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GEOMETRICAL WAVE EQUATION AND THE CAUCHY-LIKE THEOREM FOR OCTONIONS

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Abstract: Riemann surfaces, cohomology and homology groups, Cartan's spinors and triality, octonionic projective geometry, are all well supported by Complex Structures [1], [2], [3], [4]. Furthermore, in Theoretical Physics, mainly in General Relativity, Supersymmetry and Particle Physics, Complex Theory Plays a Key Role [5], [6], [7], [8]. In this context it is expected that generalizations of concepts and main results from the Classical Complex Theory, like conformal and quasiconformal mappings [9], [10] in both quaternionic and octonionic algebra, may be useful for other fields of research, as for graphical computing enviromment [11]. In this Note, following recent works by the autors [12], [13], the Cauchy Theorem will be extended for Octonions in an analogous way that it has recentely been made for quaternions [14]. Finally, will be given an octonionic treatment of the wave equation, which means a wave produced by a hyper-string with initial conditions similar to the one-dimensional case.

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1. Introduction and Motivation

Regarded as an 8-dimensional mon-associative and non commutative extension of the quaternions, octonions, whose algebra is denoted by O, are standing on the widest possible normed division algebra over the real numbers that can be obtained from the Cayley-Dickson Construction. They are related to a number of symmetries in Mathematics, called exceptional, as the exceptional Lie Groups. In Mathematical and Theoretical Physics, Octonions appear in different issues as in Relativity Theory, Quantum Mechanics, Strings, Supersymmetry, among others fields. Supersymmetry is deeply rooted to division algebras [3]: Non abelian Yang-Mills fields minimally coupled to massles spinors are supersymetry if and only if the dimension of space-time is 3, 4, 6 or 10. The same happened to Green-Scharwarz superstring. Supersymmetry relies on the vanishing of a Certain expression involving a spinor-field, and the reason is the existence of normed division algevras 1, 2, 4 and 8 - real numbers, complex numbers, quaternions and octonions, respectively. Moreover, in General Relativity the Octonionic Geometry (gravity) developed long ago by Colber Oliveira and S. Marques, J. Math. Phys. (1985), ref. [6], has recently been extended to noncommutative and nonassociative Space-time coordinates associated with octonionic-valued coordinates and momenta [8].

Despite being a link together many important phenomena whose connections are still misterious, octonions remanin a remarkable and fascinating mathematical object in their own right.

Attempts to set up an octonionic analogue of the theory of analytic functions, as for instance works on the role of Jordan pairs, Jordan triple Systems and Freedenthal triple Systems in the construction of exceptional Lie Groups, are in progress [5]. Even being non associative and non commutative, it has recentely been obtained that octonions preserve properties of complex numbers like periodicity and regularity [12] and [13].

In this article, as an extension of recent authors' efforts in obtaining a quaternionic version of well known results of classical complex theory, it will be worked out an extension of the Cauchy-Theorem for octonions, on the same way it has been obtained for quaternions. This octonionic case is relevant in this own right and also for eventual applications in Physics and other branches of research[5].

In this regard an octonionic treatment for the wave equation is also given here with. It is indicated that some integrals appearing along the way might be calculated by using the Cauchy Theorem for octonions (Theorem 3, equation (12)).

2. Octonionic Exponential Function

In this section it will be discussed the octonionic exponential function. That is essential for performing the demonstration of the extended Cauchy Theorem for octonions. Let $O = (o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8)$ be an octonionic number. It follows that:

$$e^o = e^{o_1}\{\cos|\vec{o}| + \vec{o}(\frac{\sin|\vec{o}|}{|\vec{o}|})\}, \tag{1}$$

where

$$\vec{o} = o_2 i + o_3 j + o_4 k + o_5 l + o_6 li + o_7 lj + o_8 lk$$

and

$$|\vec{o}| = \sqrt{o_2^2 + o_3^2 + o_4^2 + o_5^2 + o_6^2 + o_7^2 + o_8^2}.$$

Then we can rewrite the

$$e^q = e^{q_1}\{\cos|\vec{q}| + q_2\frac{\sin|q|}{|q|} + q_3\frac{\sin|q|}{|q|} + q_4\frac{\sin|q|}{|q|}\} \tag{2}$$

or yet, using the coordinates notation

$$e^q = (e^{q_1}\cos|\vec{q}|, e^{q_1}q_2\frac{\sin|q|}{|q|}, e^{q_1}q_3\frac{\sin|q|}{|q|}, e^{q_1}q_4\frac{\sin|q|}{|q|}) \tag{3}$$

Once it is an octonion whose coordinates depend on the $\sin|\vec{q}|$ and $\cos|\vec{q}|$, one can investigate the relationship in terms of the computation of $|e^q|$. The result is shown in the theorem below.

Theorem 1. *If e^q is a octonionic exponential function, then*

$$|e^q| = e^{q_1} \tag{4}$$

Demonstration: By the definition of an exponential function of octonionic type

$$|e^o| = |e^{o_1}||\{\cos|\vec{o}| + \vec{o}(\frac{\sin|\vec{o}|}{|\vec{o}|})\}| \tag{5}$$

$$|e^o| = |e^{o_1}|\{\cos^2|\vec{o}| + \sum_{n=2}^{8} q_n^2(\frac{\sin|\vec{o}|}{|\vec{o}|})^2\} \tag{6}$$

$$|e^o| = |e^{o_1}|\{\cos^2|\vec{o}| + (o_2^2 + o_3^2 + o_4^2 + o_5^2 + o_6^2 + o_7^2 + o_8^2)(\frac{\sin|\vec{o}|}{|\vec{o}|})^2\} \tag{7}$$

$$|e^o| = |e^{o_1}|\{\cos^2|\vec{o}| + |\vec{o}|^2(\frac{\sin|\vec{o}|}{|\vec{o}|})^2\} \tag{8}$$

$$|e^o| = |e^{o_1}|\{\cos^2|\vec{o}| + \sin^2|\vec{o}|\} \tag{9}$$

But $|\vec{o}|$ *is a real number, and it follows that* $\cos^2|\vec{o}| + \sin^2|\vec{o}| = 1$. *Then*

$$|e^o| = e^{o_1}.$$

Note that $|e^{o_1}|$ *is a real number.*

It becomes important to notice that the octonionic exponential function can be written as follows:

$$e^o = \rho\{\cos|\vec{o}| + \vec{o}(\frac{\sin|\vec{o}|}{|\vec{o}|})\} = \rho e^{\vec{o}} \tag{10}$$

3. Cauchy-Like Theorem for Octonions

Let us write, for instance, the classical Cauchy Theorem of Complex Analysis.

Theorem 2. *(Cauchy's Integral Formula) Let f(z) be analytic and univocal in a single connected domain D, then for any point* z_0 *and any closed path in D that encloses* z_0

$$\int_C \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0) \tag{11}$$

That encloses z_0*, then the following integral holds.*

Now, we shall proceed in order to obtain a Cauchy-like Theorem for octonions. It will be performed in similar way to the quaternionic case [14].

Theorem 3. *(Cauchy Integral formula for octonions) Be* Ω *a domain simply connected inside a eight dimensional space and* $f(o)$ *a regular function in* Ω*. Then*

$$\int_\varphi \frac{f(o)}{o - o_0} dq = \pi(i + j + k + l + li + lj + 2lk) f(o_0) \tag{12}$$

Where φ *is a single closed hypersurface in* Ω *and* o_0 *is any point in* φ.

Demonstration: Either φ_0 *a hypersphere with center at the point* 0_0*, i.e.;* $|o - o_0| = r_0$*, where* r_0 *is small enough as to allow* φ_0 *be contained inside* φ*. The function* $\frac{f(o)}{o - o_0}$ *is regular in* Ω/o_0*, it follows that*

$$\int_\varphi \frac{f(q)}{q - q_0} dq = \int_{\varphi_0} \frac{f(q)}{q - q_0} dq \tag{13}$$

Using the identity $f(o) = f(o_0) + f(o) - f(o_0)$, *it follows that*

$$\int_{\varphi} \frac{f(o)}{o - o_0} dq = \int_{\varphi_0} \frac{f(o)}{o - o_0} dq$$

$$\int_{\varphi} \frac{f(o)}{o - o_0} dq = \int_{\varphi_0} \frac{f(o_0)}{o - o_0} dq + \int_{\varphi_0} \frac{f(o) - f(o_0)}{o - o_0} do$$

$$\int_{\varphi} \frac{f(o)}{o - o_0} dq = f(o_0) \int_{\varphi_0} \frac{do}{o - o_0} + \int_{\varphi_0} \frac{f(o) - f(o_0)}{o - o_0} do$$

The octonion $o - o_0$ *can be written as*

$$e^o = \rho e^{\vec{o'}}$$

where $\vec{o'} = (o_2 - o'_2, o_3 - o'_3, o_4 - o'_4, o_5 - o'_5, o_6 - o'_6, o_7 - o'_7, o_8 - o'_8)$. *Now writing*

$$\vec{o'} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7) = \theta_1 i + \theta_2 j + \theta_3 k + \theta_4 l + \theta_5 li + \theta_6 lj + \theta_7 lk$$

$e^o = e^{o_1} e^{\theta_1 i+\theta_2 j+\theta_3 k+\theta_4 l+\theta_5 li+\theta_6 lj+\theta_7 lk}$, *then we have* $o - o_0$ *can be written as*

$$o - o_0 = \rho e^{\theta_1 i+\theta_2 j+\theta_3 k+\theta_4 l+\theta_5 li+\theta_6 lj+\theta_7 lk}$$

where $\rho > 0$, $-\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}$, $-\frac{\pi}{2} < \theta_2 < \frac{\pi}{2}$, $-\frac{\pi}{2} < \theta_3 < \frac{\pi}{2}$, $-\frac{\pi}{2} < \theta_4 < \frac{\pi}{2}$, $-\frac{\pi}{2} < \theta_5 < \frac{\pi}{2}$, $-\frac{\pi}{2} < \theta_6 < \frac{\pi}{2}$ *and* $0 < \theta_7 < 2\pi$. *In order to solve the first integral that appears in the second member of* (17) *we proceed to change the variable* $u = o - o_0$. *Thus*

$$\begin{aligned} du = do = d(o - o_0) &= d(\rho e^{\theta_1 i+\theta_2 j+\theta_3 k+\theta_4 l+\theta_5 li+\theta_6 lj+\theta_7 lk}) \\ &= \rho d(e^{\theta_1 i+\theta_2 j+\theta_3 k+\theta_4 l+\theta_5 li+\theta_6 lj+\theta_7 lk}), \end{aligned}$$

where

$$\begin{aligned} & d(e^{\theta_1 i+\theta_2 j+\theta_3 k+\theta_4 l+\theta_5 li+\theta_6 lj+\theta_7 lk}) \\ &= e^{\theta_1 i+\theta_2 j+\theta_3 k+\theta_4 l+\theta_5 li+\theta_6 lj+\theta_7 lk}(d\theta_1 i+d\theta_2 j+d\theta_3 k+d\theta_4 l+d\theta_5 li+d\theta_6 lj+d\theta_7 lk). \end{aligned}$$

Applying that result in the first right hand integral:

$$\begin{aligned} & \int_{\varphi_0} \frac{do}{o - o_0} \\ &= \int_{\varphi_0} \frac{\rho e^{\theta_1 i+\theta_2 j+\theta_3 k+\theta_4 l+\theta_5 li+\theta_6 lj+\theta_7 lk}}{\rho e^{\theta_1 i+\theta_2 j+\theta_3 k+\theta_4 l+\theta_5 li+\theta_6 lj+\theta_7 lk}}(d\theta_1 i+d\theta_2 j+d\theta_3 k+d\theta_4 l+d\theta_5 li+d\theta_6 lj+d\theta_7 lk) \end{aligned}$$

$$\int_{\varphi_0} \frac{do}{o-o_0} = \int_{\varphi_0} (d\theta_1 i + d\theta_2 j + d\theta_3 k + d\theta_4 l + d\theta_5 li + d\theta_6 lj + d\theta_7 lk)$$

$$\int_{\varphi_0} \frac{do}{o-o_0} = i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\theta_1 + j\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\theta_2 + k\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\theta_3 + l\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\theta_4 + li\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\theta_5 + lj\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\theta_6 + lk\int_{0}^{2\pi} d\theta_3$$

$$\int_{\varphi_0} \frac{do}{o-o_0} = \pi(i+j+k+l+li+lj+2lk).$$

On the other hand using the fact that f *be continuous at a point* q, *for a given* $\varepsilon > 0$, *exists* $\delta > 0$ *such that* $|o-o_0| < \delta$, *what implicates that* $|f(o)-f(o_0)| < \varepsilon = \frac{\varepsilon_0}{10\pi}$

$$|\int_{\varphi_0} \frac{f(o)-f(o_0)}{o-o_0} do| = \int_{\varphi_0} \frac{|f(o)-f(o_0)|}{|o-o_0|}|do| \quad (14)$$

$$|\int_{\varphi_0} \frac{f(q)-f(q_0)}{o-o_0} do| < \varepsilon|\int_{\varphi_0} (id\theta_1) + jd\theta_2 + kd\theta_3| = \varepsilon|i+j+2k| \quad (15)$$

$$|\int_{\varphi_0} \frac{f(q)-f(q_0)}{o-o_0} do| < 10\pi\varepsilon = \frac{10\pi}{10\pi}\varepsilon_0 = \varepsilon_0. \quad (16)$$

Once ε *can be taken with a value as small as we wish, it follows that*

$$\int_{\varphi_0} \frac{f(o)-f(o_0)}{o-o_0} dq = 0$$

Then,

$$\int_{\varphi} \frac{f(o)}{o-o_0} do = \pi(i+j++k+l+li+lj+2lk)f(o_0). \quad (17)$$

This completes the proof.

4. Wave Equation in the Octonionic Case

Extensions of the wave equation for more general cases are important with regard to applications to modern physical theories such as M-theory, op. cit. [3],[5]. Therefore, it is necessary to show that a general case, when particularized to an extent, lies in the classical solution well known in the literature [15].

Let us for instance consider the following general case for an octonionic function $u(t, \vec{x})$, where t is a scalar which may be associated with the Physical time, and $\vec{x}$ is a spatial vector lying in the space octonions. For good reasons, we shall regard the method of separation of variables for the treatment of $u(t, \vec{x})$ and its derivatives in the context of a general wave equation. The following statements will be made:

(i) $u(t, \vec{x}) = T(t) * \vec{X}(\vec{x}) : R \times R^7 \to O$ (octonionic space); $u(t, \vec{x})$ is the octonionic function, and $*$ the octonionic product. $T(t)$, $\vec{X}(\vec{x})$ are functions depending only in t and $\vec{x}$, respectively;

(ii) $u(t, \vec{x}) = u(F(t), \vec{x})$;

(iii) The derivatives of $\vec{u}$ are given by

$$\frac{\partial^2 u(t, \vec{x})}{\partial \vec{x}^2} = T(t) * \vec{X}''(\vec{x})$$

and

$$\frac{\partial^2 u(t, \vec{x})}{\partial T^2} = T''(t) * \vec{X}(\vec{x})$$

(iv) For simplicity is considered here, $u(t, \vec{x}) = T(t) * \vec{X}(\vec{x}) = T(t) \cdot \vec{X}(\vec{x})$.

Considering now the classical wave equation for the case of one dimension, it follows that:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{18}$$

Moreover, the initial conditions for the problem in question are given below:

$$u(0,t) = 0 \tag{19}$$

$$u(L,t) = 0 \tag{20}$$

$$u(x,0) = f(x) \tag{21}$$

$$\frac{\partial u(x,0)}{\partial t} = g(x) \tag{22}$$

the conditions given in (19) and (20) requires that the string is fixed at the origin and $x = L$. The conditions given in (21) and (22), show the initial deflection of the rope and the initial velocity in instance $t = 0$.

The equation (18), can be solved by separation of variables, or Method of Fourier. Thus, it follows that:

$$u(x,t) = X(x) \cdot T(t) \tag{23}$$

.	1	i	j	k	l	li	lj	lK
1	1	i	j	k	l	li	lj	lk
i	i	-1	k	$-j$	$-li$	l	$-lk$	lj
j	j	$-k$	-1	i	$-lj$	lk	l	$-li$
k	k	j	$-i$	-1	$-lk$	$-lj$	li	l
l	l	li	lj	lk	-1	$-i$	$-j$	$-k$
li	li	$-l$	$-lk$	lj	i	-1	$-k$	j
lk	lk	$-lj$	li	l	k	$-j$	i	-1

Table 1: Octonionic multiplication table

which leads to the solution of (18) which is given by:

$$u(x,t) = \sum_{n=1}^{\infty}(C_n \cos\frac{an\pi}{L}t + D_n \sin\frac{an\pi}{L}t)\sin\frac{n\pi}{L}x \tag{24}$$

with values of C_n and D_n given by:

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}xdx \tag{25}$$

and

$$D_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi}{L}xdx. \tag{26}$$

determined using the initial conditions (21) and (22).

The problem now is to solve the equation (18) for the octonionic case that means (18) is given by:

$$\frac{\partial^2 u(t,\vec{x})}{\partial t^2} = a^2\frac{\partial^2 u(t,\vec{x})}{\partial \vec{x}^2}, \tag{27}$$

where,

$$u(t,\vec{x}) = u(t, x_1 i + x_2 j + x_3 k + x_4 l + x_5 li + x_6 lj + x_7 lk)$$

is a function of variables t and $\vec{x}$, and (27) is the called wave equation, and $1, i, j, k, l, li, lj$, and lk follow the octonionic multiplication table given by Table 1:

Performing now the separation of variables, there is a product of a scalar by a vector function, given by:

$$u(t,\vec{x}) = T(t) * \vec{X}(\vec{x}) = T(t)\cdot\vec{X}(\vec{x}) \tag{28}$$

Then,

$$\frac{\partial^2 u(t,\vec{x})}{\partial t^2} = T''(t)\vec{X}(\vec{x})$$

and $\frac{\partial^2 u(t,\vec{x})}{\partial \vec{x}^2} = T(t)\vec{X}''(\vec{x})$, with initial conditions of the problem:

$$u(t,\vec{0}) = 0 \tag{29}$$

$$u(t,L,0,0,0,0,0,0) = 0 \tag{30}$$

$$u(t,0,L,0,0,0,0,0) = 0 \tag{31}$$

$$u(t,0,0,L,0,0,0,0) = 0 \tag{32}$$

$$u(t,0,0,0,L,0,0,0) = 0 \tag{33}$$

$$u(t,0,0,0,0,L,0,0) = 0 \tag{34}$$

$$u(t,0,0,0,0,0,L,0) = 0 \tag{35}$$

$$u(t,0,0,0,0,0,0,L) = 0 \tag{36}$$

$$u(0,\vec{x}) = f(\vec{x}), \tag{37}$$

$$\frac{\partial u(0,\vec{x})}{\partial t} = g(\vec{x}) \tag{38}$$

where the condition (29) fixed hyper-string at the origin, while the conditions given in (30) − (36) show that the hyper-rope is fixed at the ends of L. The conditions (37) and (38) relate to initial deflection and initial velocity respectively.

Using the method of separation of variables in (27), we have that:

$$\frac{T''(t)}{a^2T(t)} = \frac{\vec{X}''(\vec{x})}{\vec{X}(\vec{x})} = -\lambda, \tag{39}$$

obtaining the differential equations:

$$\vec{X}''(\vec{x}) + \lambda\vec{X}(\vec{x}) = 0 \tag{40}$$

$$T''(t) + \lambda a^2 T(t) = 0. \tag{41}$$

The equation (40) is of vector type, while the equation (41) is an ordinary differential equation in the variable t. The solutions are given by:

$$\vec{X}(\vec{x}) = A\cos\sqrt{\lambda}\vec{x} + B\sin\sqrt{\lambda}\vec{x} \tag{42}$$

$$T(t) = C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t \tag{43}$$

with A, B, C and D arbitrary constants.

Using the initial conditions of the problem and the treatment for trigonometric functions seen in [12], and considering that the function $T(t) \neq 0$, it follows that:

$$u(t,\vec{x}) = (A\cos\sqrt{\lambda}\vec{x} + B\sin\sqrt{\lambda}\vec{x})(C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t) \tag{44}$$

and

$$u(t,\vec{0}) = A\cos\sqrt{\lambda}0 + B\sin\sqrt{\lambda}0 = 0 \tag{45}$$

the conditions $(30) - (36)$ are always equal to

$$u(t,L_i) = A\cos\sqrt{\lambda}L + B\sin\sqrt{\lambda}L = 0, \quad i = 1,\dots,7; \tag{46}$$

to $A \neq 0$, it follows that:

$$B\sin\sqrt{\lambda}L = 0 \tag{47}$$

$$\sqrt{\lambda} = \frac{n\pi}{L}. \tag{48}$$

Having now

$$\vec{X}(\vec{x}) = B\sin\frac{n\pi}{L}\vec{x} \tag{49}$$

in $T(t)$,

$$T(t) = C\cos\frac{n\pi}{L}t + D\sin\frac{n\pi}{L}t. \tag{50}$$

Defining now the function $u_n(t,\vec{x})$ given by:

$$u_n(t,\vec{x}) = \sin\frac{n\pi}{L}x(C_n\cos\frac{an\pi}{L}t + D_n\sin\frac{an\pi}{L}t), \tag{51}$$

where each value of C and D are given depending on n. Thus,

$$u_n(t,\vec{x}) = \sum_{n=1}^{\infty}(C_n\cos\frac{n\pi a}{L}t + D_n\sin\frac{n\pi a}{L}t)\sin\frac{n\pi}{L}\vec{x}. \tag{52}$$

Using the initial condition (37) and (38), C_n and D_n are given by:

$$u(0,\vec{x}) = \sum_{n=1}^{\infty}C_n\sin\frac{n\pi}{L}\vec{x} = f(\vec{x}), \tag{53}$$

where

$$C_n = \frac{2}{L}\int_{\Lambda} f(\vec{x})\sin\frac{n\pi}{L}\vec{x}d\vec{x}. \tag{54}$$

Moreover,

$$\frac{\partial(0,\vec{x})}{\partial t} = \frac{n\pi a}{L} D_n \sin\frac{n\pi}{L}\vec{x} = 0 \tag{55}$$

with,

$$D_n = \frac{2}{an\pi}\int_\Lambda g(\vec{x})\sin\frac{n\pi}{L}\vec{x}d\vec{x}, \tag{56}$$

where the integrals are calculated from the initial point $A = (0,0,0,0,0,0,0)$ to $B = (L,L,L,L,L,L,L)$.

5. Conclusion

In this article it is shown that the equation obtained in (17) , is similar to the well known formula of the theory of a Complex Variable (11).This equation which is used for the determination of octonionic derivatives [16], is relevant for the treatment of the wave equations of octonionic types which appear in Quantum Mechanics.

In this context it is given besides the Cauchy type integral, a solution to the wave equation of octonionic type where the integrals appearing in (55) and (57) can be calculated through the equation (17).

On the other hand, constants C_n and D_n if you use the one-dimensional case, are given as in the classical case, resulting the same general solution found in the literature [15].

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