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NEW GRAMMIAN SOLUTIONS FOR A VARIABLE-COEFFICIENT MKPI EQUATION

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Abstract: In this paper, Grammian solutions of a variable-coefficient mKPI equation are obtained by employing the bilinear formalism.

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1. Introduction

During the past several years, the study of coupled nonlinear evolution equations (NEEs) has played an important role in explaining many interesting phenomena such as the fluid dynamics, plasma physics and so on. For understanding those nonlinear mechanism, numerous work has been done on solitary wave solutions to NEEs, see [1]-[5]. The Wronskian technique is an efficient direct way to search for exact solutions of NEEs, see [6]. The main steps in this technique are to first transform the NEEs into bilinear form and then to express the solutions of these bilinear equations as Wronskian, Casorati or Grammian type determinant. In the early 1990s, based on the Wronskian technique, Hirota and Ohta (see [1]) developed a procedure to produce new coupled systems of equations with solutions in the form of Pfaffians. The key points involved in this procedure, which we now call Pfaffianization, are to first express N-soliton solutions of an un-Pfaffianized equation in the form of Wronskian, Casorati or Grammian type determinant, then to replace the determinant with a Pfaffian

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whose elements satisfy the Pfaffianized form of the dispersion relation given in the determinant solutions of the un-Pfaffianized equation, and finally to construct a new coupled system whose solutions are these Pfaffians.

In this paper, we would like to apply the Pfaffianization procedure to the variable-coefficient mKPI (vcmKPI) equation. We will first present the N-soliton solutions in terms of Grammian type determinant by employing the bilinear formalism. Then by applying the method of Pfaffianization to the vcmKPI case, we derive the bilinear form of the Pfaffianized vcmKPI equation. Finally by the dependent variable transformation, a new integrable, coupled vcmKPI system in nonlinear form is derived.

2. Bilinear Form and Grammian Solution

In this section, we will use the bilinear approach to derive the Grammian solutions for the vcmKPI equation.

The vcmKPI equation is written as (see [7])

$$4u_t + y(u_{xxx} - 6u^2u_x + 6u_x\partial^{-1}u_y + 3\partial^{-1}u_{yy}) + 2xu_y - u^2 + 3\partial^{-1}u_y + (vw)_x = 0. \quad (1)$$

With the help of the dependent variable transformation

$$u = (\ln \frac{\tau}{\tau'})_x, \quad v = \frac{h}{\tau}, \quad w = \frac{s}{\tau'}, \quad (2)$$

equation (1) can be transformed into the bilinear form

$$D_y\tau\cdot\tau' - D_x^2\tau\cdot\tau' = 0, \quad (3a)$$

$$4D_t\tau\cdot\tau' + y(D_x^3 + 3D_xD_y)\tau\cdot\tau' + 2xD_y\tau\cdot\tau' + \tau_x\tau' + \tau\tau'_x = 0, \quad (3b)$$

$$D_yh\cdot\tau' - D_x^2h\cdot\tau' = 0, \quad (3c)$$

$$D_ys\cdot\tau + D_x^2s\cdot\tau = 0, \quad (3d)$$

where D is the well-known Hirota bilinear operator

$$D_x^mD_t^na\cdot b = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^na(x,t)b(x',t')|_{x'=x,t'=t}. \quad (4)$$

In the following we show that the bilinear equations (3) have solutions in the Grammian form

$$\tau = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1N} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NN} \end{vmatrix} = |\mathcal{G}|, \quad (5a)$$

$$\tau' = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1N} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NN} \end{vmatrix} - \begin{vmatrix} 0 & \int^x \Psi_1 & \cdots & \int^x \Psi_N \\ -\Phi_1 & g_{11} & \cdots & g_{1N} \\ \vdots & \vdots & \cdots & \vdots \\ -\Phi_N & g_{N1} & \cdots & g_{NN} \end{vmatrix}. \tag{5b}$$

For simplicity, we denote

$$\tau' = \tau - \begin{vmatrix} 0 & \int^x \Psi dx \\ -\Phi & \mathcal{G} \end{vmatrix} = \tau + [\int^x \Psi dx|\Phi] = \tau + [\Psi^{(-1)}|\Phi]. \tag{6}$$

$$h = \tau, \quad s = \tau - [\Psi^{(-1)}|\Phi], \tag{7}$$

where g_{ij} satisfies

$$g_{ij} = c_{ij} + \int^x \Phi_i \Psi_j dx \qquad (i, j = 1, 2, \cdots, N), \tag{8}$$

$$\frac{\partial \Phi_i}{\partial y} = -\frac{\partial^2 \Phi_i}{\partial x^2}, \quad \frac{\partial \Psi_j}{\partial y} = \frac{\partial^2 \Psi_j}{\partial x^2}, \tag{9a}$$

$$\frac{\partial}{\partial t}\Phi_i = -y\Phi_{i,xxx} + \frac{1}{2}x\Phi_{i,xx} + \frac{1}{4}\Phi_{i,x}, \tag{9b}$$

$$\frac{\partial}{\partial t}\Psi_j = -y\Psi_{j,xxx} - \frac{1}{2}x\Psi_{j,xx} - \frac{3}{4}\Psi_{j,x}. \tag{9c}$$

By use of the property of the bordered determinant (see [9]) and the dispersion relation (9), we can easily obtain various derivatives of τ and τ' as

$$\tau_x = -[\Psi|\Phi], \tag{10a}$$

$$\tau_{xx} = -[\Psi|\Phi^{(1)}] - [\Psi^{(1)}|\Phi], \tag{10b}$$

$$\tau_{xxx} = -[\Psi|\Phi^{(2)}] - 2[\Psi^{(1)}|\Phi^{(1)}] - [\Psi^{(2)}|\Phi], \tag{10c}$$

$$\tau_y = [\Psi|\Phi^{(1)}] - [\Psi^{(1)}|\Phi], \tag{10d}$$

$$\tau_{xy} = -[\Psi^{(2)}|\Phi] + [\Psi|\Phi^{(2)}], \tag{10e}$$

$$\begin{aligned}\tau_t =& y([\Psi|\Phi^{(2)}] - [\Psi^{(1)}|\Phi^{(1)}] + [\Psi^{(2)}|\Phi]) \\ &+ \frac{1}{2}x([\Psi^{(1)}|\Phi] - [\Psi|\Phi^{(1)}]) + \frac{1}{4}[\Psi|\Phi],\end{aligned} \tag{10f}$$

$$\tau'_x = [\Psi^{(-1)}|\Phi^{(1)}], \tag{10g}$$

$$\tau'_{xx} = [\Psi|\Phi^{(1)}] + [\Psi^{(-1)}|\Phi^{(2)}] - [\Psi^{(-1)}, \Psi|\Phi^{(1)}, \Phi], \tag{10h}$$

$$\tau'_{xxx} = [\Psi^{(1)}|\Phi^{(1)}] + 2[\Psi|\Phi^{(2)}] + [\Psi^{(-1)}|\Phi^{(3)}] - 2[\Psi^{(-1)}, \Psi|\Phi^{(2)}, \Phi] - [\Psi^{(-1)}, \Psi^{(1)}|\Phi^{(1)}, \Phi], \tag{10i}$$

$$\tau'_{y} = [\Psi|\Phi^{(1)}] - [\Psi^{(-1)}|\Phi^{(2)}] + [\Psi^{(-1)}, \Psi|\Phi, \Phi^{(1)}], \tag{10j}$$

$$\tau'_{xy} = [\Psi^{(1)}|\Phi^{(1)}] - [\Psi^{(-1)}|\Phi^{(3)}] - [\Psi^{(-1)}, \Psi^{(1)}|\Phi, \Phi^{(1)}], \tag{10k}$$

$$\tau'_{t} = y([\Psi|\Phi^{(2)}] - [\Psi^{(1)}|\Phi^{(1)}] - [\Psi^{(-1)}|\Phi^{(3)}] + [\Psi^{(-1)}, \Psi|\Phi, \Phi^{(2)}] - [\Psi^{(-1)}, \Psi^{(1)}|\Phi, \Phi^{(1)}]) - \frac{1}{2}x([\Psi|\Phi^{(1)}] - [\Psi^{(-1)}|\Phi^{(2)}]) + \frac{1}{4}[\Psi^{(-1)}|\Phi^{(1)}]. \tag{10l}$$

Substituting (10) into equation (3a), we have

$$2([\Psi^{(-1)}|\Phi] \times [\Psi|\Phi^{(1)}] - [\Psi^{(-1)}|\Phi^{(1)}] \times [\Psi|\Phi] - [\Psi^{(-1)}, \Psi|\Phi, \Phi^{(1)}] \times \tau) = 0. \tag{11}$$

This is nothing but the Plücker relation for determinants. Therefore, we have shown that τ and τ' satisfy (3a). In the same way, substituting (10) into (3b), we see (3b) reduces to the following Plücker identity

$$-6y([\Psi^{(-1)}|\Phi] \times [\Psi^{(1)}|\Phi^{(1)}] - [\Psi^{(-1)}|\Phi^{(1)}] \times [\Psi^{(1)}|\Phi] + [\Psi^{(-1)}, \Psi^{(1)}|\Phi, \Phi^{(1)}] \times \tau) + 6y([\Psi^{(-1)}|\Phi] \times [\Psi|\Phi^{(2)}] - [\Psi^{(-1)}|\Phi^{(2)}] \times [\Psi|\Phi] + [\Psi^{(-1)}, \Psi|\Phi, \Phi^{(2)}] \times \tau) = 0. \tag{12}$$

Thus equation (3b) holds. In the same way, we can prove that equations (3c) and (3d) also hold.

3. Conclusion

The Grammian solutions of a variable-coefficient mKPI equation are obtained by employing the bilinear formalism. The method can also be used to solve other variable-coefficient nonlinear partial differential equations.

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