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REMARKS ON $K(l)_*SO(2^{l+1}+1)$

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Abstract: We study the connected and periodic l-th Morava K-theories of $SO(2^{l+1}+1)$, using the methods of the previous papers.

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1. Introduction and Preliminaries

In earlier papers, the additive structure of the (periodic) Morava K-theories of special orthogonal groups was calculated and some progress was made analyzing the algebra structure. However, this is difficult due the problems in passing between ordinary homology and periodic theories. It might be better to use the connective theories as an intermediary. In an earlier paper ([9]), we studied $K(2)_*SO(9)$ and its connective version $k(2)_*SO(9)$. In this paper, we generalize the results to $K(l)_*SO(2^{l+1}+1)$ and $k(l)_*SO(2^{l+1}+1)$.

Throughout this paper, we work at the prime 2. In particular, BP will denote the 2-primary Brown-Peterson theory and $k(l)$ and $K(l)$ will denote respectively the connective and periodic lth Morava K-theories at the prime 2. Also, H_*X will denote the ordinary homology of X with $\mathbb{Z}/(2)$ coefficients.

Next, we recall some of the earlier results and set up our notation.

The source for this paragraph is [3]. $G_n = SO(n+2)/(SO(2)\times SO(n))$, the generating variety for the homology of $\Omega_0 SO(n+2)$, has torsion-free homology [1]. Note that $G_\infty = \mathbb{C}P^\infty$. Let x be the image of the standard gen-

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erator of $MU^*\mathbb{C}P^\infty$ in MU^*G_n. Then $MU\mathbb{Q}^*G_{2n-1} = MU\mathbb{Q}^*[x]/(x^{2n})$. Let $\{\beta_0', \beta_1, \dots \beta_{2n-1}\}$ be the basis of $MU\mathbb{Q}_*G_{2n-1}$ that is dual to $\{1, x, \dots, x^{2n-1}\}$. Let $\beta_0 \in \widetilde{MU}_0\Omega SO(2n+1)$ be the unique element such that $\beta_0^2 = 2\beta_0$. Define $\alpha_i = \sum_{j=0}^{i} c_{i-j}\beta_j$ where c_j is the coefficient of t^{j+1} in the MU-[2]-series. If $1 \le i \le 2n-1$ and $1 \le j \le n-1$, then α_i and β_j are in $\widetilde{MU}_*\Omega SO(2n+1)$.

If h is an MU-algebra theory, we will denote the images of the β's and α's in $h_*\Omega SO(2n+1)$ by the same symbols. These elements are independent of n in the sense that if $0 \le i < n < q$ and $j < 2n$, then the map $h_*\Omega SO(2n+1) \to h_*\Omega SO(2q+1)$ induced by inclusion preserves β_i and α_j. The image of u under homology suspension $\widetilde{h}_*\Omega SO(2n+1) \to h_{*+1}SO(2n+1)$ will be denoted by $\overline{u}$.

Fix a ground ring of characteristic 2. Let $\Gamma_k(u)$ denote the divided power algebra of height k on u. This is the dual of the primitively generated truncated polynomial algebra $R[v]/(v^{2^k})$. The full divided power algebra will be denoted by $\Gamma(u)$. The j-th divided power of u will be denoted by $\gamma_j(u)$.

We need some facts concerning the bar spectral sequence; for the details, see [4]*Section 3. If G is a compact connected Lie group and h a BP-algebra theory, then the bar spectral sequence

$$E^2_{**}(G,h) = \mathrm{Tor}^{h_*\Omega G}_{**}(h_*, h_*) \Rightarrow h_*G$$

is a spectral sequence of commutative algebras. If $E^r_{**}(G,h)$ is free over h_* for all r, then it a spectral sequence of bicommutative, biassociative Hopf algebras.

By [4]*Theorem 1.1, if $l \le s \le \infty$, then

$$E^\infty_{**}\left(SO(2^{l+1}-1), P(s)\right) = \bigotimes_{i=0}^{2^l-2} \Gamma_{p(i)+1}(\overline{\beta}_i)$$

where $p(i)$ is defined by $2^l \le 2^{p(i)}(2i+1) < 2^{l+1}$. There are no Hopf algebra extension problems if $s > l$.

Hence there are unique elements $\gamma_{ij} \in P(l)_{2j(2i+1)}SO(2^{l+1}-1)$ which project to the $(2j)$th divided power of $\overline{\beta}_i$ in $H_{2j(2i+1)}SO(2^{l+1}-1)$ for $0 \le i \le 2^l-2$, $1 \le j < 2^{p(i)}$. We will use the same symbols to denote the images of these elements in $h_*SO(N)$ for $N \ge 2^{l+1}$, where h may be $P(l)$, $k(l)$ or $K(l)$.

Let A be an algebra over a commutative ring R. A is said to be simply generated by $g_1, g_2, \dots$ over R if A is a free R-module with basis

$$\{1\} \cup \{g_{i_1}g_{i_2}\dots g_{i_s} \mid i_1 < i_2 < \dots < i_s\}.$$

As we do not assume that A is commutative, the order of the elements may be important.

2. The Calculations

Given elements α and β of bidegrees $(c, 2^k(a+b)-c-1)$ and (a,b) respectively, define a spectral sequence of Hopf algebras by

$$E^r_{**}(\beta,\alpha) = \begin{cases} E(\alpha)\otimes\Gamma(\beta) & r \le 2^k a - c \\ \Gamma_k(\beta) & r > 2^k a - c \end{cases}$$

$$d^r(\gamma_j(\beta)) = \begin{cases} 0 & r \ne 2^k a - c \text{ or } j < 2^k \\ \alpha\gamma_{j-2^k}(\beta) & r = 2^k a - c \text{ and } j \ge 2^k. \end{cases}$$

Define a spectral sequence of algebras with E^r-term given by

$$\begin{array}{lr} E(\overline{\alpha}_{2^{l+1}-1})\otimes E(\overline{\beta}_0)\otimes\Gamma(\gamma_{0,1}) & 2 \le r < 2^{l+1} \\ E(\overline{\alpha}_{2^{l+1}-1})\otimes\Gamma_l(\gamma_{0,1})\otimes P(l)_*\langle 1, \overline{\beta}_0\gamma_{0,2^l j}\rangle/(v_l\overline{\beta}_0\gamma_{0,2^l j}=0) & \\ & 2^{l+1} \le r < 2^{l+2} \\ \Gamma_l(\gamma_{0,1})\otimes P(l)_*\langle 1, \overline{\beta}_0.\overline{\alpha}_{2^{l+1}-1}, \overline{\beta}_0\gamma_{0,2^l}\rangle/(v_l\overline{\beta}_0=0, v_l\overline{\beta}_0\gamma_{0,2^l}=0) & \\ & 2^{l+2} \le r \le \infty \end{array}$$

Here $\overline{\beta}_0$ has bidegree $(1,0)$, $\gamma_{0,j}$ continues $\gamma_{2j}(\overline{\beta}_0)$ and has bidegree $(0,2j)$, and $\overline{\alpha}_{2^{l+1}-1}$ has bidegree $(1, 2^{l+2}-2)$. The non-trivial differentials are determined by

$$d^{2^{l+1}-1}(\gamma_{0,2^l j}) = v_l\overline{\beta}_0\gamma_{0,2^l(j-1)}$$
$$d^{2^{l+2}-1}(\overline{\beta}_0\gamma_{0,2^l j}) = \overline{\alpha}_{2^{l+1}-1}\overline{\beta}_0\gamma_{0,2^l(j-2)}$$

Proposition 2.1. *The bar spectral sequence converging to* $k(l)_*SO(2^{l+1}+1)$ *is given by*

$$\bigotimes_{i=2^{l-1}}^{2^l} E(\overline{\beta}_i) \quad\otimes\quad \bigotimes_{i=1}^{2^{l-1}-2} E^*_{**}(\overline{\beta}_i, \overline{\alpha}_{2^{p(i)}(2i+1)}) \quad\otimes\quad \widetilde{E}^*_{**}$$

Proof. Using the description in [4]**pp.* 54–55, we see that the Petrie complex for $k(l)_*\Omega SO(2^{l+1}+1)$ has trivial differential and is same as the E^2-term of the spectral sequence above. Thus the description holds at the E^2-term. Also, for degree reasons, $d^{2k}=0$ for all k. Also, note that all differentials must vanish on $\overline{\beta}_i$ and $\overline{\alpha}_j$.

By [4]*Theorem 1.1, the Bss converging to $P(l)_*SO(2^{l+1}-1)$ is

$$\bigotimes_{i=2^{l-1}-1}^{2^l} E(\overline{\beta}_i) \quad \otimes \quad \bigotimes_{i=0}^{2^{l-1}-2} E^*_{**}(\overline{\beta}_i, \overline{\alpha}_{2^{p(i)}(2i+1)})$$

Mapping this into the Bss converging to $P(l)_*SO(2^{l+1}+1)$, we see that the differentials on $\gamma_j(\overline{\beta}_i)$ are as described above for $i>0$, and that $d^r(\gamma_j(\overline{\beta}_0))=0$ for $2\le r<2^{l+1}-1$.

Using the fact that $\alpha_{2^l-1}=v_l\beta_0$ in $P(l)_*\Omega SO(2^{l+1}-1)$ and the values of $d^{2^{l+1}-1}$ on the Bss for $P(l)_*SO(2^{l+1}-1)$, we see that $d^{2^{l+1}-1}(\gamma_{0,j})=v_l\overline{\beta}_0\gamma_{0,j-2^l}$. It follows that the $E^{2^{l+1}}$-term is as described.

Note that the reduction to mod-2 homology maps the submodule of $E^{2^{l+1}}_{**}$ generated by $\langle\overline{\beta}_0\gamma_{0,2^li}\rangle$ monomorphically into the mod-2 homology Bss. It follows that $d^r=0$ for $2^{l+1}\le r<2^{l+2}-1$, and that $d^{2^{l+2}-1}(\overline{\beta}_0\gamma_{0,2^lj})=\overline{\beta}_0\overline{\alpha}_{2^{l+1}-1}\gamma_{0,2^l(j-2)}$. An easy calculation shows that the $E^{2^{l+2}}$-term is as described. For degree reasons, all further differentials must be trivial. □

It follows from [8]*Theorem 3.1 that there is a unique element ζ_{2^l} in $k(l)_*SO(2^{l+1}+1)$ whose reduction to mod-2 homology is $\overline{\beta}_0\gamma_{0,2^{l+1}}$ and which satisfies $v_l\zeta_{2^l}=\overline{\alpha}_{2^{l+1}-1}$.

Proposition 2.2. *The following elements simply generate a subalgebra of $k(l)_*SO(2^{l+1}+1)$:*

$$\{\overline{\beta}_1,\overline{\beta}_2,\ldots,\overline{\beta}_{2^l-1}\}\cup\{\gamma_{i,2^s}\mid 0\le i<2^{l-2}, 1\le s<p(i)\}$$
$$\cup\{\gamma_{i,2^{p(i)}}\mid 0\le i<2^{l-2}\}$$

As a right module over that subalgebra, $k(l)_*SO(2^{l+1}+1)$ is generated by 1, $\overline{\beta}_0$ and ζ_{2^l} subject to the relation $v_l\overline{\beta}_0=0$.

Note. The elements of the first set commute with each other, as do the elements of the second set. But other commutators need not be trivial. For definiteness, we will consider the elements ordered as listed.

Proof. Combining the above calculation of the Bss with the fact that ζ_{2^l} projects to $\overline{\beta}_0\gamma_{0,2^l}$ in $H_*SO(2^{l+1}+1)$ implies that ζ_{2^l} is represented by $\overline{\beta}_0\gamma_{0,2^l}$ in the Bss. As $\alpha_{2^l-1}=v_l\beta_0$ in $P(l)_*\Omega SO(2^{l+1}+1)$ and $\overline{\alpha}_{2^l-1}=0$ in $P(l)_*SO(2^{l+1}-1)$, $v_l\overline{\beta}_0=0$ in $k(l)_*SO(2^{l+1}+1)$. Finally $v_l\zeta_{2^l}=\overline{\alpha}_{2^{l+1}-1}$. Comparison with $k(l)_*SO(2^{l+1}-1)$ shows that these relations solve all additive extension problems in the Bss converging to $k(l)_*SO(2^{l+1}+1)$. □

Recall that for finite complexes X and Y,

$$(k(l)_*X/v_l\text{-torsion}) \otimes (k(l)_*Y/v_l\text{-torsion}) \to k(l)_*(X \times Y)/v_l\text{-torsion}$$

is an isomorphism. Since $k(l)$ is not commutative, this is not enough to make $k(l)_*X/v_l$-torsion a Hopf algebra when X is an H-space. However, the obstruction to the commutativity of $k(l)$ is divisible by v_l. Adapting the discussion in [6]*Section 1, we can show that if X is an H-space, then $(k(l)_*X/v_l\text{-torsion}) \otimes \mathbb{Z}/2$ is a cocommutative Hopf algebra. Alternatively, note that

$$(k(l)_*X/v_l\text{-torsion}) \otimes \mathbb{Z}/2$$

is the E^∞-term of the Bockstein spectral sequence, and this is a spectral sequence of Hopf algebras if X is an H-space [2]*p. 116.

The next lemma is a consequence of Theorem 3.1 and Lemma 3.4 of [8].

Lemma 2.3. *As Hopf algebras*

$$(k(l)_*SO(2^{l+1}+1)/v_l\textit{-torsion}) \otimes \mathbb{Z}/2 \cong E(\zeta_{2^l}) \otimes \Gamma_l(\gamma_{0,1}) \otimes E(\{\overline{\beta}_{2^l-1}) \otimes \bigotimes_{j=1}^{2^l-2} \Gamma_{p(j)+1}(\overline{\beta}_j)$$

Lemma 2.4. *The module of primitives of $K(l)_*SO(2^{l+1}+1)$ is free on the basis $\{\overline{\beta}_j \mid 1 \le j < 2^l\} \cup \{\zeta_{2^l}, \gamma_{01}\}$.*

Proof. The listed elements are independent by 2.2. Since the homology suspension of any element of $BP_*\Omega SO(2^{l+1}+1)$ is primitive, $\overline{\beta}_i$'s are primitive. So is $\overline{\alpha}_{2^l-1}$, and hence $\zeta_{2^l} = v_l^{-1}\overline{\alpha}_{2^{l+1}-1}$ is also primitive. Finally, γ_{01} is primitive because $\Delta_*(\gamma_{01})$ equals $\gamma_{01} \otimes 1 + \overline{\beta}_0 \otimes \overline{\beta}_0 + 1 \otimes \gamma_{01}$ in $k(l)_*SO(2^{l+1}-1)$, and $\overline{\beta}_0$ is v_2-torsion in $k(l)_*SO(2^{l+1}+1)$.

Denote by A the $k(l)_*$-module generated by by the elements listed in the statement of the lemma, and by $\overline{\Delta}$ the reduced diagonal $SO(2^{l+1}+1) \to SO(2^{l+1}+1) \wedge SO(2^{l+1}+1)$.

Let x be a primitive of $K(l)_*SO(2^{l+1}+1)$. We can write $x = v_l^s y$, where y is the image of an element in $k(l)_*SO(2^{l+1}+1)$ that is not divisible by v_l and $\overline{\Delta}_*y$ is v_l-torsion. We will show by induction on its degree that y is the sum of an element of A and a v_l-torsion element. Note that this is vacuous if the degree of y is less than two.

The image of y in $(k(l)_*SO(2^{l+1}+1)/v_l\text{-torsion}) \otimes \mathbb{Z}/2$ is primitive. So by Lemma 2.3, it is in the image of A. Hence we can write $y = y' + v_l^t z + y''$ where

$y' \in A$, $t \geq 1$, $z \in k(l)_*SO(2^{l+1}+1)$ is not divisible by v_l, and y'' is v_l-torsion. Then z must be primitive in $K(l)_*SO(2^{l+1}+1)$, that is $\overline{\Delta}_* z$ must be v_l-torsion. By our induction assumption, $z \in A$. □

The values of the Milnor Bocksteins Q_{l-1} on the generators listed in 2.2, except for $Q_{l-1}(\zeta_l)$, follow from [8]*p. 427.

Lemma 2.5. $Q_{l-1}(\zeta_{2^l}) = \overline{\beta}_0 \overline{\beta}_{2^{l-1}}$.

Proof. Now $Q_{l-1}(v_l \zeta_{2^l}) = Q_{l-1}(\overline{\alpha}_{2^l-1}) = 0$ because $\overline{\alpha}_{2^l-1}$ comes from $BP_*SO(2^{l=1}+1)$. So $Q_{l-1}(\zeta_{2^l})$ is v_l-torsion. In $H_*SO(2^{l+1}+1)$, $Q_{l-1}(\zeta_{2^l}) = Q_{l-1}(\overline{\beta}_0 \gamma_{0,2^l}) = \overline{\beta}_0(\overline{\beta}_0 \gamma_{0,2^l} + \overline{\beta}_{2^{l-1}})$. □

The commutators of the generators in 2.2 except those involving $\overline{\beta}_{2^l-1}$ or ζ_{2^l} follow from the results of [8]. Commutators involving ζ_{2^l} will be studied in a latter paper.

Proposition 2.6. *In $K(l)_*SO(2^{l+1}+1)$, $[\gamma_{i,1}, \overline{\beta}_{2^l-1}] = v_l \overline{\beta}_{2i+1}$ and*

$$[\gamma_{i,2^j}, \overline{\beta}_{2^l-1}] = [\gamma_{i,2^{j-1}}, \overline{\beta}_{2^l-1}]\gamma_{i,2^{j-1}} + v_l \overline{\beta}_{2^{j-1}(2i+1)}$$

for $0 \leq i \leq 2^{l-2}$ and $1 \leq j < p(i)$.

Proof. This is proved along the same lines as in [7]*Section 5. So we will limit ourselves to an outline.

First note that these commutators must be divisible by v_l because $H_*SO(2^{l+1}+1)$ is commutative. Next, because $Q_{l-1}(\overline{\beta}_{2^l-1}) = 0$, $[-, \overline{\beta}_{2^l-1}]$ is both an algebra derivation and a coalgebra derivation. So $x = [\gamma_{i,1}, \overline{\beta}_{2^l-1}]$ is primitive in $K(l)_*SO(2^{l+1}+1)$. Degree considerations and Lemma 2.4 imply that x is either 0 or $v_l \overline{\beta}_{2i+1}$.

If $x = 0$, then $\gamma_{i,1} \otimes \overline{\beta}_{2^l-1} - \overline{\beta}_{2^l-1} \otimes \gamma_{i,1}$ would be in the image of

$$\begin{aligned} \widetilde{k}(l)_{*-2}P \to \widetilde{H}_{*-2}P \to \widetilde{H}_*(SO(2^{l+1}+1) \wedge SO(2^{l+1}+1)) \\ \cong \widetilde{H}_*SO(2^{l+1}+1) \otimes \widetilde{H}_*SO(2^{l+1}+1) \quad (1) \end{aligned}$$

where P is the projective plane of $SO(2^{l+1}+1)$. This contradicts [7]*Proposition 4.1

For the second set of commutators, we reason the same way, using

$$[\gamma_{i,2^j}, \overline{\beta}_{2^l-1}] - [\gamma_{i,2^{j-1}}, \overline{\beta}_{2^l-1}]\gamma_{i,2^{j-1}}.$$ □

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