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ON CONVOLUTION OF GENERALIZED REPUNITS

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Abstract: We will concentrate on properties of the generalized repunits $R_n(k)$, where k is any nonnegative integer and n is any positive integer greater than 1. In this paper some results on congruences of generalized repunits are stated. Further the generating function for generalized repunits is found, some relations for them are proved using this generating function and $m-$fold convolution formula is derived.

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1. Introduction

The term *repunit* was coined by Beiler [1] in 1966. A repunit R_n is any integer written in decimal form as a string of 1's. The numbers 1, 11, 111, 1111, 11111, etc., are examples of repunits. Thus repunits have the form $R_n = \frac{10^n-1}{9}$. A repunit prime is a repunit that is also a prime number. Reuschle [11] found the factorization of all repunits up to R_{16} and many divisors of larger ones. Hoppe [7] proved R_{19} to be prime in 1916 and Lehmer [10] and Kraitchik [9] independently found R_{23} to be prime in 1929. R_{317} was found to be a probable prime circa 1967 and Brillhart and Williams was proved to be prime in 1978.

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R_{1031} was found to be probable prime in 1978 by Williams and Seah and proved prime by Williams, Dubner [15] in 1986. In recent time four probably prime repunits have known. In 1999 Dubner [5] found R_{49081}, Baxter discovered R_{86453} in 2000, Dubner found R_{109297} in 2007 and Voznyy and Budnyy [14] found R_{270343} in 2007. For more information about repunits see Yates [16]. Snyder [13] extended the repunit for arbitrary integer $b \geq 2$ by this way

$$R_n(b) = \frac{b^n - 1}{b - 1}. \tag{1}$$

They are called as *generalized repunits* or *repunits to base b and* consist of a string of 1's when written in base b. Some facts on the divisibility and primality of $R_n(b)$ can be found in Jaroma [8], Dubner [4] and tables of factorizations of the numerator of $R_n(b)$ e. g. in [2] and [3]. For the simplicity of notation we will write $M_n(k)$ instead of $R_n(k+1)$ in the rest of text. It is obvious that the numbers $M_n(k)$ are connected with the binomial theorem, concretely with the identity $(k+1)^n = \sum_{i=0}^{n} \binom{n}{i} k^{n-i}$.

It is easy to realize that all the numbers $M_n(k)$ are nonnegative integers for arbitrary positive integers k and n from the binomial theorem. Hence we can also write the numbers $M_n(k)$ in the form

$$M_n(k) = \sum_{i=0}^{n-1} \binom{n}{i} k^{n-1-i} \tag{2}$$

with $M_0(k) = 0$, $M_1(k) = 1$.

2. The Main Results

The main results established in this paper concern a congruence for the numbers $M_n(k)$ and a $m-$fold convolution formula is derived.

Theorem 1. *Let a, b, l be any positive integers. Then*

$$M_n(al+b) \equiv \begin{cases} n \pmod{a}, & \text{iff} \quad a \mid b; \\ M_n(b) \pmod{a}, & \text{iff} \quad a \nmid b. \end{cases} \tag{3}$$

Theorem 2. *Let k, $m \geq 2$, n be any positive integers. Then*

$$\sum_{\substack{n_1, n_2, \ldots n_m \\ n_1+n_2+\cdots+n_m=n}} M_{n_1}(k)\, M_{n_2}(k) \cdots M_{n_m}(k) =$$

$$= \sum_{l=0}^{n-m} \binom{l+m-1}{m-1} \binom{n-l-1}{m-1} (k+1)^{n-m-l}.$$

3. Some Preliminary Results

(i) Recurrence relations

With respect to (1) the numbers $M_n(k)$ satisfy a homogeneous and nonhomogeneous linear difference equation of the second order, respectively, and the roots of the characteristic equation must be $\lambda_1 = k+1$ and $\lambda_2 = 1$. Hence the recurrence for the numbers $M_n(k)$ has the form

$$M_{n+2}(k) - (k+2)M_{n+1}(k) + (k+1)M_n(k) = 0.$$

By an analogous procedure we can find for example recurrence

$$M_{n+1}(k) - (k+1)\, M_n(k) = 1, M_0(k) = 0\ .$$

(ii) The generating function for the numbers $M_n(k)$.

After arrangements we have for the generating function $m(x)$ of $M_n(k)$ by (1)

$$\begin{aligned} m(x) &= \sum_{n=0}^{\infty} \frac{(k+1)^n - 1}{k} x^n \\ &= \sum_{n=0}^{\infty} (k+1)^n x^n - \frac{1}{k} \sum_{n=0}^{\infty} x^n \\ &= \frac{1}{k} \left(\frac{1}{1-(k+1)x} - \frac{1}{1-x} \right) \\ &= \frac{x}{(1-(k+1)x)(1-x)}\ . \end{aligned} \tag{4}$$

4. The Proofs of the Main Theorems

Proof of Theorem 1. Firstly, let $b \mid a$. Then

$$M_n(al+b) = M_n(am) = \frac{(am+1)^n - 1}{am} = \frac{\sum_{i=1}^{n} \binom{n}{i} (am)^i}{am}$$

$$= n + \sum_{i=2}^{n} \binom{n}{i} (am)^{i-1} \equiv n \pmod{a} .$$

Let $b \nmid a$. Then $al + b \equiv b \pmod{a}$ and

$$\begin{aligned} M_n(al+b) &= \frac{(al+b+1)^n - 1}{al+b} \\ &= \frac{\left(a \sum_{i=0}^{n-1} \binom{n}{i} a^{n-i-1} l^{n-i} (b+1)^i + (b+1)^n - 1\right)}{al+b} \\ &\equiv \frac{(b+1)^n - 1}{b} \pmod{a} . \end{aligned}$$

Proof of Theorem 2. We use the following well–known fact on the generating function. If any sequence $\langle a_n \rangle$ has the generating function $A(x)$ (for example see [6], p. 355) then the m–fold convolution of the sequence $\langle a_n \rangle$ with itself has nth term equal to

$$\sum_{\substack{n_1, n_2, \dots n_m \\ n_1 + n_2 + \dots + n_m = n}} a_{n_1} a_{n_2} \cdots a_{n_m}$$

and its generating function is $A^m(x)$. Thus, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\sum_{\substack{n_1, n_2, \dots n_m \\ n_1 + n_2 + \dots + n_m = n}} M_{n_1}(k)\ M_{n_2}(k) \cdots M_{n_m}(k) \right) x^n \\ &\qquad = \left(\frac{x}{(1-x)(1-(k+1)x)} \right)^m . \end{aligned}$$

On the right side we get

$$\begin{aligned} &\frac{x^m}{(1-x)^m (1-(k+1)x)^m} = x^m \frac{1}{(1-x)^m} \frac{1}{(1-(k+1)x)^m} \\ &= x^m \left(\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} x^n \right) \left(\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} (k+1)^n \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{l+m-1}{m-1} \binom{n-l+m-1}{m-1} (k+1)^{n-l} \right) x^{n+m} \\ &= \sum_{n=m}^{\infty} \sum_{l=0}^{n-m} \binom{l+m-1}{m-1} \binom{n-m-l+m-1}{m-1} (k+1)^{n-m-l} x^n \end{aligned}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n-m} \binom{l+m-1}{m-1} \binom{n-l-1}{m-1} (k+1)^{n-m-l} \right) x^n \ .$$

Therefore

$$\sum_{\substack{n_1, n_2, \ldots n_m \\ n_1+n_2+\cdots+n_m=n}} M_{n_1}(k)\ M_{n_2}(k) \cdots M_{n_m}(k) =$$

$$= \sum_{l=0}^{n-m} \binom{l+m-1}{m-1} \binom{n-l-1}{m-1} (k+1)^{n-m-l}.$$

References

[1] A.H. Beiler, *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains*, New York, Dover (1966).

[2] J. Brillhart, D.H. Lehmer, J. Selfridge, S.S. Wagstaff Jr., B. Tuckerman, *Factorizations of* $b^n + / - 1$, $b = 2, 3, 5, 6, 7, 10, 11, 12$ *Up to High Powers*, Rev. ed. Providence, RI: Amer. Math. Soc. (1988).

[3] R.P. Brent, P.L. Montgomery, J.J. Herman te Riele, Factorizations of Cunningham numbers with bases 13 to 99, Submitted on 19 Apr. 2010, arXiv:1004.3169v1 [math.NT].

[4] H. Dubner, Generalized repunit primes, *Math. Comput.*, **61** (1993), 927-930.

[5] H. Dubner, repunit R49081 is a probable prime, *Math. Comput.*, **71**, No. 238 (2002), 833-835.

[6] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Second Edition, Addison-Wesley Publishing Company (1994).

[7] O. Hoppe, Proof of the primality of $n = (10^1 9 - l)/9$, *Proc. Lond. Math. Soc.*, Records of Meetings of Dec. 6, 1917 and Feb. 14 (1918).

[8] J. Jaroma, Factoring generalized repunits, *Irish Math. Soc. Bulletin,* **59** (2007), 29-35.

[9] M. Kraitchik, *Theorie des Nombres*, Tome II, Gauthiers-Villars, Paris, France (1926).

[10] D.H. Lehmer, On the number $n = (10^2 3 - l)/9$, *Bull. Amer. Math. Soc.*, **35**, No. 3 (1929), 349-350.

[11] C.G. Reuschle, *Mathematische Abhandlung, Enthaltend: Neue Zahlentheoretische Tabellen. Koniglichen*, Gymnasium zu Stuttgart (1856).

[12] W. Sierpínski, *Elementary Theory of Numbers*, Pa'nstwowe Wydawnictwo Naukowe, Warszawa (1964).

[13] W.M. Snyder, Factoring repunits, *Am. Math. Monthly*, **89** (1982), 462-466.

[14] M. Voznyy, A. Budnyy, New PRP repunit R(270343), 15 Jul 2007; http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0707&L=nmbrthry&T=0&P=1086.

[15] H.C. Williams, H. Dubner, The Primality of R1031, *Math. Comput.*, **47** (1986), 703-711.

[16] S. Yates, *Repunits and Repetends*, Star Publ. Co., Boynton Beach, FL (1982).