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BOUNDED POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

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Abstract: This paper is considered with the second order nonlinear neutral difference equations

$$\Delta^2(y_n \pm y_{n-\tau}) + f(n+1, y_{n+1-\sigma}) = 0, \quad \forall n \geq n_0.$$

The existence results of bounded positive solutions for the equations are proved by using the Rothe fixed point theorem. Two nontrivial examples are included.

AMS Subject Classification: 39A10, 39A20
Key Words: second order nonlinear neutral difference equation, bounded positive solution, Rothe fixed point theorem

1. Introduction

Migda and Migda [3] discussed the asymptotic behavior of the second order neutral difference equation

$$\Delta^2(y_n + py_{n-k}) + f(n, y_n) = 0, \quad \forall n \geq 1. \tag{1.1}$$

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The aim of this paper is to study the existence of the second order nonlinear neutral difference equations

$$\Delta^2(y_n + y_{n-\tau}) + f(n+1, y_{n+1-\sigma}) = 0, \quad \forall n \geq n_0, \tag{1.2}$$

$$\Delta^2(y_n - y_{n-\tau}) + f(n+1, y_{n+1-\sigma}) = 0, \quad \forall n \geq n_0, \tag{1.3}$$

where $\tau, \sigma \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $f \in C\big(\mathbb{N}_{n_0} \times \mathbb{R}, \mathbb{R}\big)$. By using the Rothe fixed point theorem, we prove the existence of a bounded positive solution of Eq.(1.2) and Eq.(1.3), respectively. Two examples are constructed to illuminate our results.

2. Preliminaries

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the sets of all integers, positive integers and nonnegative integers, respectively,

$$\beta = \min\{n_0 - \tau,\, n_0 + 1 - \sigma\}, \quad \mathbb{Z}_\beta = \{n : n \in \mathbb{Z} \text{ with } n \geq \beta\},$$
$$\mathbb{N}_{n_0} = \{n : n \in \mathbb{N}_0 \text{ with } n \geq n_0\},$$

l^∞_β denotes the Banach space of all bounded sequences $y = \{y_n\}_{n \in \mathbb{Z}_\beta}$ with the norm

$$\|y\| = \sup_{n \in \mathbb{Z}_\beta} |y_n| \quad \text{for } y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in l^\infty_\beta$$

and

$$B(\{M\}_{n \in \mathbb{Z}_\beta}, \{N\}_{n \in \mathbb{Z}_\beta})$$
$$= \big\{\{y_n\}_{n \in \mathbb{Z}_\beta} \in l^\infty_\beta : \|\{y_n\}_{n \in \mathbb{Z}_\beta} - \{M\}_{n \in \mathbb{Z}_\beta}\| < N\big\}, \quad \forall M, N \in \mathbb{R}.$$

For the sake of convenience, we denote $B(\{M\}_{n \in \mathbb{Z}_\beta}, \{N\}_{n \in \mathbb{Z}_\beta})$ by $B(M, N)$. It is easy to see that $B(M, N)$ is a bounded open convex subset of l^∞_β.

By a solution of Eq.(1.2) (resp., Eq.(1.3)), we mean a sequence $\{y_n\}_{n \in \mathbb{Z}_\beta} \in l^\infty_\beta$ with a positive integer $T \geq n_0 + \tau + \sigma + |\beta|$ such that Eq.(1.2) (resp., Eq.(1.3)) is satisfied for all $n \geq T$.

Lemma 2.1. (Discrete Arzela-Ascoli's Theorem [1]) *A bounded, uniformly Cauchy subset Y of l^∞_β is relatively compact.*

Lemma 2.2. (Rothe Fixed Point Theorem [2]) *Let D be a bounded convex open subset of a Banach space E and $A : \overline{D} \to E$ be a condensing mapping, and $A(\partial D) \subseteq \overline{D}$. Then A has a fixed point in $\overline{D}$.*

3. Main Results

Now we prove the existence of bounded positive solutions for Eq.(1.2) and Eq.(1.3).

Theorem 3.1. *Assume that there exist two constants* N *and* M *with* $M > N > 0$ *and a nonnegative sequence* $\{p_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying*

$$|f(n,u)| \le p_n, \quad \forall (n,u) \in \mathbb{N}_{n_0} \times [N,M]; \tag{3.1}$$

$$\sum_{s=n_0}^{\infty}\sum_{i=s}^{\infty} p_{i+1} < +\infty. \tag{3.2}$$

Then Eq.(1.2) *has a bounded positive solution.*

Proof. We prove that there exists a mapping $S : \overline{B(M,N)} \to l_\beta^\infty$ with $S(\partial B(M,\, N)) \subseteq \overline{B(M,N)}$ such that S has a fixed point $y \in \overline{B(M,N)}$, which is also a bounded positive solution of Eq.(1.2).

(3.2) ensures that there exists $T \ge n_0 + \tau + \sigma + |\beta|$ satisfying

$$\sum_{s=T+\tau}^{\infty}\sum_{i=s}^{\infty} p_{i+1} < \frac{N}{2}. \tag{3.3}$$

Define a mapping $S : \overline{B(M,N)} \to l_\beta^\infty$ as follows

$$(Sy)_n = \begin{cases} M - \sum_{l=1}^{\infty}\sum_{s=n+(2l-1)\tau}^{n+2l\tau-1}\sum_{i=s}^{\infty} f(i+1, y_{i+1-\sigma}), & n \ge T \\ (Sy)_T, \quad \beta \le n < T \end{cases} \tag{3.4}$$

for each $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$. In view of (3.1), (3.3) and (3.4), we get that for every $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \partial B(M,N) \subseteq \overline{B(M,N)}$ and $n \ge T$

$$\begin{aligned} |(Sy)_n - M| &= \left| -\sum_{l=1}^{\infty}\sum_{s=n+(2l-1)\tau}^{n+2l\tau-1}\sum_{i=s}^{\infty} f(i+1, y_{i+1-\sigma})\right| \\ &\le \sum_{l=1}^{\infty}\sum_{s=n+(2l-1)\tau}^{n+2l\tau-1}\sum_{i=s}^{\infty} |f(i+1, y_{i+1-\sigma})| \\ &\le \sum_{s=T+\tau}^{\infty}\sum_{i=s}^{\infty} p_{i+1} < \frac{N}{2}, \end{aligned}$$

which yields that

$$\|Sy - M\| \le \frac{N}{2} < N,$$

that is, $S(\partial B(M,N)) \subseteq \overline{B(M,N)}$.

Now we prove that S is a continuous and condensing mapping in $\overline{B(M,N)}$. Let $y^\omega = \{y_n^\omega\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$ for each $\omega \in \mathbb{N}$ and $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$ with $\lim_{\omega\to\infty} y^\omega = y$. (3.2) and the continuity of f guarantee that for each $\varepsilon > 0$ there exist $T_1, T_2, T_3 \in \mathbb{N}$ with $T_1 > T$ and $T_2 > T_1 + \tau - 1$ satisfying

$$\max\left\{ \sum_{s=T_1+\tau}^{\infty} \sum_{i=s}^{\infty} p_{i+1},\ (T_1 - T) \sum_{i=T_2}^{\infty} p_{i+1} \right\} < \frac{\varepsilon}{16}; \tag{3.5}$$

$$\sum_{s=T+\tau}^{T_1+\tau-1} \sum_{i=s}^{T_2-1} \left| f(i+1, y_{i+1-\tau}^\omega) - f(i+1, y_{i+1-\tau}) \right| < \frac{\varepsilon}{16}, \quad \forall \omega \ge T_3. \tag{3.6}$$

By virtue of (3.1) and (3.4)-(3.6), we infer that for any $\omega \ge T_3$

$$\begin{aligned}
&\left\|Sy^\omega - Sy\right\| \\
&= \sup_{n\ge T} \left| \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \sum_{i=s}^{\infty} \left[f(i+1, y_{i+1-\sigma}^\omega) - f(i+1, y_{i+1-\sigma}) \right] \right| \\
&\le \max\left\{ \sup_{n\ge T_1} \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \sum_{i=s}^{\infty} \left[|f(i+1, y_{i+1-\sigma}^\omega)| + |f(i+1, y_{i+1-\sigma})| \right], \right. \\
&\qquad \left. \sup_{T\le n\le T_1-1} \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \sum_{i=s}^{\infty} \left| f(i+1, y_{i+1-\sigma}^\omega) - f(i+1, y_{i+1-\sigma} \right| \right\} \\
&\le \max\left\{ 2 \sum_{s=T_1+\tau}^{\infty} \sum_{i=s}^{\infty} p_{i+1},\ \sum_{s=T+\tau}^{\infty} \sum_{i=s}^{\infty} \left| f(i+1, y_{i+1-\sigma}^\omega) - f(i+1, y_{i+1-\sigma}) \right| \right\} \\
&\le \max\left\{ \frac{\varepsilon}{8}, \sum_{s=T_1+\tau}^{\infty} \sum_{i=s}^{\infty} \left| f(i+1, y_{i+1-\sigma}^\omega) - f(i+1, y_{i+1-\sigma}) \right| \right. \\
&\qquad \left. + \sum_{s=T+\tau}^{T_1+\tau-1} \sum_{i=s}^{\infty} \left| f(i+1, y_{i+1-\sigma}^\omega) - f(i+1, y_{i+1-\sigma}) \right| \right\} \\
&\le \max\left\{ \frac{\varepsilon}{8}, 2 \sum_{s=T_1+\tau}^{\infty} \sum_{i=s}^{\infty} p_{i+1} \right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s=T+\tau}^{T_1+\tau-1}\sum_{i=T_2}^{\infty}\left|f(i+1,y^{\omega}_{i+1-\sigma})-f(i+1,y_{i+1-\sigma})\right| \\
&+\sum_{s=T+\tau}^{T_1+\tau-1}\sum_{i=s}^{T_2-1}\left|f(i+1,y^{\omega}_{i+1-\sigma})-f(i+1,y_{i+1-\sigma})\right|\Bigg\} \\
\le{}& \max\left\{\frac{\varepsilon}{8},\frac{\varepsilon}{8}+2\sum_{s=T+\tau}^{T_1+\tau-1}\sum_{i=T_2}^{\infty}p_{i+1}+\frac{\varepsilon}{16}\right\} \\
\le{}& \max\left\{\frac{\varepsilon}{8},\frac{3\varepsilon}{16}+\frac{\varepsilon}{8}\right\}<\varepsilon,
\end{aligned}
$$

which implies that $\lim_{\omega\to\infty} Sy^{\omega}=Sy$, that is, S is continuous in $\overline{B(M,N)}$.

Using (3.1), (3.3) and (3.4), we deduce that for any $y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\overline{B(M,N)}$

$$
\begin{aligned}
\|Sy\| &= \sup_{n\ge T}\left|M-\sum_{l=1}^{\infty}\sum_{s=n+(2l-1)\tau}^{n+2l\tau-1}\sum_{i=s}^{\infty}f(i+1,y_{i+1-\sigma})\right| \\
&\le M+\sum_{s=T+\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1}<M+\frac{N}{2},
\end{aligned}
$$

which means that $S(\overline{B(M,N)})$ is uniformly bounded.

Given $\varepsilon>0$. (3.2) implies that there exists $T^*>T$ satisfying

$$
\sum_{s=T^*+\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1}<\frac{\epsilon}{2},
$$

which together with (3.1) and (3.4) ensures that for all $y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\overline{B(M,N)}$ and $t_2>t_1\ge T^*$

$$
\begin{aligned}
|(Sy)_{t_2}-(Sy)_{t_1}| &= \Bigg|\sum_{l=1}^{\infty}\sum_{s=t_2+(2l-1)\tau}^{t_2+2l\tau-1}\sum_{i=s}^{\infty}f(i+1,y_{i+1-\sigma}) \\
&\qquad -\sum_{l=1}^{\infty}\sum_{s=t_1+(2l-1)\tau}^{t_1+2l\tau-1}\sum_{i=s}^{\infty}f(i+1,y_{i+1-\sigma})\Bigg| \\
&\le \sum_{l=1}^{\infty}\sum_{s=t_2+(2l-1)\tau}^{t_2+2l\tau-1}\sum_{i=s}^{\infty}|f(i+1,y_{i+1-\sigma})|
\end{aligned}
$$

$$+\sum_{l=1}^{\infty}\sum_{s=t_1+(2l-1)\tau}^{t_1+2l\tau-1}\sum_{i=s}^{\infty}|f(i+1,y_{i+1-\sigma})|$$
$$\le 2\sum_{s=T^*+\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1}<\epsilon,$$

which yields that $S(\overline{B(M,N)})$ is uniformly Cauchy. Thus Lemma 2.1 ensures that $S(\overline{B(M,N)})$ is relatively compact. Consequently S_L is condensing in $\overline{B(M,N)}$.

Lemma 2.2 implies that S has a fixe point $y=\{y_n\}_{n\imath\mathbb{Z}_\beta}\in\overline{B(M,N)}$, that is,

$$y_n=M-\sum_{l=1}^{\infty}\sum_{s=n+(2l-1)\tau}^{n+2l\tau-1}\sum_{i=s}^{\infty}f(i+1,y_{i+1-\sigma}),\quad \forall n\ge T,$$

which yields that

$$y_n+y_{n-\tau}=2M-\sum_{s=n}^{\infty}\sum_{i=s}^{\infty}f(i+1,y_{i+1-\sigma}),\quad \forall n\ge T+\tau$$

and

$$\Delta(y_n+y_{n-\tau})=\sum_{i=n}^{\infty}f(i+1,y_{i+1-\sigma}),\quad \forall n\ge T+\tau,$$

which gives that

$$\Delta^2(y_n+y_{n-\tau})=-f(n+1,y_{n+1-\sigma}),\quad \forall n\ge T+\tau,$$

that is, Eq.(1.2) has a bounded positive solution $y\in\overline{B(M,N)}$. This completes the proof. □

Theorem 3.2. *Assume that there exist two constants N and M with $M>N>0$ and a nonnegative sequence $\{p_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.1) *and*

$$\sum_{l=1}^{\infty}\sum_{s=n_0+l\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1}<+\infty. \tag{3.7}$$

Then Eq.(1.3) *has a bounded positive solution.*

Proof. We prove that there exists a mapping $S:\overline{B(M,N)}\to l_\beta^\infty$ with $S(\partial B(M,N))\subseteq\overline{B(M,N)}$ such that S has a fixed point $y\in\overline{B(M,N)}$, which is also a bounded positive solution of Eq.(1.3). In view of (3.7), there exists a positive integer $T \geq n_0 + \tau + \sigma + |\beta|$ such that

$$\sum_{l=1}^{\infty} \sum_{s=T+l\tau}^{\infty} \sum_{i=s}^{\infty} p_{i+1} < \frac{N}{2}. \tag{3.8}$$

Define a mapping $S : \overline{B(M,N)} \to l_\beta^\infty$ as follows

$$(Sy)_n = \begin{cases} M + \sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \sum_{i=s}^{\infty} f(i+1, y_{i+1-\sigma}), & n \geq T \\ (Sy)_T, \quad \beta \leq n < T \end{cases} \tag{3.9}$$

for each $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$. It follows from (3.1), (3.8) and (3.9) that for every $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \partial B(M,N) \subseteq \overline{B(M,N)}$ and $n \geq T$

$$\begin{aligned} |(Sy)_n - M| &= \left| -\sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \sum_{i=s}^{\infty} f(i+1, y_{i+1-\sigma}) \right| \\ &\leq \sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \sum_{i=s}^{\infty} |f(i+1, y_{i+1-\sigma})| \\ &\leq \sum_{l=1}^{\infty} \sum_{s=T+l\tau}^{\infty} \sum_{i=s}^{\infty} p_{i+1} < \frac{N}{2}, \end{aligned}$$

which means that

$$\|Sy - M\| \leq \frac{N}{2} < N,$$

that is, $S(\partial\overline{B(M,N)} \subseteq \overline{B(M,N)}$.

Now we prove that S is a continuous and condensing mapping in $\overline{B(M,N)}$. Put $y^\omega = \{y_n^\omega\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$ for each $\omega \in \mathbb{N}$ and $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$ with $\lim_{\omega\to\infty} y^\omega = y$. Let $\varepsilon > 0$. Using (3.9) and the continuity of f, we conclude that there exist $T_1, T_2, T_3, T_4 \in \mathbb{N}$ with $T_3 > T$ and $T_2 > T_3 + T_1\tau$ satisfying

$$\begin{aligned} &\max\left\{ \sum_{l=1}^{\infty} \sum_{s=T_3+l\tau}^{\infty} \sum_{i=s}^{\infty} p_{i+1}, \sum_{l=T_1}^{\infty} \sum_{s=T+l\tau}^{T_3+l\tau-1} \sum_{i=s}^{\infty} p_{i+1}, \right. \\ &\left. \sum_{l=1}^{T_1-1} \sum_{s=T+l\tau}^{T_3+l\tau-1} \sum_{i=T_2}^{\infty} p_{i+1} \right\} < \frac{\varepsilon}{16}; \end{aligned} \tag{3.10}$$

$$\begin{aligned}&\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\sum_{i=s}^{T_2-1}\left|f(i+1,y_{i+1-\tau}^{\omega})-f(i+1,y_{i+1-\tau})\right|\\&\quad<\frac{\varepsilon}{16},\quad \forall\omega\geq T_4.\end{aligned}\tag{3.11}$$

Using (3.1) and (3.9)-(3.11), we get that for $\omega \geq T_4$

$$\begin{aligned}&\left\|Sy^{\omega}-Sy\right\|\\
&=\sup_{n\geq T}\left|\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\sum_{i=s}^{\infty}\left[f(i+1,y_{i+1-\sigma}^{\omega})-f(i+1,y_{i+1-\sigma})\right]\right|\\
&\leq\max\left\{\sup_{n\geq T_3}\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\sum_{i=s}^{\infty}\left[|f(i+1,y_{i+1-\sigma}^{\omega})|+|f(i+1,y_{i+1-\sigma})|\right],\right.\\
&\qquad\left.\sup_{T\leq n\leq T_3-1}\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\sum_{i=s}^{\infty}\left|f(i+1,y_{i+1-\sigma}^{\omega})-f(i+1,y_{i+1-\sigma})\right|\right\}\\
&\leq\max\left\{2\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1},\right.\\
&\qquad\left.\sum_{l=1}^{\infty}\sum_{s=T+l\tau}^{\infty}\sum_{i=s}^{\infty}\left|f(i+1,y_{i+1-\sigma}^{\omega})-f(i+1,y_{i+1-\sigma})\right|\right\}\\
&\leq\max\left\{\frac{\varepsilon}{8},\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\sum_{i=s}^{\infty}\left|f(i+1,y_{i+1-\sigma}^{\omega})-f(i+1,y_{i+1-\sigma})\right|\right.\\
&\qquad\left.+\sum_{l=1}^{\infty}\sum_{s=T+l\tau}^{T_3+l\tau-1}\sum_{i=s}^{\infty}\left|f(i+1,y_{i+1-\sigma}^{\omega})-f(i+1,y_{i+1-\sigma})\right|\right\}\\
&\leq\max\left\{\frac{\varepsilon}{8},2\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1}\right.\\
&\qquad+\sum_{l=T_1}^{\infty}\sum_{s=T+l\tau}^{T_3+l\tau-1}\sum_{i=s}^{\infty}\left|f(i+1,y_{i+1-\sigma}^{\omega})-f(i+1,y_{i+1-\sigma})\right|\\
&\qquad\left.+\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\sum_{i=s}^{\infty}\left|f(i+1,y_{i+1-\sigma}^{\omega})-f(i+1,y_{i+1-\sigma})\right|\right\}\\
&\leq\max\left\{\frac{\varepsilon}{8},\frac{\varepsilon}{8}+2\sum_{l=T_1}^{\infty}\sum_{s=T+l\tau}^{T_3+l\tau-1}\sum_{i=s}^{\infty}p_{i+1}\right.\end{aligned}$$

$$+\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+\tau-1}\sum_{i=T_2}^{\infty}\left|f(i+1,y^{\omega}_{i+1-\sigma})-f(i+1,y_{i+1-\sigma})\right|$$
$$\left.+\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+\tau-1}\sum_{i=s}^{T_2-1}\left|f(i+1,y^{\omega}_{i+1-\sigma})-f(i+1,y_{i+1-\sigma})\right|\right\}$$
$$\leq \max\left\{\frac{\varepsilon}{8},\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+2\left(\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+\tau-1}\sum_{i=T_2}^{\infty}p_{i+1}\right)+\frac{\varepsilon}{16}\right\}<\varepsilon,$$

which yields that $\lim_{\omega\to\infty} Sy^{\omega} = Sy$, that is, S is continuous in $\overline{B(M,N)}$.

It follows from (3.1), (3.8) and (3.9) that for any $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$

$$\|Sy\| = \sup_{n\geq T}\left|M+\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\sum_{i=s}^{\infty} f(i+1,y_{i+1-\sigma})\right|$$
$$\leq M+\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1} < M+\frac{N}{2},$$

that is, $S(\overline{B(M,N)})$ is uniformly bounded.

Let $\varepsilon > 0$. It follows from (3.7) that there exists $T^* > T$ satisfying

$$\sum_{l=1}^{\infty}\sum_{s=T^*+l\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1} < \frac{\varepsilon}{2},$$

which together with (3.1) and (3.9) yields that for all $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$ and $t_2 > t_1 \geq T^*$

$$\left|(Sy)_{t_2}-(Sy)_{t_1}\right|$$
$$=\left|\sum_{l=1}^{\infty}\sum_{s=t_2+l\tau}^{\infty}\sum_{i=s}^{\infty}f(i+1,y_{i+1-\sigma})-\sum_{l=1}^{\infty}\sum_{s=t_1+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}f(i+1,y_{i+1-\sigma})\right|$$
$$\leq\sum_{l=1}^{\infty}\sum_{s=t_2+l\tau}^{\infty}\sum_{i=s}^{\infty}\left|f(i+1,y_{i+1-\sigma})\right|+\sum_{l=1}^{\infty}\sum_{s=t_1+l\tau}^{\infty}\sum_{i=s}^{\infty}\left|f(i+1,y_{i+1-\sigma})\right|$$
$$\leq 2\sum_{l=1}^{\infty}\sum_{s=T^*+l\tau}^{\infty}\sum_{i=s}^{\infty}p_{i+1}<\varepsilon,$$

which implies that $S(\overline{B(M,N)})$ is uniformly Cauchy. Hence Lemma 2.1 implies that $S(\overline{B(M,N)})$ is relatively compact, that is, S is condensing in $\overline{B(M,N)}$.

Thus Lemma 2.2 ensures that S has a fixed point $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$, that is,

$$y_n = M + \sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \sum_{i=s}^{\infty} f(i+1, y_{i+1-\sigma}), \quad \forall n \geq T,$$

which leads to

$$y_n - y_{n-\tau} = -\sum_{s=n}^{\infty}\sum_{i=s}^{\infty} f(i+1, y_{i+1-\sigma}), \quad \forall n \geq T+\tau$$

and

$$\Delta(y_n - y_{n-\tau}) = \sum_{i=n}^{\infty} f(i+1, y_{i+1-\sigma}), \quad \forall n \geq T+\tau,$$

which yields that

$$\Delta^2(y_n - y_{n-\tau}) = -f(n+1, y_{n+1-\sigma}), \quad \forall n \geq T+\tau,$$

that is, Eq.(1.3) has a bounded positive solution in $\overline{B(M,N)}$. This completes the proof. □

4. Examples

Now we construct two examples to show the applications of the results presented in Section 3.

Example 4.1. Consider the second order nonlinear neutral difference equation

$$\Delta^2(y_n + y_{n-\tau}) + \frac{\sin^2 n + \sqrt{n} y_{n+1-\sigma} - 3n^3 y_{n+1-\sigma}^4 + (y_{n+1-\sigma} - 1)^{\frac{4}{5}}}{n^6 + \ln^5(n^3+2) + |y_{n+1-\sigma} - n^2|^3} = 0, \quad \forall n \geq 1, \tag{4.1}$$

where $\tau, \sigma \in \mathbb{N}$ are fixed. Let $n_0 = 1$, $N = 2$, $M = 3$, $\beta = \min\{1-\tau, 2-\sigma\}$ and

$$f(n,u) = \frac{\sin^2 n + \sqrt{n}u - 3n^3u^4 + (u-1)^{\frac{4}{5}}}{n^6 + \ln^5(n^3+2) + |u-n^2|^3},$$
$$p_n = \frac{3 + 3\sqrt{n} + 243n^3}{n^6}, \quad \forall (n,u) \in \mathbb{N}_{n_0} \times \mathbb{R}.$$

Clearly (3.1) and (3.2) hold. Thus Theorem 3.1 implies that Eq.(4.1) has a bounded positive solution.

Example 4.2. Consider the second order nonlinear neutral difference equation

$$\Delta^2(y_n - y_{n-\tau}) + \frac{n^2(y_{n+1-\sigma} - 2)^9 - (n^3 - 4n + 1)y^3_{n+1-\sigma}}{n^9 + (y^4_{n+1-\sigma} - 3n)^4 + \cos(n^2 + y_{n+1-\sigma})} = 0, \quad \forall n \geq 2, \tag{4.2}$$

where $\tau, \sigma \in \mathbb{N}$ are fixed. Let $n_0 = 2$, $N = 1$, $M = 3$, $\beta = \min\{2-\tau, 3-\sigma\}$ and

$$f(n, u) = \frac{n^2(u-2)^9 - (n^3 - 4n + 1)u^3}{n^9 + (u^4 - 3n)^4 + \cos(n^2 + u)},$$
$$p_n = \frac{n^2 + 27(n^3 + 4n + 1)}{n^9}, \quad \forall (n, u) \in \mathbb{N}_{n_0} \times \mathbb{R}.$$

Obviously (3.1) and (3.7) hold. Thus Theorem 3.2 gives that Eq.(4.2) has a bounded positive solution.

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