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FINITE ABELIAN GROUPS BASED ON JR-2CN

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Abstract: This paper is concerned with establishing new type of finite Abelian groups based on the infinite Abelian group JR-2CN. JR-2CN is the set of infinite integer numbers that can be represented as a summation of two signed cubic numbers. We designate each finite Abelian group generated from JR-2CN as $2\mathrm{JR}_n$, where n is used to determine the order of the group $2\mathrm{JR}_n$. The addition binary operations that are applied to construct these finite Abelian groups are originally based on addition operation of JR-2CN, but under the concept of the arithmetic modulo. Since the elements of JR-2CN are ordered pairs, therefore, it is crucial to apply the modulo on each component for each ordered pair. Theorems and propositions related to three essential parts of this study are stated, and proved. The first part is determining the nature of the elements. The second part is concerned with the critical status of the order of each set of $2\mathrm{JR}_n$, and last part is the Abelian group proof.

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1. Introduction

The study is done in this paper concerned with constructing finite sets then turn them to finite Abelian groups based on the infinite Abelian Group JR-2CN. JR-2CN as an infinite set of numbers has been introduced in [2]-Definition (1),

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where we represented the integers as ordered pairs as follows,

JR-2CN is the family of all the integers N that can be represented as a sum of two signed cubic numbers $j^3 + r^3$, where,

$$\{N = j^3 + r^3 : \ j = \alpha + x, \ r = \alpha - x; \ \forall \ \alpha, x \in Z\} \tag{1}$$

Each member of JR-2CN is formed as an ordered pair whose components are the two signed cubes j and r. Therefore, each integer in JR-2CN is in the form of (j, r). Consequently, each integer in JR-2CN, $j^3 + r^3$ is a point on the quadratic polynomial $6\alpha x^2 + 2\alpha^3$.

The aim of this research is to create finite Abelian groups based on the infinite Abelian group JR-2CN. In [1]-Definition (2.1), we have defined the addition $\oplus_{JR2CN}$ on JR-2CN, such that for $i = 1, 2$, let $j_i = \alpha_i + x_i$ and $r_i = \alpha_i - x_i$, $\forall (j_1, r_1), (j_2, r_2) \in$ JR-2CN then,

$$\begin{aligned} (j_1, r_1) \oplus_{JR2CN} (j_2, r_2) &= (j_1 + j_2, r_1 + r_2) \text{ iff} \\ (j_1 + j_2)^3 + (r_1 + r_2)^3 &= 6(\alpha_1 + \alpha_2)(x_1 + x_2)^2 \\ &\quad +2(\alpha_1 + \alpha_2)^3. \end{aligned} \tag{2}$$

We denote the finite subsets of JR-2CN as $2\mathrm{JR}_n$, which later; they will be the finite Abelian groups $2\mathrm{JR}_n$. The 2 in $2\mathrm{JR}_n$ refers to the sum of the two cubic numbers and n refers to the modulo and determines the order of the set. Considering the abnormality of JR-2CN, the construction of the finite subsets must start by giving the procedure of collecting the ordered pairs of each $2\mathrm{JR}_n$. For this purpose, we are using the arithmetic modulo concept on ordered pairs instead of the ordinary use of the modulo on integers. The definition will be given in the next section.

The method of collecting the elements is based on modulo n. We have found that it is easier to describe it by giving examples on constructing the first few finite Abelian groups under modulo 1, 2, 3, 4, 5 and 6, before we give the theoretical construction. Since the elements of JR-2CN are in the shape of ordered pairs, then modulo n is bound to be applied on both components whenever it is necessary to keep the ordered pair in JR-2CN, and it is applied on one component otherwise. This critical situation of applying the arithmetic modulo on an ordered pairs as one unit has no reference in number theory, the usual situation is applying modulo on integers. Nevertheless, we referred to [3]-section 31:3 for covering the basics of this topic. It is important to mention here that it is well expected for $2\mathrm{JR}_n$ to have ordered pairs whose either or both components are negative because, after all, $2\mathrm{JR}_n$ is a subset of JR-2CN. Thus, for each n, each ordered pair in $2\mathrm{JR}_n$ is taking its components from the set $\{0, \pm 1, \cdots, \pm(n-1)\}$. After illustrating all the possible ordered pairs, there will be two steps of eliminating, the first one is removing the ordered pairs that do not belong to JR-2CN. Deducing these ordered pairs is easy by following the definition of JR-2CN. It is crucial not to have two congruent ordered pairs in the same set; therefore, the second eliminating is applied on congruent ordered pairs.

In order to transform the finite set 2JR_n into finite Abelian group, a commutative addition binary operation on 2JR_n is defined analogous to the commutative addition binary operation that has been defined on JR-2CN but under modulo n. Each addition binary operation on 2JR_n is denoted by $\oplus_{2_\mathsf{n}}$ where, 2 refers to the sum of the two signed cubic numbers and n refers to the modulo.

The paper is organized as follows. Besides the introduction, section two describes the method of collecting the ordered pairs through giving examples for the first six finite sets, we refer to [4] and [5]. We will start by giving a general definition for 2JR_n in order to define the addition binary operation. We think it is necessary–even though it is not different in concept–to define the identity and the inverse elements of the groups. Section three is allocated for the theoretical background of constructing the finite set 2JR_n, and we refer to [7] and [3]. Proving that 2JR_n is finite Abelian group for all n will be given in section four, we preferred to refer to [6] to cover this topic.

2. Elimination of the Ordered Pairs of 2JR_n

The finite Abelian groups we are about to construct in the following sections are driven from the infinite Abelian group of the summation of two signed cubic number JR-2CN, see [1]. These finite sets 2JR_n cannot be considered as subgroups of JR-2CN, because the binary operation $\oplus_{2_\mathsf{n}}$ associate with each 2JR_n is not the same binary operation $\oplus_{JR-2CN}$ that has been defined on JR-2CN, in fact, $\oplus_{2_\mathsf{n}}$ is modulo n of $\oplus_{JR-2CN}$. The definition will be given next, we refer to [4] and [5]. We will start with constructing the first few finite groups, step by step, to give a better understanding of the method of selecting the elements. We will give the theoretical part in the next section.

Definition 2.1. Let 2JR_n be the set of all ordered pairs in JR-2CN modulo n whose components' absolute values are no larger than $n-1$. In other word,

$$\begin{aligned} 2JR_n &= \{(j,r) mod\, n : (j,r) \in JR\text{-}2CN, \text{ where} \\ &\qquad |j| \leq n-1, |r| \leq n-1\} \end{aligned} \tag{3}$$

Definition 2.2. Let $\oplus_{2_n}$ be a relation defined from $2JR_n \times 2JR_n$ into $2JR_n$, such that, $\forall (j,r),(k,s) \in 2JR_n$, we have

$$\begin{aligned}(j,r) \oplus_{2_n} (k,s) &\equiv (j+k, r+s) mod\, n \\ &\equiv ((j+k) mod\, n, (r+s) mod\, n).\end{aligned}$$

Proposition 2.1. *The relation $\oplus_{2_n}$ is well defined binary operation, closed, associative and commutative on $2JR_n$.*

The proof can be deduced by following the concept of the arithmetic modulo n, definition (2.2) and ([1]-definition 2.1, propositions 2.1-2.3).

Definition 2.3. Under the addition binary operation $\oplus_{2_n}$, the identity element is $(0,0)$ and the inverse element of (j,r) is $(j,r)^{-1}$, where, $(j,r) \oplus_{2_n} (j,r)^{-1} \equiv (0,0) mod\, n$

1. **Modulo 1, $\mathbf{2JR_1}$**

   Obviously we have only one ordered pair modulo 1 belongs to JR-2CN, which is $(0,0)$. All the group's conditions have been satisfied under the binary operation $\oplus_{2_1}$. Therefore,

$$2JR_1 = \{(0,0)\} \tag{4}$$

For the next five examples, we will see quite clear the differences between $2JR_n$ when n is odd and when it is even. Basically, we have to look at the ordered pairs of JR-2CN, [2], then we select which pairs modulo n to be included in $2JR_2$. However, this path will be impractical as hard as it be to gather all the ordered pairs of JR-2CN, especially when n is large. To resolve this issue, we have- at the beginning- to enclose all possible ordered pairs whose components from the finite set $\{0, \pm 1, \cdots, \pm(n-1)\}$, regardless whether or not the ordered pair is in JR-2CN. The purpose of this step is to ensure not to miss any ordered pair that might belong to $2JR_n$ during the collecting process. Next, the eliminating process comes, which is, actually, the way to set up each $2JR_n$. As we mentioned in the previous section, the eliminating procedure consists of two steps, eliminating the ordered pairs that do not belong to JR-2CN, where, the definition of JR-2CN in [2] states $\{(j,r) \in JR{-}2CN$ when $j = x + \alpha$ and $r = \alpha - x,\ \forall\ \alpha, x \in Z\}$. Therefore, it is obvious, which ordered pairs should be eliminated, and then, eliminating the congruent ordered pairs.

2. **Modulo 2, $\mathbf{2JR_2}$**

   Based on definition (2.1), the components of the ordered pairs modulo 2 must take their values from the set $\{0, \pm 1\}$. Therefore, it is very much easy to list the possible ordered pairs that might belong to $2\mathrm{JR}_2$:

$$\{(-1,-1),(-1,0),(-1,1),(0,-1),(0,0),(0,1),\\ (1,-1),(1,0),(1,1)\} \tag{5}$$

   The ordered pairs $(0,1),(1,0),(0,-1)$, and $(-1,0)$ do not belong to JR-2CN, so they must be eliminated. As for the rest of the pairs, by giving a close glance, we find,

$$\{(-1,1),(1,-1),(-1,-1)\} \equiv (1,1) \bmod 2$$

   The congruent, above, suggests eliminating these three pairs from the set. As in modulo on integer numbers, all the four ordered pairs are congruent to each other; therefore, we can eliminate any three of them, but it is more convenient keeping $(1,1)$ in the set and eliminate the others, because both of its components are positive. At the end, $2\mathrm{JR}_2$ contains only 2 elements,

$$2JR_2 = \{(0,0),(1,1)\} \tag{6}$$

   The set $2\mathrm{JR}_2$ with the binary operation $\oplus_{2_2}$ formulate a finite Abelian group because all the conditions are satisfied. The identity element is $(0,0)$, and the inverse is,

$$(1,1)^{-1} = (1,1).$$

   Moreover, the binary operation $\oplus_{2_2}$ is closed, associative and Abelian on $2\mathrm{JR}_2$.

3. **Modulo 3, $\mathbf{2JR_3}$**

   As for this set, all the possible ordered pairs can be listed as follows.

$$\{(-2,-2),(-2,-1),(-2,0),(-2,1),(-2,2),(-1,-2),\\ (-1,-1),(-1,0),(-1,1),(-1,2),(0,-2),(0,-1),\\ (0,0),(0,1),(0,2),(1,-2),(1,-1),(1,0),\\ (1,1),(1,2),(2,-2),(2,-1),(2,0),(2,1),(2,2)\} \tag{7}$$

We have to eliminate the ordered pairs that are not in JR-2CN, which they are $(0,1)$, $(1,0)$, $(0,-1)$, $(-1,0)$, $(1,2)$, $(2,1)$, $(1,-2)$, $(-2,1)$, $(-1,2), (2,-1)$, $(-1,-2)$, and $(-2,-1)$. Last step, the congruent ordered pairs are,

$$\begin{aligned} (-1,-1) &\equiv (2,2) \bmod 3, \\ (-2,-2) &\equiv (1,1) \bmod 3, \\ (2,-2) &\equiv (-1,1) \bmod 3, \text{ and,} \\ (-2,2) &\equiv (1,-1) \bmod 3 \end{aligned}$$

Therefore, the ordered pairs $(-1,-1), (2,-2), (-2,2)$, and $(-2,-2)$ must be eliminated (The method of which ordered pairs to be eliminated and which to be kept will be given in the next section). Therefore, $2\mathrm{JR}_3$ is,

$$\begin{aligned} 2JR_3 = \{&(0,0),(0,2),(2,0),(0,-2),(-2,0),(1,1), \\ &(-1,1),(1,-1),(2,2)\} \end{aligned} \tag{8}$$

The set $2\mathrm{JR}_3$ with the binary operation $\oplus_{2_3}$ formulate a finite Abelian group because all the conditions are applicable. The identity element is $(0,0)$, and the inverses are,

$$\begin{aligned} (0,2)^{-1} &= (0,-2), \quad & (2,0)^{-1} &= (-2,0), \\ (1,1)^{-1} &= (2,2), \quad & (1,-1)^{-1} &= (-1,1). \end{aligned}$$

The binary operation $\oplus_{2_3}$ is associative, Abelian and closed over $2\mathrm{JR}_3$. However, we have to mention that it might be producing an ordered pair that does not belong to JR-2CN and consequently, to $2\mathrm{JR}_3$, during applying $\oplus_{2_3}$ on two ordered pairs. If this case happened, we can safely say, we always can use the congruence to convert this ordered pair to an ordered pair that does belong to the set. For instance,

$$(0,2) \oplus_{2_3} (0,2) \equiv (0,1) \bmod 3$$

Which cannot be accepted because $(0,1)$ does not belong to $2\mathrm{JR}_3$. In this case, using the congruent modulo 3 again on $(0,1)$ produces $(0,-2)$.

According to the example above, the list down contains the congruent ordered pairs that are not in $2\mathrm{JR}_3$, which they might occur during the calculations.

$$\begin{array}{rclrcl}
(0,1) & \equiv & (0,-2) \bmod 3, & (1,0) & \equiv & (-2,0) \bmod 3,\\
(-1,0) & \equiv & (2,0) \bmod 3, & (0,-1) & \equiv & (0,2) \bmod 3,\\
(1,2) & \equiv & (1,-1) \bmod 3, & (2,1) & \equiv & (-1,1) \bmod 3,\\
(-1,2) & \equiv & (2,2) \bmod 3, & (1,-2) & \equiv & (1,1) \bmod 3,\\
(-1,-2) & \equiv & (-1,1) \bmod 3, & (-2,-1) & \equiv & (1,-1) \bmod 3.
\end{array}$$

Important note to mention here, we tend to keep the ordered pairs whose components are positive or whose components are smaller under the absolute value whenever it is possible. This procedure will be introduced in the next section under the concept of the "simplest form." For example, $(-2,2) \equiv (1,-1) \bmod 3$, we chose keeping $(1,-1)$ and removing $(-2,2)$. Of course the set 2JR_3 will stay a finite Abelian group if we go all the way around. At any case, it is crucial not to keep two congruent ordered pairs together inside the set because this will negate being a group.

4. **Modulo 4, $2\mathbf{JR}_4$**

Following the same procedure in the three groups above with keeping in mind that we are now dealing with modulo 4. The set 2JR_4 originally before eliminating should be,

$$\begin{aligned}
&\{(-3,-3),(-3,-2),(-3,-1),(-3,0),(-3,1),(-3,2),\\
&(-3,3),(-2,-3),(-2,-2),(-2,-1),(-2,0),(-2,1),\\
&(-2,2),(-2,3),(-1,-1),(-1,0),(-1,1),(-1,2),(-1,3),\\
&(0,-3),(0,-2),(0,-1),(0,0),(-1,-3),(-1,-2),(0,1),(0,2),\\
&(0,3),(1,-3),(1,-2),(1,-1),(1,0),(1,1),(1,2),(1,3),(2,-3),\\
&(2,-2),(2,-1),(2,0),(2,1),(2,2),(2,3),(3,-3),(3,-2),\\
&(3,-1),(3,0),(3,1),(3,2),(3,3)\}
\end{aligned} \tag{9}$$

Now by eliminating all the ordered pairs that do not belong to JR-2CN, The set will be,

$$\begin{aligned}
&\{(-3,-3),(-3,-1),(-3,1),(-3,3),(-2,-2),(-2,0),(-2,2),\\
&(-1,-3),(-1,-1),(-1,1),(-1,3),(0,-2),(0,0),(0,2),(1,-3),\\
&(1,-1),(1,1),(1,3),(2,-2),(2,0),(2,2),(3,-3),(3,-1),(3,1),\\
&(3,3)\}
\end{aligned}$$

The final step, eliminating all the congruent ordered pairs and keeping the ones whose components are as much positive and small as it could be. Down here the equivalent ordered pairs of this set,

$$\begin{aligned}
\{(1,-3),(-3,1),(-3,-3)\} &\equiv (1,1) \bmod 4\\
\{(2,-2),(-2,2),(-2,-2)\} &\equiv (2,2) \bmod 4\\
\{(3,-1),(-1,3),(-1,-1)\} &\equiv (3,3) \bmod 4\\
(0,-2) &\equiv (0,2) \bmod 4\\
(-2,0) &\equiv (2,0) \bmod 4\\
(1,-1) &\equiv (1,3) \bmod 4\\
(-1,1) &\equiv (3,1) \bmod 4
\end{aligned}$$

This elimination process leaves us with,

$$2JR_4 = \{(0,0),(0,2),(2,0),(1,1),(1,3),(3,1),(2,2),(3,3)\} \tag{10}$$

The identity element is $(0,0)$. The inverses are,

$$\begin{aligned}
(0,2)^{-1} &= (0,2), & (2,0)^{-1} &= (2,0), & (1,1)^{-1} &= (3,3),\\
(1,3)^{-1} &= (3,1), & (2,2)^{-1} &= (2,2).
\end{aligned}$$

The binary operation $\oplus_{2_4}$ is closed, associative, and Abelian over $2\mathrm{JR}_4$.

5. **Modulo 5, $\mathbf{2JR_5}$**

listing all the possible elements when $n = 5$,

$$\begin{aligned}
&\{(-4,-4)(-4,-3),(-4,-2),(-4,-1),(-4,0),(-4,1),\\
&(-4,2),(-4,3),(-4,4),(-3,-4),(-3,-3),(-3,-2),\\
&(-3,-1),(-3,0),(-3,1),(-3,2),(-3,3),(-3,4),\\
&(-2,-4),(-2,-3),(-2,-2),(-2,-1),(-2,0),(-2,1),\\
&(-2,2),(-2,3),(-2,4),(-1,-4),(-1,-3),(-1,-2),\\
&(-1,-1),(-1,0),(-1,1),(-1,2),(-1,3),(-1,4),(0,-4),\\
&(0,-3),(0,-2),(0,-1),(0,0),(0,1),(0,2),(0,3),(0,4),(1,-4),\\
&(1,-3),(1,-2),(1,-1),(1,0),(1,1),(1,2),(1,3),(1,4),(2,-4),\\
&(2,-3),(2,-2),(2,-1),(2,0),(2,1),(2,2),(2,3),(2,4),(3,-4),\\
&(3,-3),(3,-2),(3,-1),(3,0),(3,1),(3,2),(3,3),(3,4),(4,-4),
\end{aligned}$$

$$(4,-3),(4,-2),(4,-1),(4,0),(4,1),(4,2),(4,3),(4,4)\} \quad (11)$$

Eliminating the ordered pairs that do not belong to JR-2CN leaves the set with,

$$\begin{aligned}&\{(-4,-4),(-4,-2),(-4,0),(-4,2),(-4,4),(-3,-3),\\&(-3,-1),(-3,1),(-3,3),(-2,-4),(-2,-2),(-2,0),(-2,2),\\&(-2,4),(-1,-3),(-1,-1),(-1,1),(-1,3),(0,-4),(0,-2),\\&(0,0),(0,2),(0,4),(1,-3),(1,-1),(1,1),(1,3),(2,-4),\\&(2,-2),(2,0),(2,2),(2,4),(3,-3),(3,-1),(3,1),(3,3),\\&(4,-4),(4,-2),(4,0),(4,2),(4,4)\} \quad (12)\end{aligned}$$

And finally, check the equivalent ordered pairs in the above set and remove them, keeping in mind to eliminate the ordered pairs with the negative components if it is possible and to keep the ones with positive and small components. We find that,

$$\begin{array}{rclrcl}(-4,-4) & \equiv & (1,1) \bmod 5, & (-4,-2) & \equiv & (1,3) \bmod 5,\\ (-4,4) & \equiv & (1,-1) \bmod 5, & (-3,-3) & \equiv & (2,2) \bmod 5,\\ (-3,-1) & \equiv & (2,4) \bmod 5\ , & (-3,3) & \equiv & (2,-2) \bmod 5,\\ (-2,-4) & \equiv & (3,1) \bmod 5, & (-2,-2) & \equiv & (3,3) \bmod 5,\\ (-2,4) & \equiv & (3,-1) \bmod 5, & (-1,-3) & \equiv & (4,2) \bmod 5\\ (-1,-1) & \equiv & (4,4) \bmod 5, & (3,-3) & \equiv & (-2,2) \bmod 5,\\ (4,-4) & \equiv & (-1,1) \bmod 5, & (4,-2) & \equiv & (-1,3) \bmod 5,\\ (2,-4) & \equiv & (-3,1) \bmod 5, & (-4,2) & \equiv & (1,-3) \bmod 5.\end{array}$$

Eliminating these ordered pairs gives the final shape of the set $2\mathrm{JR}_5$,

$$\begin{aligned}2JR_5 \;=\; &\{(0,0),(0,2),(2,0),(0,-2),(-2,0),(2,-2),(-2,2),(4,0),\\&(0,4),(-4,0),(0,-4),(1,1),(1,-1),(-1,1),(1,3),(3,1),\\&(1,-3),(-3,1),(-1,3),(3,-1),(2,2),(2,4),(4,2),(3,3),\\&(4,4)\} \quad (13)\end{aligned}$$

The identity element is $(0,0)$. The inverses are,

$$\begin{array}{rclrclrcl}(0,2)^{-1} & = & (0,-2), & (2,0)^{-1} & = & (-2,0), & (4,0)^{-1} & = & (-4,0),\\ (0,4)^{-1} & = & (0,-4), & (1,1)^{-1} & = & (4,4), & (1,-1)^{-1} & = & (-1,1),\\ (1,3)^{-1} & = & (4,2), & (1,-3)^{-1} & = & (-1,3), & (3,1)^{-1} & = & (2,4),\\ (-3,1)^{-1} & = & (3,-1), & (2,2)^{-1} & = & (3,3), & (2,-2)^{-1} & = & (-2,2).\end{array}$$

The binary operation $\oplus_{25}$ is closed, associative, and Abelian over $2\mathrm{JR}_5$.

6. **Modulo 6, $\mathbf{2JR_6}$**

   This is the last example we introduce in this section where we followed the same procedures in the above examples, especially, in modulo 2 and 4.

$$\begin{aligned} 2JR_6 &= \{(0,0),(0,2),(2,0),(4,0),(0,4),(1,1),(1,3),(3,1),(1,5),\\ &\quad (5,1),(2,2),(2,4),(4,2),(3,3),(5,3),(3,5),(4,4),(5,5)\} \end{aligned}$$

   The identity element is $(0,0)$. The inverses are,

$$\begin{array}{llllll} (0,2)^{-1} = (0,4), & (2,0)^{-1} = (4,0), & (1,1)^{-1} = (5,5),\\ (1,3)^{-1} = (5,3), & (3,1)^{-1} = (3,5), & (1,5)^{-1} = (5,1),\\ (2,2)^{-1} = (4,4), & (2,4)^{-1} = (4,2), & (3,3)^{-1} = (3,3). \end{array}$$

   The binary operation $\oplus_{2_6}$ is closed, associative, and Abelian over $2JR_6$.

3. Construction of The Finite Sets $\mathbf{2JR}_n$

In this section, we present the theorems that construct the finite sets 2JR_n, and to answer the question, of how we can generate the ordered pairs under the arithmetic modulo n. We need to discuss the cases when n is even and odd separately. In the general concept, and in addition to the related references, we also refer to [7] and [3].

3.1. Constructing $\mathbf{2JR}_n$ when n is Even

Theorem 3.1. *The ordered pair (j,r) belongs to the set $2JR_n$ under the arithmetic modulo n when n is even, if the following conditions are satisfied:*

1. *The sum of j and r is an even number.*
2. *The components j and r are non negative.*

Proof. The first condition is obviously held because it follows the definition of JR-2CN, $j = \alpha + x$ and $r = \alpha - x$, where we easily can see that $j + r$ must be an even number, see [2].

As for the second condition, the case when either component is zero, then the other one must be even to satisfy condition (1). Other than that case, the statement of this condition can be rephrased as:

Every order pair in $2JR_n$ whose either or both components are negative, is congruent modulo n to an ordered pair whose components are positive.

To prove this statement, assume that $2\mathrm{JR}_n$ has an ordered pair (j, r) such that $j < 0$ (Later, the same proof can be applied on the cases where $r < 0$ and $j, r < 0$). Then we have the following:

If j is odd(even): In this case, r must be odd(even) as well to satisfy condition (1). Thus, $j \bmod n$ is an odd(even) positive number, $j \bmod n$ is exactly $j + n$ because j is negative, moreover, $j + n \leq n - 1$. We already have $j + r$ is even number, therefore, $(j + n) + r$ is an even number too, and that means, $(j \bmod n, r) \in 2JR_n$. □

Condition (2) of theorem (3.1) gives an absolute conclusion that every ordered pair in $2\mathrm{JR}_n$ has nonnegative components.

Next theorem is given to ensure that there are no ordered pairs congruent modulo n left in $2\mathrm{JR}_n$ after applying theorem (3.1).

Theorem 3.2. *When n is even, for any two different ordered pairs $(j_k, r_k), (j_l, r_l) \in 2JR_n$, we have $(j_k, r_k) \not\equiv (j_l, r_l) \bmod n$.*

Proof. Assume that there exist $(j_k, r_k), (j_l, r_l) \in 2JR_n$ such that $(j_k, r_k) \neq (j_l, r_l)$ and $(j_k, r_k) \equiv (j_l, r_l) \bmod n$. The only result for $(j_l, r_l) \bmod n$ is (j, r), where j (respectively r) is 0 only if j_l (respectively r_l) is zero, otherwise j and r are negative. Keeping in mind that the components of any ordered pair of any $2\mathrm{JR}_n$ cannot exceed the range $-(n-1)$ to $n-1$. However, this is impossible, when n is even, $2\mathrm{JR}_n$ contains only ordered pairs whose components are nonnegative by condition (2) of theorem (3.1). Thus, there is no $(j_k, r_k) \in 2JR_n$ such that $(j_k, r_k) \equiv (j_l, r_l) \bmod n$ for $(j_k, r_k) \neq (j_l, r_l)$. □

Theorem 3.3. *The order of $2JR_n$ when n is even, is $\frac{n^2}{2}$*

Proof. Basically, the total number of the ordered pairs in $2\mathrm{JR}_n$ must be

$$|2JR_n|_{all\ the\ possible\ ordered\ pairs} = P_2^{2n-1} = (2n-1)^2 \tag{14}$$

This result is followed directly by using the permutations. We have modulo n, and we have the actual set to choose the components from is $\{0, \pm 1, \pm 2, \cdots, \pm(n-1)\}$.

Based on theorem (3.1, condition 2), we can remove safely all the ordered pairs with negative components because they are congruent to ordered pairs whose components are positive. That means the components of the ordered pairs must be picked from the set $\{0, 1, 2, \cdots, (n-1)\}$, so that the permutations is P_2^n. Therefore, the order of $2JR_n$ after eliminating all the ordered pairs whose either or bother components are negative is,

$$|2JR_n|_{ordered\ piars\ with\ positive\ components} = P_2^n = n^2 \tag{15}$$

Now, based on theorem (3.1, condition 1), we have to eliminate each ordered pair (j, r) whose components' summation is odd, which means half of the ordered pairs. Thus, the order of the set is,

$$|2JR_n| = \frac{n^2}{2} \tag{16}$$

□

3.2. Constructing $\mathbf{2JR}_n$ when n is odd

We need the following definition before we start with the theorems.

Definition 3.1. Let $(j_1, r_1), (j_2, r_2) \in 2JR_n$, and let $(j_2, r_2) \equiv (j_1, r_1)\ mod\, n$. We say (j_1, r_1) is a **simplest form** of (j_2, r_2) when at least two of the following statements are hold,

1. j_1 and r_1 are positive,
2. $j_1 < j_2$ or $|j_1| < |j_2|$,
3. $r_1 < r_2$ or $|r_1| < |r_2|$,
4. $|j_1| + |r_1| < |j_2| + |r_2|$,
5. Either or both of j_1 and r_1 are negative and $|j_1| + |r_1| < n$

And in this case (j_2, r_2) must be eliminated.

We would assume that the ordered pair whose components are positive then it is already in the simplest form. Other than that, let us see an instance, under modulo 5, the finite set $2JR_5$ in section (2, item 5), originally, has the ordered pair $(-2, -4)$, but it is congruent modulo 5 to $(3, 1)$. Another example in the same set, the ordered pair $(-4, 2)$ is congruent modulo 5 to $(1, -3)$. From these two examples; we can see that -2 and -4 are congruent to 3 and 1 which they are positive, so we removed $(-2, -4)$ from the set. The same with $(-4, 2)$, we easily can see that 1 and -3 are smaller than -4 and 2 in absolute value. Moreover, the summations of the components in these examples are smaller under the absolute value, and most importantly, the summation of the absolute values of the components of $(3, 1)$ and $(1, -3)$ is less than 5.

Theorem 3.4. *When n is odd, the simplest form of the ordered pair (j, r) in $2JR_n$ satisfies the following conditions:*

1. *The sum of j and r is even number.*
2. *Every ordered pair whose components are negative is congruent modulo n to an ordered pair whose components are positive.*
3. *$2JR_n$ must has ordered pairs whose at least one of the components is negative.*
4. *If either j or r is negative and the summation of their absolute values is grater than n then (j, r) is congruent modulo n to an ordered pair whose summation's absolute values of its components is less than n.*

Proof. The first condition followed the definition of the set JR-2CN where there are no ordered pairs whose components' summation is odd. Now, to prove condition (2), assume that (j, r) is an ordered pair in $2\mathrm{JR}_n$, where both j and r are negative.

If j, r are both odd(even), then, as n is odd, we have both of $j \bmod n$ and $r \bmod n$ are positive even(odd) numbers. Therefore (j, r) is congruent modulo n to an ordered pair whose components are positive and whose components' summation is even.

To prove condition (3), assume that, $(j, r) \in 2JR_n$ with $j < 0, r > 0$ and j is odd. Normally, r would be odd as well to satisfy condition (1). In this case, taking modulo n on j again to force it to be positive, produces the even number $j \bmod n$. But, $j \bmod n + r$ is an odd number which contradicts condition (1). Therefore, when n is odd, $2\mathrm{JR}_n$ must contain ordered pairs in the simplest form with either component is negative. The same argument would be applied when j is even. Probably, one might wonder, why modulo n was applied only on j and not on r to avoid the contradiction with condition (1). The answer for that is easy, if we apply modulo n on both components, then the result again would be an ordered pair with one component is positive, and the other is negative.

To prove condition (4), assume that, $(j, r) \in 2JR_n$ with $|j| + |r| > n$, let $j < 0, r > 0$, that means

$$|j| + r > n. \tag{17}$$

If both component are even, then $j \bmod n$ is an odd positive number. In this case, we must apply modulo n over r to turn it to an odd number so that the components can satisfy condition (1) of the theorem. Therefore, $r \bmod n$ is an odd negative number. All what we need to prove now is $j \bmod n + |r \bmod n| < n$. From equation (17), we have $|j| > n - r$, but, $n - r = |n - r|$, and,

$|n-r| = |r-n|$, therefore, $|j| > |r-n|$. We must notice that $|r-n| = r \bmod n$ because $r > 0$ and it must be kept in the range of $\{-(n-1), (n-1)\}$, so that we have,

$$|r \bmod n| < |j|. \tag{18}$$

Now,

$$\begin{aligned} j \bmod n + |r \bmod n| &< j \bmod n + |j| \\ &= j + n + |j|, \text{ for } j \bmod n = j + n \\ &= n, \text{ for } j + |j| = 0, \text{ therefore,} \\ j \bmod n + |r \bmod n| &< n \end{aligned} \tag{19}$$

□

The only question yet remained unanswered regarding the elements of $2\mathrm{JR}_n$ is- Are there any congruent ordered pairs in $2\mathrm{JR}_n$? The following theorem will prove that there are not.

Theorem 3.5. *When n is odd, for any two different ordered pairs (j_k, r_k), $(j_l, r_l) \in 2JR_n$, we have $(j_k, r_k) \not\equiv (j_l, r_l) \bmod n$*

Proof. Assume that, there exist $(j_k, r_k), (j_l, r_l) \in 2JR_n$ such that $(j_k, r_k) \neq (j_l, r_l)$ and $(j_k, r_k) \equiv (j_l, r_l) \bmod n$, then we have the following cases.

Case 1. The ordered pairs of the form $(j_l, 0)$(respectively $(0, r_l)$). The only possibility we have for this form is when $|j_l|$ (respectively $|r_l|$) is even, otherwise $(j_l, 0)$(respectively $(0, r_l)$) is not in $2\mathrm{JR}_n$, (see the def. of JR-2CN). In this case, $|j_l| \bmod n$ (respectively $|r_l| \bmod n$) is odd number. Which is impossible to be included in $2\mathrm{JR}_n$.

Case 2. The ordered pairs of the form (j_l, r_l) when j_l and r_l are both positive. For this case, $(j_l, r_l) \bmod n$ is an ordered pair whose both components are negative. Which is, again, impossible because we already have excluded these ordered pairs from $2\mathrm{JR}_n$ during the setting up. See theorem (3.4-condition 2).

Case 3. The ordered pairs of the form (j_l, r_l) when $j_l < 0$ and $r_l > 0$ or vise–versa. Of course, we have the liberty to apply the modulo on either component if it is possible. Nevertheless, we cannot do that in this case because, applying the modulo on either component would give one component is even and the other is odd, which is impossible to be included in $2\mathrm{JR}_n$. Therefore, this choice will be excluded. Now, applying the modulo on both components produces an ordered pair not in the simplest form (see theorem (3.4-condition 4)), therefore, this case also cannot be happened.

For the above three cases, $\forall (j_k, r_k), (j_l, r_l) \in 2JR_n$,
$(j_k, r_k) \not\equiv (j_l, r_l) \mod n$ if, $(j_k, r_k) \neq (j_l, r_l)$. □

Theorem 3.6. *The order of 2JR$_n$ when n is odd, is n^2*

Proof. Referring to equation (14), the original number of ordered pairs that might be in 2JR$_n$ is,

$$|2JR_n|_{all\ the\ possible\ ordered\ piars} = (2n-1)^2 \tag{20}$$

(We have used the same argument we used in theorem (3.3)).

First, we have to eliminate every ordered pair that does not belong to JR-2CN, following theorem (3.4), condition (1), it means, eliminating the ordered pairs whose components' summation is odd. Therefore, the first eliminating is,

$$|Ordered\ piars\ not\ in\ JR-2CN| = \frac{(2n-1)^2-1}{2} \tag{21}$$

The second eliminating is by following theorem (3.4), condition (2). We have to eliminate all the ordered pairs whose both components are negative. Again, we have to use the permutations. The first component must be chosen from the set $\{-(n-1), \cdots, -2, -1\}$, there is a place for only one choice, so that we have $P_1^{n-1} = n-1$. As for the second component, must also be chosen from the same set but with avoiding the choices that result odd summation for the components. For instance, if $n = 7$, and $j = -6$, then r must be chosen from $\{-6, -4, -2\}$. If $j = -3$, then r must be chosen from $\{-5, -3, -1\}$. Therefore, the number of the elements in that set of the second components will be $\left(\frac{n-1}{2}\right)$. As long as there is a place for only one choice, then the permutation will be $P_1^{\frac{n-1}{2}} = \frac{n-1}{2}$. As a result $|Ordered\ pairs\ whose\ both\ components\ are\ negative|$

$$\begin{aligned} &= P_1^{n-1} \cdot P_1^{\frac{n-1}{2}} \\ &= \frac{(n-1)^2}{2} \end{aligned} \tag{22}$$

The third eliminating depends on definition (3.1). Basically, we need to eliminate the ordered which are not in the simplest form. Following statement (5) in the definition and theorem (3.4), condition (4), will be enough to apply the eliminating. We need to notice that, the ordered pairs with positive components and the ordered pairs with either component is zero are already in the simplest form, so that we will exclude those from the eliminating process. We already have removed the ordered pairs whose both components are negative and whose components' summation is odd. All what left to be checked are the ordered pairs whose either component is negative. First, we need to know how many ordered pairs fall in that category; we mean the category of the ordered pairs whose non of the components are zero and at least one component is negative. We can follow the same analysis of the second elimination but with some changes. The first component takes its value from the set $\{-(n-1), \cdots, -2, -1\}$, there is a place only for one choice; therefore the permutation is $P_1^{n-1} = n-1$. The second component takes its value from the set $\{1, 2, \cdots, (n-1)\}$ with avoiding the choice that makes the summation of the two components odd, for instance, if $n = 5, j = -2$, then r must take its value from $\{2, 4\}$ when actually, the set is $\{1, 2, 3, 4\}$. Therefore, the set of choices for the second component will be divided by two. Again, there is a place only for one choice, so that, the permutation is $P_1^{\frac{n-1}{2}} = \frac{n-1}{2}$. Whatever the result is, must be multiplied by 2 because whenever (j, r) exists then (r, j) exists too. As a result,

$|Ordered\ pairs\ whose\ either\ components\ is\ negative|$

$$\begin{aligned} &= 2 \cdot P_1^{n-1} \cdot P_1^{\frac{n-1}{2}} \\ &= (n-1)^2 \end{aligned} \tag{23}$$

Now, half these ordered pairs are not in the simplest form, and they are congruent to the other half which they are in the simplest form and follows theorem (3.4), condition (4). Therefore,

$|Ordered\ pairs\ that\ are\ not\ in\ the\ simplest\ form|$

$$= \frac{(n-1)^2}{2} \tag{24}$$

Finally, by subtracting the results of the equations (21–24) from equation (20) we get,

$$|2JR_n| = n^2 \tag{25}$$

□

4. $(2JR_n, \oplus_{2_n})$ is a Finite Abelian Group

As we have seen in section (2), every 2JR_n was Abelian group, for $n = 1, 2, 3, 4$, 5, 6, but what we have shown in that section is only for specific n. In this section, we will show that $(2JR_n, \oplus_{2_n})$ is Abelian group in general for any n.

Proposition (2.1) shows that the binary operation $\oplus_{2_n}$ is well defined, closed, associative and commutative over 2JR_n. What is left to show the identity and there is a unique inverse for every ordered pair in $2\mathrm{JR}_n$. As for the inverses, we have to discuss the case where n is odd and even separately. For that purpose, the proposition below is discussing the case when n is even, and the proof will be mainly concerned with uniqueness of the inverses. As we mentioned above, the closure property has already been proved. However, in order to dismiss any misunderstanding, a quick proof sketch for the closure property is given here. Let $(j, r), (k, s) \in 2JR_n$, then $(j, r) \oplus_{2_n} (k, s)$ *mod* $n \in 2JR_n$ for the following reasons. According the the definition of $2\mathrm{JR}_n$, j and r must be both even or both odd, The same is applied on (k, s). Therefore we have two case to discuss. The first case when all of these components are even(the same would be applied if all of them are odd), then $(j, r) \oplus_{2_n} (k, s)$ *mod* n, again, gives ordered pair with even components in $2\mathrm{JR}_n$. It is worth to mention that the signs of the components will not effect the closuring because *mod* n will keep the components inside the set. For example, under mod 5, we might have $(2, 4) \oplus_{2_5} (0, -2)$ *mod* 5, which results, $(2, 2)$ which is obviously in $2\mathrm{JR}_5$, see the set (13). The second case is when j and r are both even but k and s are both odd (and vise versa). In this case, $(j, r) \oplus_{2_n} (k, s)$ *mod* n gives an ordered pair whose both components are odd. Again, still inside $2\mathrm{JR}_n$. For example, $(4, 0) \oplus_{2_5} (3, -1)$ results $(7, -1)$ under mod 5, gives $(2, 4)$. Of course here we had to apply mod 5 on both components to satisfy the definition of the set. Therefore, $\oplus_{2_n}$ is closed over $2\mathrm{JR}_n$.

Proposition 4.1. *For n is even, $(2JR_n, \oplus_{2_n})$ is finite Abelian group with identity $(0, 0)$ and a unique inverse element $(n-j, n-r) \bmod n$ for every element (j, r).*

Proof. To prove that $(n-j, n-r) \bmod n$ is the inverse element of (j, r). As long as $(j, r) \in 2JR_n$ then j and r are either both even or both odd, keeping in mind that we are dealing with the case when n is even. Thus, if j and r are both even (odd), then $n - j$ and $n - r$ are both even (odd), and $n - j + n - r$ is even and so that it is in $2\mathrm{JR}_n$.

For both two cases, the even and the odd components, $(n - j, n - r) \bmod n$ is the inverse of (j, r), because, $(j, r) \oplus_{2_n} (n - j, n - r) \equiv (0, 0) \bmod n$.

To prove the uniqueness part. The only other ordered pairs that can be inverses to the ordered pair (j, r) are $(-j, -r)$, $(-j, n - r)$ and $(n - j, -r)$. However, all of these ordered pairs are impossible to be in $2JR_n$, since n is even, then every ordered pair in $2\mathrm{JR}_n$ must have its both components are positive as we have shown in sections (2 and 3).

Finally, $(0, 0)$ is the identity element in $2\mathrm{JR}_n$. Therefore, when n is even, $(2JR_n, \oplus_{2_n})$ is finite Abelian group. □

Proposition 4.2. *For n is odd, $(2JR_n, \oplus_{2_n})$ is finite Abelian group with identity $(0,0)$ and a unique inverse element $(n-j, n-r) \bmod n$ for (j,r) if both of j and r are positive. And a unique inverse element $(-j,-r) \bmod n$ for (j,r) if either j or r is negative or zero.*

Proof. It is easy to prove that $(0,0)$ represents the identity element in this group.

To prove that $(n-j, n-r)$ *mod* n is the inverse of (j,r), if both of j and r are positive, then we only need to follow the same argument in the proposition (4.1).

Now, if either j or r is negative or zero, then $(n-j, n-r)$ *mod* n cannot be an inverse because in this case, the ordered pair $(n-j, n-r) \notin 2JR_n$, where the summation of its components modulo n is odd. Moreover, $(n-j, n-r) \equiv (-j,-r) \bmod n$. Therefore, the inverse of (j,r) is $(-j,-r)$, regardless that j and r were both odd or both even.

To prove the uniqueness. For the case of both j and r are positive, the only other possible inverses are $(-j,-r)$, $(-j, n-r)$ and $(n-j,-r)$. However, $(-j,-r) \notin 2JR_n$ and for the other two, the sum of the component modulo n is odd, so that, they are, again, not in $2\mathrm{JR}_n$.

Now, for the case of either j or r is negative or zero, the only other possible inverses are $(-j, n-r)$ and $(n-j,-r)$, but both are, actually, congruent modulo n to $(-j,-r)$, and they were already been removed.

Therefore, when n is odd, $(2JR_n, \oplus_{2_n})$ is finite Abelian group. □

5. Conclusion

In this paper, we construct finite Abelian groups based on the summation of two signed cubic numbers. The method of collecting the points, and constructing the finite groups is explained in details in section two through giving examples for the first six Abelian groups. The theoretical method of collecting the points, in general, for every n in the sets $2\mathrm{JR}_n$ is given in section three. Moreover, proofs are given to determine the order of each set. We proved that every finite set of $2\mathrm{JR}_n$ is Abelian group, we studied the cases when n is odd and even separately.

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