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APPLICATION OF TRIANGULAR INTUITIONISTIC FUZZY NUMBERS IN BI-MATRIX GAMES

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Abstract: The basic aim of this paper is to an application of triangular intuitionistic fuzzy numbers(TIFNs) to a single non-cooperative bi-matrix games. Firstly, the concept of TIFNs and their cut sets are introduced as well as the inequality relations between two TIFNs. Secondly, a bi-matrix game with payoffs of TIFNs is considered and an attempt is made to conceptualize the meaning of a Nash-equilibrium solution for such games. Practical investigations have been discussed for selling a product in a market share for two different companies.

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1. Introduction

Game theory is the study of the ways in which strategic interactions among rational players produce outcomes with respect to the preferences(or utilities) of those players, none of which might have been intended by any of them. A two person game where two players are defined as decision makers, is a simplest case of game theory. In real game situations, usually players are not able to

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evaluate exactly the outcomes of game due to lack of information. Therefore, fuzzy game theory studied by various researchers [1, 2, 3, 4, 5, 6] provides an efficient framework which solves the real-life conflict problems with fuzzy information [12, 15, 16, 17, 18] and has achieved a success. Recently much more attention has been focused on bi-matrix games with fuzzy payoffs, this means that elements of the payoff matrix are fuzzy numbers, Dubois and Prade [7], where it is assumed that there membership functions, indicating the degree of belongingness, are known. In this case the degree of non-belongingness is just automatically the complement to 1. Thus the fuzzy set theory is no means to incorporate the hesitation degree. Atanassov [8] introduced the concept of an intuitionistic fuzzy set(IF-set), characterized by two functions expressing the degree of membership and the degree of non-membership respectively which is meant to reflect the fact that the degree of non-membership is not always equal to 1 minus the degree of membership, while there may be some hesitation degree. The IF-set may express information more abundant and flexible than the fuzzy set when the uncertain information is involved. The IF-set has been applied to some areas [9, 10, 11, 12]. It is essential and possible to apply IF-set to game problem as the players have some degree of hesitation or uncertainty about payoffs. The bi-matrix game theory, take care of problems that involve vagueness. Bustin and Burillo [13] pointed out that the notion of vague sets is that of IFS. Hladik [14], Nayak and Pal [15] studied interval valued bi-matrix games. Maeda [16] discussed about the characterization of the equilibrium strategies. Nishizaki and Sakawa [17] defined the concept of equilibrium solution in multi-objective bi-matrix games with fuzzy goals and fuzzy payoffs. Vidyottama et. al. [18] studied bi-matrix games with fuzzy goals and fuzzy payoffs. The intuitionistic fuzzy number has not been yet applied to bi-matrix game. In this paper we formulate a bi-matrix game with payoffs of TIFNs. Based on inequality relations on TIFNs, we define the concept of equilibrium strategy for such intuitionistic fuzzy bi-matrix game(IFBG). We shall show that this equilibrium strategy is characterized as Nash equilibrium strategy, Nash [19].

This paper is organized as follows: In Section 2, The concept of TIFNs with cut sets are introduced. Furthermore inequality relations between two TIFNs are defined. In Section 3, the concept of bi-matrix game with payoffs of TIFNs is introduced and defined the concept of equilibrium strategy for such games. In Section 4, a computational procedure is illustrated via a application to a market share problem.

2. Intuitionistic Fuzzy Sets

The intuitionistic fuzzy set introduced by Atanassov [8] is characterized by two functions expressing the degree of belongingness and the degree of non-belongingness respectively.

Definition 1. Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universal set. An intuitionistic fuzzy set $\widetilde{A}$ in a given universal set U is an object having the form

$$\widetilde{A} = \Big\{ \langle x_i, \mu_{\widetilde{A}}(x_i), \nu_{\widetilde{A}}(x_i) \rangle : x_i \in U \Big\}, \tag{1}$$

where the functions

$$\begin{aligned} & \mu_{\widetilde{A}} : U \to [0,1]; \quad i.e., x_i \in U \to \mu_{\widetilde{A}}(x_i) \in [0,1] \\ \text{and} \qquad & \nu_{\widetilde{A}} : U \to [0,1] \quad i.e., x_i \in U \to \nu_{\widetilde{A}}(x_i) \in [0,1] \end{aligned}$$

define the degree of membership and the degree of non-membership of an element $x_i \in U$, such that they satisfy the following conditions :

$$0 \leq \mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) \leq 1, \forall x_i \in U$$

which is known as intuitionistic condition. The degree of acceptance $\mu_{\widetilde{A}}(x)$ and of non-acceptance $\nu_{\widetilde{A}}(x)$ can be arbitrary.

Definition 2. ((α, β)-Cuts) A set of (α, β)-cut, generated by IFS $\widetilde{A}$, where $\alpha, \beta \in [0,1]$ are fixed numbers such that $\alpha + \beta \leq 1$ is defined as

$$\widetilde{A}_{\alpha,\beta} = \begin{cases} \Big\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x) \Big\rangle; & x \in U \\ \mu_{\widetilde{A}}(x) \geq \alpha, \nu_{\widetilde{A}}(x) \leq \beta; & \alpha, \beta \in [0,1], \end{cases}$$

where (α, β)-cut, denoted by $\widetilde{A}_{\alpha,\beta}$, is defined as the crisp set of elements x which belong to $\widetilde{A}$ at least to the degree α and which does not belong to $\widetilde{A}$ at most to the degree β.

2.1. Triangular Intuitionistic Fuzzy Number

According to Seikh et. all [20] we introduce TIFNs as a special type of intuitionistic fuzzy number as follows:

Definition 3. (Triangular Intuitionistic Fuzzy Number) A triangular intuitionistic fuzzy number(TIFN) denoted by, $\widetilde{a} = \langle a, l, r; w_a, u_a \rangle$ is a special intuitionistic fuzzy set on a real number set $\Re$, whose membership function and non-membership functions are defined as follows (Fig. 1):

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x-a+l}{l} w_a; & a-l \leq x < a \\ \dfrac{a+r-x}{r} w_a; & a \leq x \leq a+r \\ 0; & \text{otherwise} \end{cases}$$

and $$\nu_{\tilde{a}}(x) = \begin{cases} \dfrac{(a-x)+u_a(x-a+l)}{l}; & a-l \leq x < a \\ \dfrac{(x-a)+u_a(a+r-x}{r}; & a \leq x \leq a+r \\ 1; & \text{otherwise} \end{cases}$$

where l, r are called spreads and a is called mean value. w_a and u_a

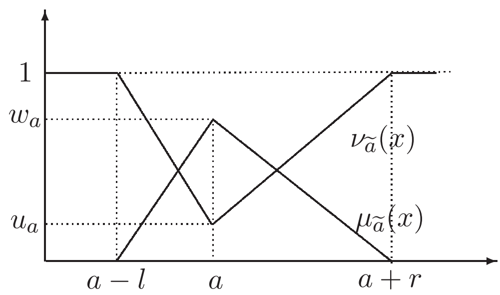

Figure 1: Triangular intuitionistic fuzzy number

represent the maximum degree of membership and minimum degree of non-membership respectively such that they satisfy the condition

$$0 \leq w_a \leq 1, 0 \leq u_a \leq 1 \text{ and } 0 \leq w_a + u_a \leq 1.$$

It is easily shown that $\mu_{\tilde{a}}(x)$ is convex and $\nu_{\tilde{a}}(x)$ is concave for $x \in U$. Further, for $a - l \leq x < a$,

$$\begin{aligned} \mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) &= \frac{x-a+l}{l} w_a + \frac{(a-x)+u_a(x-a+l)}{l} \\ &= \frac{a-x}{l}(1-u_a-w_a) + (u_a+w_a). \end{aligned}$$

When, $x = a-l, \mu_{\tilde{a}}(x)+\nu_{\tilde{a}}(x) = 1$ and when $x = a, \mu_{\tilde{a}}(x)+\nu_{\tilde{a}}(x) = w_a+u_a \leq 1$. When $a \leq x \leq a+r$,

$$\mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) = \frac{a+r-x}{r} w_a + \frac{(x-a)+u_a(a+r-x)}{r}$$

$$= \frac{x-a}{r}(1-u_a-w_a)+(u_a+w_a).$$

When, $x=a, \mu_{\widetilde{a}}(x)+\nu_{\widetilde{a}}(x)=w_a+u_a \leq 1$ and when $x=a+r, \mu_{\widetilde{a}}(x)+\nu_{\widetilde{a}}(x)=1$. Therefore, $0 \leq \mu_{\widetilde{a}}(x)+\nu_{\widetilde{a}}(x) \leq 1$. The quantity

$$\Pi_{\widetilde{a}}(x)=1-\mu_{\widetilde{a}}(x)-\nu_{\widetilde{a}}(x),$$

is called the measure of uncertaincy. The set of all these fuzzy numbers is denoted by TIFN($\Re$). The basic arithmetic operations are described in Seikh et. all [20].

2.2. (α, β)-Cut Set of TIFN

Definition 4. A (α, β)-cut set of a TIFN $\widetilde{a}=\langle a, l_a, r_a; w_a, u_a\rangle$ is a crisp subset of $\Re$, which is defined as

$$\widetilde{a}_{\alpha,\beta}=\{x : \mu_{\widetilde{a}}(x) \geq \alpha, \nu_{\widetilde{a}}(x) \leq \beta\},$$

where $0 \leq \alpha \leq w_a, u_a \leq \beta \leq 1$ and $0 \leq \alpha+\beta \leq 1$. A α-cut set of a TIFN $\widetilde{a}$ is a crisp subset of $\Re$, which is defined as

$$\widetilde{a}_\alpha=\{x : \mu_{\widetilde{a}}(x) \geq \alpha\};\ \text{where, } 0 \leq \alpha \leq w_a.$$

According to the definition of TIFN it can be easily shown that $\widehat{a}_\alpha=\{x : \mu_{\widetilde{a}}(x) \geq \alpha\}$ is a closed interval, defined by

$$\widehat{a}_\alpha=[a_L(\alpha), a_R(\alpha)] \tag{2}$$

$$\text{where } a_L(\alpha)=(a-l_a)+\frac{l_a\alpha}{w_a};\ \text{and } a_R(\alpha)=(a+r_a)-\frac{r_a\alpha}{w_a}.$$

Similarly a β-cut set of a TIFN $\widetilde{a}=\langle a, l_a, r_a; w_a, u_a\rangle$ is a crisp subset of $\Re$, which is defined as

$$\widetilde{a}_\beta=\{x : \nu_{\widetilde{a}}(x) \leq \beta\} \text{ where } u_a \leq \beta \leq 1.$$

It follows from definition that $\widetilde{a}_\beta$ is a closed interval, denoted by

$$\widehat{a}_\beta=[A_L(\beta), A_R(\beta)],$$

which can be calculated as

$$\widehat{a}_\beta=[a_L(\beta), a_R(\beta)] \tag{3}$$

where $a_L(\beta) = (a - l_a) + \dfrac{(1-\beta)l_a}{1-u_a}$; and $a_R(\beta) = (a + r_a) - \dfrac{(1-\beta)r_a}{1-u_a}$.

It can be easily proven that for $\widetilde{a} = \langle a, l_a, r_a; w_a, u_a\rangle \in \text{TIFN}(\Re)$ and for any $\alpha \in [0, w_a]$and $\beta \in [u_a, 1]$ where $0 \leq \alpha + \beta \leq 1$

$$\widetilde{a}_{\alpha,\beta} = \widehat{a}_\alpha \wedge \widehat{a}_\beta, \tag{4}$$

where the symbol "$\wedge$" denotes the minimum between $\widetilde{a}_\alpha$ and $\widetilde{a}_\beta$. Thus from equations (2),(3) and (4) we have following relations

$$\widetilde{a}_{\alpha,\beta} = \begin{cases} \widehat{a}_\beta; & \text{if } \alpha < \frac{1-\beta}{1-u_a} w_a \\ \widehat{a}_\alpha; & \text{if } \alpha > \frac{1-\beta}{1-u_a} w_a \\ \widehat{a}_\beta \text{ or } \widehat{a}_\alpha; & \text{if } \alpha = \frac{1-\beta}{1-u_a} w_a. \end{cases} \tag{5}$$

Example 1. Let $\widetilde{a} = \langle 2, 1, 0.5; 0.6, 0.3\rangle$ be a TIFN. Then for $\alpha \in (0, 0.6]$, we get following α-cut as

$$\widehat{a}_\alpha = \left[1 + \frac{\alpha}{0.6}, 2.5 - \frac{\alpha}{1.2}\right]$$

and for $\beta \in [0.3, 1]$, we get following β-cut as

$$\widehat{a}_\beta = \left[1 + \frac{1-\beta}{0.7}, 2.5 - \frac{1-\beta}{1.4}\right].$$

Thus for $\alpha = 0.1$ and $\beta = 0.4$ such that $0 \leq 0.5 \leq 1$, and using (4) we have

$$\widetilde{a}_{0.1,0.4} = \widehat{a}_{0.1} \wedge \widehat{a}_{0.4} = \widehat{a}_{0.4} = [1.8571, 2.071].$$

2.3. Inequality Relations of TIFNs

The raking order relation between two TIFNs is a difficult problem. However, ITFNs must be ranked before the action is taken by the decision maker. In this section we describe a new ranking order relation between two TIFNs by defining average ranking index. Assume that $\widetilde{a} = \langle a, l_a, r_a; w_a, u_a\rangle$ and $\widetilde{b} = \langle b, l_b, r_b; w_b, u_b\rangle$ be two TIFNs, $\widehat{a}_\alpha, \widehat{b}_\alpha$ and $\widehat{a}_\beta, \widehat{b}_\beta$ be their α-cuts and β-cuts respectively. Let $m_a(\widehat{a}_\alpha)$ and $m_a(\widehat{a}_\beta)$ be mean values of the intervals $\widehat{a}_\alpha$ and $\widehat{a}_\beta$, respectively i.e.,

$$m_a(\widehat{a}_\alpha) = \frac{2aw_a + (w_a - \alpha)(r_a - l_a)}{2w_a}$$

$$\text{and} \qquad m_a(\widehat{a}_\beta) = \frac{2a(1-u_a) + (\beta - u_a)(r_a - l_a)}{2(1-u_a)}.$$

$m_b(\widehat{b}_\alpha)$ and $m_b(\widehat{b}_\beta)$ can be defined similarly. We define average raking index of the membership function $S_\mu(\widetilde{a}, \widetilde{b})$ and average raking index of the non-membership function $S_\nu(\widetilde{a}, \widetilde{b})$ for the TIFNs $\widetilde{a}$ and $\widetilde{b}$ as follows:

$$S_\mu(\widetilde{a}, \widetilde{b}) = \int_0^{\min\{w_a, w_b\}} \left[m_b(\widehat{b}_\alpha) - m_a(\widehat{a}_\alpha)\right] d\alpha$$
$$= \begin{cases} \frac{1}{4}\left[4(b-a) + (r_b - l_b)\left(2 - \frac{w_a}{w_b}\right) - (r_a - l_a)\right] w_a; & \text{if } \min\{w_a, w_b\} = w_a \\ \frac{1}{4}\left[4(b-a) + (r_b - l_b) - (r_a - l_a)\left(2 - \frac{w_b}{w_a}\right)\right] w_b; & \text{if } \min\{w_a, w_b\} = w_b, \end{cases}$$

and

$$S_\nu(\widetilde{a}, \widetilde{b}) = \int_{\max\{u_a, u_b\}}^1 \left[m_b(\widehat{b}_\beta) - m_a(\widehat{a}_\beta)\right] d\beta =$$
$$\begin{cases} \frac{1}{4}\left[4(b-a) + (r_b - l_b)\left(2 - \frac{1-u_a}{1-u_b}\right) - (r_a - l_a)\right](1-u_a); \text{ if } \max\{u_a, u_b\} = u_a \\ \frac{1}{4}\left[4(b-a) + (r_b - l_b) - (r_a - l_a)\left(2 - \frac{1-u_b}{1-u_a}\right)\right](1-u_b); \text{ if } \max\{u_a, u_b\} = u_b. \end{cases}$$

On the basis of above definition we propose the following inequality relations

(i) If $S_\mu(\widetilde{a}, \widetilde{b}) > 0$ then $\widetilde{a}$ is smaller than $\widetilde{b}$ and is denoted by $\widetilde{a} \prec \widetilde{b}$.

(ii) If $S_\mu(\widetilde{a}, \widetilde{b}) = 0$ then

   (a) If $S_\nu(\widetilde{a}, \widetilde{b}) = 0$ then $\widetilde{a}$ is equal to $\widetilde{b}$, denoted by $\widetilde{a} \asymp \widetilde{b}$;

   (b) If $S_\nu(\widetilde{a}, \widetilde{b}) > 0$ then $\widetilde{a}$ is smaller than $\widetilde{b}$, denoted by $\widetilde{a} \prec \widetilde{b}$.

The symbol "$\prec$" is an intuitionistic fuzzy version of the order relation "$<$ " in the real number set and has the linguistic interpretation as "essentially less than." The symbols "$\succ$" and "$\asymp$" are explained similarly.

Example 2. Let $\widetilde{a} = \langle 5, 1, 2; 0.6, 0.3\rangle$ and $\widetilde{b} = \langle 6, 2, 1; 0.6, 0.4\rangle$ be two TIFNs. Here we have

$$a = 5, \quad l_a = 1, \quad r_a = 2, \quad w_a = 0.6, \quad u_a = 0.3$$
$$\text{and} \qquad b = 6, \quad l_b = 2, \quad r_b = 1, \quad w_a = 0.6, \quad u_a = 0.4.$$

Then from above discussion we have $S_\mu(\widetilde{a}, \widetilde{b}) = 0.3 > 0$ and therefore $\widetilde{a} \prec \widetilde{b}$.

3. Bi-Matrix Games

A bi-matrix game can be considered as a natural extension of the matrix game. Let I, II denote two players and let $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ be the sets of all pure strategies available for players I, II respectively. By α_{ij} and γ_{ij}, we denote the pay-offs that the player I and II receive when player I plays the pure strategy i and player II plays the pure strategy j. Then we have the following pay-off matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots \alpha_{2n} \\ \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \cdots \alpha_{mn} \end{pmatrix}; B = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots \gamma_{2n} \\ \cdots & \cdots & \\ \gamma_{m1} & \gamma_{m1} & \cdots \gamma_{mn} \end{pmatrix}.$$

where we assume that each of the two players chooses a strategy, a pay-off for each of them is represented as a crisp number. We denote the game by $\Gamma = \langle \{I, II\}, A, B \rangle$.

3.1. Nash Equilibrium Solution

Nash [19] defined the concept of Nash equilibrium solutions (NES) in bi-matrix games for single pair of payoff matrices and presented methodology for obtaining them.

Definition 5. (Pure Strategy) Let I, II denote two players and let $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ be the sets of all pure strategies available for players I, II respectively. A pair of strategies $(row\ \ r, column\ \ s)$ is said to constitute a NES to a bi-matrix game Γ if the following pair of inequalities is satisfied for all $i = 1, 2, \cdots, m$ and for all $j = 1, 2, \cdots, n$:

$$\alpha_{is} \leq \alpha_{rs}; \quad \gamma_{rj} \leq \gamma_{rs}$$

Since the strategy sets are finite, these expressions may exist and in such case, bi-matrix admits a NES for pure strategy. The pair $(\alpha_{rs}, \gamma_{rs})$ is known as a *Nash equilibrium outcome* of the bi-matrix game in pure strategies. A bi-matrix game can admit more than one NES, with the equilibrium outcomes being different in each case.

Example 3. Consider a 2×2 bi-matrix game whose payoff matrices are

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It admits two Nash equilibriums ($row\ 2, column\ 1$) and ($row\ 1, column\ 2$). The corresponding equilibrium outcomes are $(2, 2)$ and $(0, 0)$ respectively.

3.2. Intuitionistic Fuzzy Bi-matrix Games(IFBGs)

The IFBG is a non-cooperative two-person, in general, a non-zero sum (in the sense of intuitionistic fuzzy set theory) game. It can be considered as a natural extension of classical game to cover situations in which the outcome of a decision process does not necessarily dictate the verdict that what one player gains and the other has to lose.

3.3. Pay-Off Matrix

Let I, II denote two players and let $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ be the sets of all pure strategies available for players I, II respectively. By the expression $\{\widetilde{a}_{ij} = \langle a_{ij}, l_{a_{ij}}, r_{a_{ij}}; w_{a_{ij}}, u_{a_{ij}}\rangle, \widetilde{b}_{ij} = \langle b_{ij}, l_{b_{ij}}, r_{b_{ij}}; w_{b_{ij}}, u_{b_{ij}}\rangle\}$ we mean the pay-off that the players I and II receive when player I plays the row pure strategy i and player II plays the column pure strategy j. Then we have the following pay-off matrices $\hat{A}$ and $\hat{B}$ whose $(i, j)^{th}$ element is

$$\{\langle a_{ij}, l_{a_{ij}}, r_{a_{ij}}; w_{a_{ij}}, u_{a_{ij}}\rangle, \langle b_{ij}, l_{b_{ij}}, r_{b_{ij}}; w_{b_{ij}}, u_{b_{ij}}\rangle\}.$$

A two-person non-zero sum bi-matrix game, comprised of two $m \times n$ dimensional matrices $(\hat{A}, \hat{B})$ can be written as

$$\hat{A} = \begin{pmatrix} \langle a_{11}, l_{a_{11}}, r_{a_{11}}; w_{a_{11}}, u_{a_{11}}\rangle & \cdots & \langle a_{1n}, l_{a_{1n}}, r_{a_{1n}}; w_{a_{1n}}, u_{a_{1n}}\rangle \\ \langle a_{21}, l_{a_{21}}, r_{a_{21}}; w_{a_{21}}, u_{a_{21}}\rangle & \cdots & \langle a_{2n}, l_{a_{2n}}, r_{a_{2n}}; w_{a_{2n}}, u_{a_{2n}}\rangle \\ \vdots & \vdots & \vdots \\ \langle a_{m1}, l_{a_{m1}}, r_{a_{m1}}; w_{a_{m1}}, u_{a_{m1}}\rangle & \cdots & \langle a_{mn}, l_{a_{mn}}, r_{a_{mn}}; w_{a_{mn}}, u_{a_{mn}}\rangle \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} \langle b_{11}, l_{b_{11}}, r_{b_{11}}; w_{b_{11}}, u_{b_{11}}\rangle & \cdots & \langle b_{1n}, l_{b_{1n}}, r_{b_{1n}}; w_{b_{1n}}, u_{b_{1n}}\rangle \\ \langle b_{21}, l_{b_{21}}, r_{b_{21}}; w_{b_{21}}, u_{b_{21}}\rangle & \cdots & \langle b_{2n}, l_{b_{2n}}, r_{b_{2n}}; w_{b_{2n}}, u_{b_{2n}}\rangle \\ \vdots & \vdots & \vdots \\ \langle b_{m1}, l_{b_{m1}}, r_{b_{m1}}; w_{b_{m1}}, u_{b_{m1}}\rangle & \cdots & \langle b_{mn}, l_{b_{mn}}, r_{b_{mn}}; w_{b_{mn}}, u_{b_{mn}}\rangle \end{pmatrix},$$

where we assume that each of the two players chooses a strategy, a pay-off for each of them is represented as a TIFN. Usually it is denoted by the symbolic notation $\widetilde{\Gamma} = \langle\{I, II\}, \hat{A}, \hat{B}\rangle$. Each pair of entries $\{\widetilde{a}_{ij}, \widetilde{b}_{ij}\}$ denotes the outcome of the game corresponding to a particular pair of decisions made by the players.

A Nash solution represents an equilibrium point when each players reacts to the other by choosing the option that given him/her the largest value preference.

Definition 6. A pair of strategies $\{row\ r, column\ s\}$ is said to constitute a Nash equilibrium solution to a bi-matrix game $\widetilde{\Gamma} = \langle\{I, II\}, \hat{A}, \hat{B}\rangle$ where $\hat{A} = \{\widetilde{a}_{ij}\}_{m\times n}, \hat{B} = \{\widetilde{b}_{ij}\}_{m\times n}$ if the following set of inequalities is satisfied for all $i = 1, 2, \cdots, m$ and for all $j = 1, 2, \cdots, n$

$$\widetilde{a}_{rs} \preceq \widetilde{a}_{is}, \quad \widetilde{b}_{rs} \preceq \widetilde{b}_{rj}. \tag{6}$$

The pair $(\widetilde{a}_{rs}, \widetilde{b}_{rs})$ is known as a Nash equilibrium outcome of the IFBG in pure strategies. An IFBG can admit more than one Nash equilibrium solution, with the equilibrium outcomes being different in each case.

4. Application to Market Share Problem

Suppose that there are two companies C_1 and C_2 aiming to enhance the market share of a product in a targeted market under the circumstances that the demand amount of the product in the targeted market is unknown. This may be represented as bi-matrix game problem. The companies C_1 and C_2 may be regarded as players I and II respectively. The framework for such game is as follows:

(i) The company C_1 produces a product to increase the market share considering two strategies, namely, 'advertisement(add)' and 'changing models frequently(cmf)'; while company C_2 produces the same product considering two other strategies viz 'launching new offers(lnf)' and 'reduce the price(rtf)'.

(ii) Although both the companies produce the same product the number of demands are unknown to them and each of them has a choice of two strategies(alternatives): 'add, cmf, lnf, rtf.' The four elements of the set U are described by $U = \{\text{add, cmf, lnf, rtf}\}$, which is the universe of discourse associated with decision space for all players.

(iii) The play of the game consists of a single move. Company C_1 and C_2 simultaneously and independently choose one of the two alternatives available to each of them. Based on demand rate of the product, they are allowed to use the different capacities of these alternatives. Thus there are two possible situations which may be considered as pure strategies M and N for the two players respectively.

  (a) Consider that company C_1 choose following two possible choices: emphasize on advertisement and trace on changing models frequently to enhance the sale of the product i.e $M = \{\text{add, cmf}\}$.

(b) Similarly for company C_2, $N = \{\text{lnf}, \text{ rtf}\}$.

Due to lack of information or imprecision of the available information, the managers of the two companies usually are not able to manipulate the exact demand rate of the product. They estimate the demand amount with certain confidence degree, but they are not so sure about it. Thus, there may be a hesitation about the estimation of the demand amount. In order to handle the uncertain situations TIFNs are used to express the demand amount of the product. The payoff matrices $\widehat{A}$ and $\widehat{B}$ for the company C_1 and C_2 are given as follows:

$$\widehat{A} = \begin{pmatrix} \langle 90, 10, 10; 0.8, 0.1\rangle & \langle 70, 8, 5; 0.6, 0.3\rangle \\ \langle 120, 8, 10; 0.4, 0.6\rangle & \langle 50, 10, 08; 0.6, 0.2\rangle \end{pmatrix};$$

$$\widehat{B} = \begin{pmatrix} \langle 120, 8, 10; 0.4, 0.6\rangle & \langle 150, 10, 10; 0.7, 0.1\rangle \\ \langle 80, 5, 10; 0.5, 0.2\rangle & \langle 40, 8, 6; 0.5, 0.3\rangle \end{pmatrix},$$

where $\langle 90, 10, 10; 0.8, 0.1\rangle$ in the matrix $\widehat{A}$ is a TIFN, which indicates that the demand amount of the product for the company C_1 is 'about 90' when the company C_1 usages the strategy '*add*' and C_2 usages the strategy 'lnf'. The maximum confidence degree of the manager is 0.8, while the minimum nonconfidence degree is 0.1. In other words the hesitation degree is 0.1. The other elements in the matrix $\widehat{A}$ and the elements of $\widehat{B}$ are explained similarly. Using the inequality relations between TIFNs as described in Section 3 and acording to the definition of Nash equilibrium solution the problem admits two NES as indicated: The first Nash equilibrium is $\{row\ 1, col\ 1\}$ and the corresoning equilibrium outcome is $\Big\{\langle 90, 10, 10; 0.8, 0.1\rangle;\ \langle 120, 8, 10; 0.4, 0.6\rangle\Big\}$. The second Nash equilibrium is $\{row\ 2, col\ 2\}$ and the corresponding equilibrium outcome is $\{\langle 50, 10, 08; 0.6, 0.2\rangle; \langle 40, 8, 6; 0.5, 0.3\rangle\}$. By comparison between their degree of hesitancy, it is clear that the Nash solution $\{\langle 90, 10, 10; 0.8, 0.1\rangle : \langle 120, 8, 10; 0.4, 0.6\rangle\}$ as the most favorable equilibrium solution than $\{\langle 50, 10, 08; 0.6, 0.2\rangle; \langle 40, 8, 6; 0.5, 0.3\rangle\}$. This solution indicates that both the company emphasizes on advertisement(add) and launching new offers(lnf) for possible increase of demands.

5. Conclusion

In this paper we have developed the concept of TIFNs, their cut sets and inequality relations. We also presented a new approach to define Nash equilibrium solution for a bi-matrix game with payoffs of TIFNs. This process helped us to automate the solution by the players of the pure strategy and their corresponding preferences needed to play a IFBG. The proposed method is illustrated with a market share problem. It may also be applied to similar competitive decision making problems. This method is simple and more accurate to deal with general two person IFBG problems. This game approach is especially appropriate for the Decision Maker, whose preference of attribute and alternatives are both unknown.

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