G_2(x, Sx, Sx)) \\ &\leq \alpha(0, 0, G_2(x, Sx, Sx)) \\ &\leq k0 \\ &= 0. \end{aligned}$$

Which gives $x = Sx$.

Let y in X be such that $y = Ty$ and $Sy = y$, then

$$\begin{aligned} G_2(x,y,y) &= G_2(Tx,Sy,Sy) \\ &\leq \alpha(G_2(x,y,y),G_2(x,Tx,Tx),G_2(y,Sy,Sy)) \\ &= \alpha(G_2(x,y,y),G_2(x,x,x),G_2(y,y,y)) \\ &= \alpha(G_2(x,y,y),0,0) \\ &\leq k0 \\ &= 0. \end{aligned}$$

Thus $x = y$.

Theorem 6.2. *Let S and T be two self maps of complete symmetric G-metric space (X,G) such that,*

$$G\,(Sx,TSy,TSy) \leq \alpha(G(x,Sx,Sx),G(Sy,TSy,TSy),G(x,Sy,Sy)),$$

for all $x,\ y\ \in X$ and for some $\alpha \in A$. Then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as $x_{2n} = Sx_{2n-1}$, $x_{2n+1} = Tx_{2n}$ for $n = 1,2,3\ldots.$

Now,

$$\begin{aligned} G(x_{2n},x_{2n+1},x_{2n+1}) &= G(Sx_{2n-1},TSx_{2n-1},TSx_{2n-1}) \\ &\leq \alpha(G(x_{2n-1},Sx_{2n-1},Sx_{2n-1}), \\ &\qquad G(Sx_{2n-1},TSx_{2n-1},TSx_{2n-1}), \\ &\qquad G(x_{2n-1},Sx_{2n-1},Sx_{2n-1})) \\ &= \alpha(G(x_{2n-1},x_{2n},x_{2n}),G(x_{2n},x_{2n+1},x_{2n+1}), \\ &\qquad G(x_{2n-1},x_{2n},x_{2n})) \\ &\leq kG(x_{2n-1},x_{2n},x_{2n}). \end{aligned}$$

Similarly, we have

$$G\,(x_{2n-1},x_{2n},x_{2n}) \leq kG(x_{2n-2},x_{2n-1},x_{2n-1}),$$

which implies that

$$G\,(x_{2n},x_{2n+1},x_{2n+1}) \leq k^2G(x_{2n-2},x_{2n-1},x_{2n-1}).$$

Proceeding in the same way, we can write

$$G\,(x_{2n},x_{2n+1},x_{2n+1}) \leq k^{2n}G(x_0,x_1,x_1).$$

In general, we can write

$$G\left(x_n, x_{n+1}, x_{n+1}\right) \leq k^n (G(x_0, x_1, x_1).$$

As $k < 1$, $G(x_n, x_{n+1}, x_{n+1}) \to 0$ as $n \to \infty$.

Now, by repeated use of the rectangular inequality of G-metric, for every integer $p > 0$, we can write

$$\begin{aligned} G(x_n, x_{n+p}, x_{n+p}) \leq & G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ & + G(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G(x_{n+p-1}, x_{n+p}, x_{n+p}). \end{aligned}$$

This gives $\lim_{n\to\infty} G(x_n, x_{n+p}, x_{n+p}) = 0$, which implies $\{x_n\}$ is a G-Cauchy sequence and since X is complete there exist $x \in X$ such that $x_n \to x$ as $n \to \infty$.

Next,

$$\begin{aligned} G(Sx, x_{2n+1}, x_{2n+1}) &= G(Sx, TSx_{2n-1}, TSx_{2n-1}) \\ &\leq \alpha(G(x, Sx, Sx), G(Sx_{2n-1}, TSx_{2n-1}, TSx_{2n-1}), \\ &\qquad G(x, Sx_{2n-1}, Sx_{2n-1})) \\ &\leq \alpha(G(x, Sx, Sx), G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x, x_{2n}, x_{2n})). \end{aligned}$$

Taking limit $n \to \infty$, we have

$$G(Sx, x, x) \leq \alpha(G(x, Sx, Sx), G(x, x, x), G(x, x, x)).$$

Since G is symmetric, we can write

$$\begin{aligned} G(x, Sx, Sx) &\leq \alpha(G(x, Sx, Sx), 0, 0) \\ &\leq k0 \\ &= 0. \end{aligned}$$

Which gives $Sx = x$. Similarly, we can show that $Tx = x$.

TO prove the uniqueness, say y is another fixed point of S and T. That is, $Sy = y$ and $Ty = y$.

So $TSy = y$.

Now,

$$\begin{aligned} G(x, y, y) &= G(Sx, TSy, TSy) \\ &\leq \alpha(G(x, Sx, Sx), G(Sy, TSy, TSy), G(x, Sy, Sy)) \\ &= \alpha(G(x, x, x), G(y, y, y), G(x, y, y)) \\ &\leq k0 \\ &= 0. \end{aligned}$$

This implies that $x = y$. Which completes the proof.

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