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ON CAYLEY ISOMORPHISMS OF CLIFFORD SEMIGROUPS

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Abstract: In this paper, we investigate the problem about determining for which Cayley graphs of a given Clifford semigroup are CI-graphs. We give sufficient conditions for Cayley graphs of Clifford semigroups to be CI-graphs and for Clifford semigroups to be CI-semigroups.

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1. Introduction

Let S be a semigroup and let A be a subset of S. The *Cayley graph* $\mathrm{Cay}(S, A)$ of S relative to A is defined as the graph with the vertex set S and the arc set $E(\mathrm{Cay}(S, A))$ consisting of those ordered pairs (x, y) such that $xa = y$ for some $a \in A$. Clearly, if A is an empty set, then $\mathrm{Cay}(S, A)$ is an empty graph.

Arthur Cayley (1821-1895) introduced Cayley graphs of groups in 1878. One of the first investigations on Cayley graphs of algebraic structures can be found in Maschke's Theorem from 1896 about groups of genus zero, that is, groups which possess a generating system such that the Cayley graph is planar, see [17].

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Cayley graphs of groups have been extensively studied and many interesting results have been obtained, see for examples [1], [2], [3], [5], [7], [8], [9], [10], and [11]. The Cayley graphs of semigroups have been considered by many authors. Many new interesting results on Cayley graphs of semigroups have appeared in various journals recently, see for examples [3], [4], [5], [6], [12], and [13]. In the investigation of the Cayley graphs of semigroups, the first of all interesting is finding the analogous of natural conditions which have been used in the group case.

A Cayley graph $\mathrm{Cay}(S, A)$ is called a *CI-graph* of S, CI stands for Cayley Isomorphism, if whenever B is a subset of S which $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, there exists an automorphism σ of S such that $\sigma(A) = B$. A semigroup S is called a *CI-semigroup* if all of its Cayley graphs are CI-graphs. The family of cyclic groups $\mathbb{Z}_p$, where p is prime, is the first known infinite family of CI-groups, see [11].

Necessary and sufficient conditions have been found for Cayley graphs of groups to be CI-graphs and for groups to be CI-groups, see for examples [9], [10], and [11]. After that it is natural to investigate Cayley graphs for semigroups which are unions of groups. A Clifford semigroup is such a union of groups. Here we investigate the conditions for Cayley graphs of Clifford semigroups enjoy the property of being CI-graphs and the conditions for Clifford semigroups enjoy the property of being CI-semigroups.

2. Basic Definitions and Results

All sets in this paper are assumed to be finite. Let S be a semigroup. The set $C(S) = \{c \in S \mid cs = sc \text{ for all } s \in S\}$ is called the *center* of S. The set of all idempotents of S is denoted by $E(S)$. An element $s \in S$ is called a *regular element* if $sxs = s$ for some $x \in S$. One calls S a *regular semigroup* if all of its elements are regular. A regular semigroup S is called a *Clifford semigroup* if $E(S) \subseteq C(S)$, i.e. idempotents of S commute with all elements of S.

If $(Y, \leq)$ is a nonempty partially ordered set such that the meet $a \wedge b$ of a and b exists for every a, b in Y, then we say that $(Y, \leq)$ is a *(lower) semilattice.* A semilattice Y is called a *chain* if, for all $x, y \in Y$, $x \leq y$ or $y \leq x$. Suppose that we have a semilattice Y and a set of groups G_α indexed by Y, and for all $\beta \leq \alpha$ in Y, there exists a group homomorphism $f_{\alpha,\beta} : G_\alpha \to G_\beta$ such that $f_{\alpha,\alpha} = id_{G_\alpha}$ is the identity mapping and for all α, β, γ with $\gamma \leq \beta \leq \alpha$, we have $f_{\beta,\gamma} f_{\alpha,\beta} = f_{\alpha,\gamma}$ where the multiplication on $S = \bigcup_{\alpha \in Y} G_\alpha$ is defined, for $x \in G_\alpha, y \in G_\beta$, by $xy = f_{\alpha,\alpha\wedge\beta}(x) f_{\beta,\alpha\wedge\beta}(y)$. It is easy to check that S is a semigroup, and called a *strong semilattice of groups.* We write $S = [Y; G_\alpha, f_{\alpha,\beta}]$. In 1941, A. H. Clifford proved that a semigroup is a Cilfford semigroup if and only if it is a strong semilattice of groups, see [16]. In the sequel, we will mainly use the term Cilfford semigroup instead of strong semilattice of groups.

The following proposition describes all automorphisms on Clifford semigroups $[Y; G_\alpha, f_{\alpha,\beta}]$.

Proposition 1. [15] *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup. Let $\eta : Y \to Y$ be an automorphism, for each $\alpha \in Y$, let $\chi_\alpha : G_\alpha \to G_{\eta(\alpha)}$ be an isomorphism, and assume that for any $\beta \le \alpha$, the diagram*

$$\begin{array}{ccc} G_\alpha & \xrightarrow{\chi_\alpha} & G_{\eta(\alpha)} \\ \big\downarrow{\scriptstyle f_{\alpha,\beta}} & & \big\downarrow{\scriptstyle f_{\eta(\alpha),\eta(\beta)}} \\ G_\beta & \xrightarrow{\chi_\beta} & G_{\eta(\beta)} \end{array}$$

commutes. Define a mapping χ on S by $\chi(a) = \chi_\alpha(a)$ if $a \in G_\alpha$. Then χ is an automorphism on S. Conversely, every automorphism on S can be so constructed.

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *(digraph) homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v))$ $\in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A (digraph) homomorphism $\varphi : (V, E) \to (V, E)$ is called an *(digraph) endomorphism.* If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective (digraph) homomorphism and φ^{-1} is also a (digraph) homomorphism, then φ is called an *(digraph) isomorphism*, we write $(V_1, E_1) \cong (V_2, E_2)$ and say that (V_1, E_1) and (V_2, E_2) are isomorphic. An (digraph) isomorphism $\varphi : (V, E) \to (V, E)$ is called an *(digraph) automorphism.*

The following lemmas describe the structure of Cayley graphs of a given Clifford semigroup.

Lemma 2. [14] *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup and $A \subseteq S$. Let $x'_\alpha \in G_\alpha,\ y'_\beta \in G_\beta$. If (x'_α, y'_β) is an arc in $\mathrm{Cay}(S, A)$, then $\beta \le \alpha$ and for each $x_\alpha \in G_\alpha$, there exists $y_\beta \in G_\beta$ such that (x_α, y_β) is an arc in $\mathrm{Cay}(S, A)$.*

Lemma 3. [14] *Let Y be a chain, $S = [Y; G_\alpha, f_{\alpha,\beta}]$ a Clifford semigroup and $A \subseteq S$. Then*

1. *the Cayley graph $\mathrm{Cay}(S, A)$ contains $|Y|$ disjoint induced subdigraphs $(G_\alpha, E_\alpha),\ \alpha \in Y$ where $(G_\alpha, E_\alpha) \cong \mathrm{Cay}(G_\alpha, A_\alpha)$ and $A_\alpha = \{f_{\gamma,\alpha}(a) \mid a \in A \cap G_\gamma,\ \alpha \le \gamma\},\ \alpha \in Y$.*

2. *for* $\alpha \neq \beta$, $x_\alpha \in G_\alpha$, $y_\beta \in G_\beta$, (x_α, y_β) *is an arc in the Cayley graph* $\mathrm{Cay}(S, A)$ *if and only if* $\beta < \alpha$ *and* $y_\beta = f_{\alpha,\beta}(x_\alpha)a$ *for some* $a \in A \cap G_\beta$.

By Lemma 3(1) and the definition of an induced subdigraph, for each $x \in G_\alpha$, $|A_\alpha|$ is the number of s in G_α such that (s, x) is an arc in $\mathrm{Cay}(S, A)$. Let (V, E) be a digraph. Recall that a directed cycle of order n in (V, E) is a sequence of vertices $(x_1, x_2, ..., x_n, x_1)$ in V such that $(x_1, x_2), ..., (x_{n-1}, x_n)$, $(x_n, x_1) \in E$. Denote the identity element of a group G_α by e_α.

Now we prove several preparatory lemmas about the Cayley graphs of Clifford semigroups.

Lemma 4. *Let* $S = [Y; G_\alpha, f_{\alpha,\beta}]$ *be a Clifford semigroup and* $A \subseteq S$. *If* $(x_1, x_2, ..., x_n, x_1)$ *is a directed cycle of order* n *in* $\mathrm{Cay}(S, A)$, *then* $x_1, x_2, ..., x_n \in G_\alpha$ *for some* $\alpha \in Y$ *such that* $|G_\alpha| \geq n$. *Moreover,* $A_\alpha \setminus \{e_\alpha\} \neq \emptyset$ *where* $A_\alpha = \{f_{\gamma,\alpha}(a) \mid a \in A \cap G_\gamma,\ \gamma \geq \alpha\}$.

Proof. Suppose that $x_i \in G_{\alpha_i}$ for all $i = 1, ..., n$. By Lemma 2, $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n \geq \alpha_1$, that is, $\alpha_1 = \alpha_2 = ... = \alpha_n$. Thus $x_1, x_2, ..., x_n \in G_{\alpha_1}$. It follows immediately that $|G_{\alpha_1}| \geq n$. By Lemma 3(1), $(x_1, x_2, ..., x_n, x_1)$ is a directed cycle of order n in $\mathrm{Cay}(G_{\alpha_1}, A_{\alpha_1})$. Then there exists $a \in A_{\alpha_1}$ such that $x_1 a = x_2$ by the definition. Since $x_1 \neq x_2$, $a \neq e_{\alpha_1}$. Hence $A_{\alpha_1} \setminus \{e_{\alpha_1}\} \neq \emptyset$. □

Given two groups G_α and G_β, the group homomorphism $f : G_\alpha \to G_\beta$ such that $f(g) = e_\beta$ for all $g \in G_\alpha$ is called a *zero-mapping.*

Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, $A \subseteq S$ and $\rho \in Y$. Then we put

$$\begin{aligned} A_\rho &= \{f_{\gamma,\rho}(a) \mid a \in A \cap G_\gamma,\ \gamma \geq \rho\} \\ Y_\rho &= \{\gamma \in Y \mid \gamma > \rho\} \\ Y_\rho^0 &= \{\gamma \in Y_\rho \mid f_{\gamma,\rho} : G_\gamma \to G_\rho \text{ is a zero-mapping}\} \\ Y_\rho^1 &= \{\gamma \in Y_\rho \mid f_{\gamma,\rho} : G_\gamma \to G_\rho \text{ is an isomorphism}\}. \end{aligned}$$

The *indegree* $\overrightarrow{d}(x)$ of a vertex x of a digraph D is the number of vertices of D that end in x.

Lemma 5. *Let* $S = [Y; G_\alpha, f_{\alpha,\beta}]$ *be a Clifford semigroup and* $A \subseteq S$. *If* Y *is a chain and all groups* G_α *are cyclic groups of order prime* p_α, *then, for all* $x \in G_\alpha$,

$$\overrightarrow{d}(x) = \begin{cases} |A_\alpha| + |Y_\alpha^1||A \cap G_\alpha| + \sum_{\gamma \in Y_\alpha^0} |G_\gamma|, & \text{if } x \in A \\ |A_\alpha| + |Y_\alpha^1||A \cap G_\alpha|, & \text{if } x \notin A. \end{cases}$$

Proof. Let $x \in G_\alpha$. By the definition, $\overrightarrow{d}(x)$ is the number of s in S such that (s, x) is an arc in $\mathrm{Cay}(S, A)$. By Lemma 2, we get that all those s must belong to G_γ for some $\gamma \geq \alpha$. We denote by $\overrightarrow{d_*}(x)$, $\overrightarrow{d_1}(x)$, $\overrightarrow{d_0}(x)$ the number of s in G_α, $\bigcup_{\gamma \in Y_\alpha^1} G_\gamma$, $\bigcup_{\gamma \in Y_\alpha^0} G_\gamma$, respectively such that (s, x) is an arc in $\mathrm{Cay}(S, A)$. Since $\{\alpha\} \cup Y_\alpha^1 \cup Y_\alpha^0 = \{\gamma \mid \gamma \geq \alpha\}$ and $\{\alpha\}$, Y_α^1, Y_α^0 are pairwise disjoint, $\overrightarrow{d}(x) = \overrightarrow{d_*}(x) + \overrightarrow{d_1}(x) + \overrightarrow{d_0}(x)$. Let $\gamma \in Y$. Consider 3 cases:

Case1. $\gamma = \alpha$. By Lemma 3(1), $\overrightarrow{d_*}(x) = |A_\alpha|$.

Case2. $\gamma \in Y_\alpha^1$. Then $\gamma > \alpha$ and $f_{\gamma,\alpha}$ is an isomorphism. Let $s \in G_\gamma$. By Lemma 3(2), we get that (s, x) is an arc in $\mathrm{Cay}(S, A)$ if and only if $x = f_{\gamma,\alpha}(s)a$ for some $a \in A \cap G_\alpha$. Hence (s, x) is an arc in $\mathrm{Cay}(S, A)$ if and only if $s = f_{\gamma,\alpha}^{-1}(x) f_{\gamma,\alpha}^{-1}(a^{-1}) = f_{\gamma,\alpha}^{-1}(xa^{-1})$. Then the number of those s in G_γ is $|A \cap G_\alpha|$. Therefore, $\overrightarrow{d_1}(x) = |Y_\alpha^1||A \cap G_\alpha|$.

Case3. $\gamma \in Y_\alpha^0$. Then $\gamma > \alpha$ and $f_{\gamma,\alpha}$ is a zero-mapping. Let $s \in G_\gamma$. By Lemma 3(2), we get that (s, x) is an arc in $\mathrm{Cay}(S, A)$ if and only if $x = f_{\gamma,\alpha}(s)a = e_\alpha a = a$ for some $a \in A \cap G_\alpha$. Therefore, for $x \in G_\alpha$, (s, x) is an arc in $\mathrm{Cay}(S, A)$ if and only if $x \in A$. Then

$$\overrightarrow{d_0}(x) = \begin{cases} \sum_{\gamma \in Y_\alpha^0} |G_\gamma|, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

□

For the Clifford semigroup $S = [Y; G_\alpha, f_{\alpha,\beta}]$ which Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings, we have $Y_\alpha^0 = Y_\alpha$ and $Y_\alpha^1 = \emptyset$ for all $\alpha \in Y$.

Lemma 6. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup where Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings. Let $A, B \subseteq S$ and $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. Then $g(G_\alpha) = G_\alpha$ for all $\alpha \in Y$ such that $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$.*

Proof. Suppose that $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$. Let $x \in G_\alpha$ and $a \in (A \cap G_\alpha) \setminus \{e_\alpha\}$, then $(x, xa, xa^2, ..., xa^{p_\alpha} = x)$ is a directed cycle of order $p_\alpha = |G_\alpha|$ in $\mathrm{Cay}(S, A)$ and $(g(x), g(xa), g(xa^2), ..., g(x))$ is also a directed cycle of order p_α in $\mathrm{Cay}(S, B)$. By Lemma 4, we have $g(x), g(xa), g(xa^2), ..., g(xa^{(p_\alpha - 1)}) \in G_\gamma$ for some $\gamma \in Y$ such that $|G_\gamma| = p_\gamma \geq p_\alpha$ and there exists $b \in B_\gamma \setminus \{e_\gamma\} = (B \cap G_\gamma) \setminus \{e_\gamma\}$. Thus $(g(x), g(x)b, g(x)b^2, ..., g(x)b^{p_\gamma} = g(x))$ is a directed cycle of order $p_\gamma = |G_\gamma|$ in $\mathrm{Cay}(S, B)$ and $(g^{-1}(g(x)) = x, g^{-1}(g(x)b), g^{-1}(g(x)b^2), ..., x)$ is also a directed cycle of order p_γ in $\mathrm{Cay}(S, A)$. Since $x \in G_\alpha$, by *Lemma 4*,

$x, g^{-1}(g(x)b), g^{-1}(g(x)b^2), ...$
$, g^{-1}(g(x)b^{(p_\gamma-1)}) \in G_\alpha$ and $p_\gamma \leq |G_\alpha| = p_\alpha \leq p_\gamma$, that is, $p_\alpha = p_\gamma$. Now we have $g(G_\alpha) = G_\gamma$. Hence the induced subdigraph with vertex set G_α in $\mathrm{Cay}(S, A)$ is isomorphic to the induced subdigraph with vertex set G_γ in $\mathrm{Cay}(S, B)$. By Lemma 3(1), we get that $\mathrm{Cay}(G_\alpha, A_\alpha) \cong \mathrm{Cay}(G_\gamma, B_\gamma)$ and thus $|A_\alpha| = |B_\gamma|$. Next we will show that $\alpha = \gamma$. If $\alpha \neq \gamma$, let we assume that $\alpha < \gamma$. By Lemma 5, in $\mathrm{Cay}(S, A)$, $\overrightarrow{d}(a) = |A_\alpha| + \sum_{\rho \in Y_\alpha} |G_\rho| = |A_\alpha| + \sum_{\alpha<\rho<\gamma} |G_\rho| + |G_\gamma| + \sum_{\rho \in Y_\gamma} |G_\rho| \geq |A_\alpha| + |G_\gamma| + \sum_{\rho \in Y_\gamma} |G_\rho| > |A_\alpha| + \sum_{\rho \in Y_\gamma} |G_\rho| = |B_\gamma| + \sum_{\rho \in Y_\gamma} |G_\rho|$. By Lemma 5 again, in $\mathrm{Cay}(S, B)$, we have $|B_\gamma| + \sum_{\rho \in Y_\gamma} |G_\rho| \geq \overrightarrow{d}(w)$ for all $w \in G_\gamma$. Then $\overrightarrow{d}(a) > \overrightarrow{d}(w)$ for all $w \in G_\gamma$, this contradicts our assumption that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Therefore, $\alpha = \gamma$. □

Lemma 7. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup where Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings. Let $A, B \subseteq S$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, then $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$ for all $\alpha \in Y$ such that $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$.*

Proof. Let $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. Suppose that $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$. By Lemma 6, $g(G_\alpha) = G_\alpha$. We obtain that $\mathrm{Cay}(G_\alpha, A_\alpha) \cong \mathrm{Cay}(G_\alpha, B_\alpha)$ and $|A_\alpha| = |B_\alpha|$. It follows easily that $\mathrm{Cay}(G_\alpha, A_\alpha \setminus \{e_\alpha\}) \cong \mathrm{Cay}(G_\alpha, B_\alpha \setminus \{e_\alpha\})$. Since all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings, $A_\alpha \setminus \{e_\alpha\} = (A \cap G_\alpha) \setminus \{e_\alpha\}$ and $B_\alpha \setminus \{e_\alpha\} = (B \cap G_\alpha) \setminus \{e_\alpha\}$. Thus $\mathrm{Cay}(G_\alpha, (A \cap G_\alpha) \setminus \{e_\alpha\}) \cong \mathrm{Cay}(G_\alpha, (B \cap G_\alpha) \setminus \{e_\alpha\})$ and hence $|(A \cap G_\alpha) \setminus \{e_\alpha\}| = |(B \cap G_\alpha) \setminus \{e_\alpha\}|$. If α is not the maximum of Y, then $\sum_{\rho \in Y_\alpha} |G_\rho| \neq 0$. Let $m = |A_\alpha| = |B_\alpha|$ and $n = |A_\alpha| + \sum_{\rho \in Y_\alpha} |G_\rho|$, then $m \neq n$. By Lemma 5, for all $x \in G_\alpha \setminus A$, $\overrightarrow{d}(x) = m$ and for all $x \in A \cap G_\alpha$, $\overrightarrow{d}(x) = n$. The increasing sequence of indegree of all elements in G_α in $\mathrm{Cay}(S, A)$ is

$$\Big(\overbrace{m, m, ..., m}^{|G_\alpha| - |(A \cap G_\alpha) \setminus \{e_\alpha\}| - 1 \quad terms}, \overrightarrow{d}(e_\alpha), \overbrace{n, n, ..., n}^{|(A \cap G_\alpha) \setminus \{e_\alpha\}| \quad terms}\Big)$$

and the increasing sequence of indegree of all elements in G_α in $\mathrm{Cay}(S, B)$ is

$$\Big(\overbrace{m, m, ..., m}^{|G_\alpha| - |(B \cap G_\alpha) \setminus \{e_\alpha\}| - 1 \quad terms}, \overrightarrow{d}(e_\alpha), \overbrace{n, n, ..., n}^{|(B \cap G_\alpha) \setminus \{e_\alpha\}| \quad terms}\Big).$$

Since $|(A \cap G_\alpha) \setminus \{e_\alpha\}| = |(B \cap G_\alpha) \setminus \{e_\alpha\}|$ and $m \neq n$, indegree of e_α in $\mathrm{Cay}(S, A)$ and in $\mathrm{Cay}(S, B)$ must be equal. Thus $e_\alpha \in A \cap G_\alpha$ if and only if $e_\alpha \in B \cap G_\alpha$. Hence $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. If α is the maximum of Y, then $A_\alpha = A \cap G_\alpha$ and $B_\alpha = B \cap G_\alpha$. Hence $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. □

Lemma 8. *Let* $S = [Y; G_\alpha, f_{\alpha,\beta}]$ *be a Clifford semigroup where* Y *is a chain, all groups* G_α *are cyclic groups of order prime* p_α *and all group homomorphisms* $f_{\alpha,\beta}$ *are zero-mappings. Let* $A, B \subseteq S$ *and* $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$*, then* $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$ *for all* $\alpha \in Y$ *such that* $A \cap G_\alpha \neq \emptyset$*.*

Proof. Let $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. Suppose that $A \cap G_\alpha \neq \emptyset$, we need to prove only 2 cases:

Case1. $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$. By Lemma 7, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$.

Case2. $A \cap G_\alpha = \{e_\alpha\}$. Clearly, $A_\alpha = \{e_\alpha\}$. If α is the maximum element of Y, then $(s, se_\alpha = s)$ is a loop in $\mathrm{Cay}(S, A)$ for all $s \in S$. It follows immediately that every vertex in $\mathrm{Cay}(S, B)$ has a loop. By Lemma 3(1), $e_\delta \in B_\delta$ for all $\delta \in Y$. Suppose that $B_\alpha = B \cap G_\alpha \neq \{e_\alpha\}$, then $(B \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$ and $|B \cap G_\alpha| > 1$. By Lemma 7, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. Thus $|A \cap G_\alpha| = |B \cap G_\alpha| > 1 = |\{e_\alpha\}|$, a contradiction. Hence $B \cap G_\alpha = \{e_\alpha\}$, that is, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. If α is not the maximum element of Y, then $Y_\alpha \neq \emptyset$ and thus $\sum_{\rho \in Y_\alpha} |G_\rho| \neq 0$. By Lemma 5, in $\mathrm{Cay}(S, A)$, $\overrightarrow{d}(e_\alpha) = 1 + \sum_{\rho \in Y_\alpha} |G_\rho|$ and $\overrightarrow{d}(x) = 1$ for all $x \in G_\alpha \setminus \{e_\alpha\}$. Thus $g(e_\alpha) \in B \cap G_\delta \subseteq B_\delta$ for some $\delta \in Y$ such that $\sum_{\rho \in Y_\alpha} |G_\rho| = \sum_{\rho \in Y_\delta} |G_\rho|$ and $|B_\delta| = 1$. Hence $\alpha = \delta$ and $\{g(e_\alpha)\} = B \cap G_\alpha = B_\alpha$. Since $A \cap G_\alpha = \{e_\alpha\}$, (e_α, e_α) is a loop in $\mathrm{Cay}(S, A)$. Obviously, $(g(e_\alpha), g(e_\alpha))$ is a loop in $\mathrm{Cay}(S, B)$. Thus $e_\alpha \in B_\alpha$, that is, $\{e_\alpha\} = \{g(e_\alpha)\} = B \cap G_\alpha = B_\alpha$. Therefore, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. □

From now on, N_0^H denotes the number of vertices u in a digraph H such that $\overrightarrow{d}(u) = 0$. Clearly, for two given digraphs H and T, if $H \cong T$, then $N_0^H = N_0^T$.

The next lemma is proved on the Clifford semigroup $S = [Y; G_\alpha, f_{\alpha,\beta}]$ which Y is a semilattice.

Lemma 9. *Let* $S = [Y; G_\alpha, f_{\alpha,\beta}]$ *be a Clifford semigroup,* ω *the minimum element of* Y*. If* $A \subseteq G_\omega$ *and* $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$*, then* $B \subseteq G_\omega$ *and* $\mathrm{Cay}(G_\omega, A) \cong \mathrm{Cay}(G_\omega, B)$*.*

Proof. Let $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. If $A = \emptyset$, then $\mathrm{Cay}(S, A)$ is an empty graph. Obviously, $\mathrm{Cay}(S, B)$ is also an empty graph, that is, $B = \emptyset \subseteq G_\omega$. Hence $\mathrm{Cay}(G_\omega, A) \cong \mathrm{Cay}(G_\omega, B)$. Let $A \neq \emptyset$. Then for all $s \in S$ and $a \in A$, $sa \in G_\omega$. Thus in $\mathrm{Cay}(S, A)$, $\overrightarrow{d}(s) = 0$ for all $s \in S \setminus G_\omega$. By Lemma 5, $\overrightarrow{d}(x) \geq |A_\omega| = |A| > 0$ for all $x \in G_\omega$. Hence $N_0^{\mathrm{Cay}(S,A)} = |S \setminus G_\omega|$. Suppose that $B \nsubseteq G_\omega$. Then there exists $b \in B \cap G_\beta$ for some $\beta > \omega$. Then

$B_\alpha \neq \emptyset$ for all $\alpha \leq \beta$. By Lemma 5 again, $\overrightarrow{d}(x) \geq |B_\alpha| > 0$ for all $x \in G_\alpha, \alpha \leq \beta$. Hence $N_0^{\mathrm{Cay}(S,B)} \leq |S \setminus \bigcup_{\alpha \leq \beta} G_\alpha| \leq |S \setminus \{G_\beta, G_\omega\}| < |S \setminus G_\omega| = N_0^{\mathrm{Cay}(S,A)}$, this contradicts our assumption that $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$. Now we claim that $B \subseteq G_\omega$. As in $\mathrm{Cay}(S,A)$, we can see that , in $\mathrm{Cay}(S,B)$, $\overrightarrow{d}(s) = 0$ for all $s \in S \setminus G_\omega$ and $\overrightarrow{d}(x) \geq |A_\omega| = |A| > 0$ for all $x \in G_\omega$, so $g(G_\omega) = G_\omega$. By Lemma 3(1), $\mathrm{Cay}(G_\omega, A) \cong \mathrm{Cay}(G_\omega, B)$. □

Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, $A \subseteq S$ and $x, y \in G_\alpha$, $z \in G_\gamma$, $\alpha \neq \gamma$. Suppose that (x,y) and (z,y) are arcs in $\mathrm{Cay}(S,A)$. By Lemma 3(1), (x,y) is an arc in $\mathrm{Cay}(G_\alpha, A_\alpha)$. Then there exists $a \in A_\alpha$ such that $y = xa$. If $a = e_\alpha$, then $y = x$ and thus (x,y) is a loop. If $a \neq e_\alpha$, then $(x, xa = y, xa^2, ..., xa^{|a|} = x)$ is a directed cycle of order $|a|$ in $\mathrm{Cay}(S,A)$. Thus (x,y) is either a loop or an arc which is contained in a directed cycle. It is easily seen that $y \neq z$, so (z,y) is not a loop. By Lemma 4, (z,y) is not contained in any directed cycles. By Lemma 3(2), $\gamma > \alpha$, that is, $\gamma \in Y_\alpha$. Let us denote by $\overrightarrow{d}_{**}(y)$ the number of vertices in $\bigcup_{\rho \in Y_\alpha} G_\rho$ that end in y and $N_{\neq 0}^{**\mathrm{Cay}(S,A)}$ the number of vertices u in $\mathrm{Cay}(S,A)$ that $\overrightarrow{d}_{**}(u) \neq 0$.

Given two subsets A, B of S. Clearly, for a graph isomorphism $g : \mathrm{Cay}(S,A) \to \mathrm{Cay}(S,B)$, it is not only $\overrightarrow{d}(y) = \overrightarrow{d}(g(y))$, but also $\overrightarrow{d}_{**}(y) = \overrightarrow{d}_{**}(g(y))$. Moreover, $N_{\neq 0}^{**\mathrm{Cay}(S,A)} = N_{\neq 0}^{**\mathrm{Cay}(S,B)}$.

Analysis similar to the proof of Lemma 5 shows that the following lemma is hold.

Lemma 10. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup and $A \subseteq S$. If Y is a chain and all groups G_α are cyclic groups of order prime p_α, then, for all $x \in G_\alpha$,*

$$\overrightarrow{d}_{**}(x) = \begin{cases} |Y_\alpha^1||A \cap G_\alpha| + \sum_{\gamma \in Y_\alpha^0} |G_\gamma|, & \text{if } x \in A \\ |Y_\alpha^1||A \cap G_\alpha|, & \text{if } x \notin A. \end{cases}$$

Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup which there exists the maximum element π in a semilattice Y. Let $A \subseteq G_\pi$ and $\rho < \pi$. Suppose that $f_{\pi,\rho}$ is a zero-mapping, then $A_\rho = \{e_\rho\}$. Thus the arc set $E(\mathrm{Cay}(G_\rho, A_\rho)) = \{(x,x) | x \in G_\rho\}$. Suppose that $f_{\pi,\rho}$ is an isomorphism. Define a mapping $\Pi_\rho : \mathrm{Cay}(G_\pi, A) \to \mathrm{Cay}(G_\rho, A_\rho)$ by $\Pi_\rho(x) = f_{\pi,\rho}(x)$ for all $x \in G_\pi$. It is clear that Π_ρ is a bijective. Let (x,y) be an arc in $\mathrm{Cay}(G_\pi, A)$, then there exists $a \in A$ such that $y = xa$. Since $f_{\pi,\rho}$ is an isomorphism, $\Pi_\rho(y) = \Pi_\rho(xa) = f_{\pi,\rho}(xa) = f_{\pi,\rho}(x) f_{\pi,\rho}(a) = \Pi_\rho(x)\Pi_\rho(a)$. Becauce $\Pi_\rho(a) = f_{\pi,\rho}(a) \in A_\rho$, $(\Pi_\rho(x), \Pi_\rho(y))$ is an arc in $\mathrm{Cay}(G_\rho, A_\rho)$. Let (z,w) be an arc in $\mathrm{Cay}(G_\rho, A_\rho)$, then there exists $a_\rho \in A_\rho = f_{\pi,\rho}(A)$ such that $w = za_\rho$. Since $f_{\pi,\rho}$ is an isomorphism, there exists unique $a_\pi \in A$ such that $f_{\pi,\rho}(a_\pi) = a_\rho$. Thus $\Pi_\rho^{-1}(w) = \Pi_\rho^{-1}(za_\rho) = f_{\pi,\rho}^{-1}(za_\rho) = f_{\pi,\rho}^{-1}(z)f_{\pi,\rho}^{-1}(a_\rho) = \Pi_\rho^{-1}(z)a_\pi$. Hence $(\Pi_\rho^{-1}(z), \Pi_\rho^{-1}(w))$ is an arc in $\mathrm{Cay}(G_\pi, A)$. Therefore, Π_ρ is a graph isomorphism. We thus get $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\rho, A_\rho)$ for all $\rho < \pi$ such that $f_{\pi,\rho}$ is an isomorphism.

Lemma 11. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, π the maximum element of Y. If $A \subseteq G_\pi$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, then $B \subseteq G_\pi$ and $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\pi, B)$.*

Proof. Let $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. Since π be the maximum element of Y, $Y_\pi^1 = Y_\pi^0 = \emptyset$. If $A \subseteq G_\pi$, then $A \cap G_\alpha = \emptyset$ for all $\alpha < \pi$. By Lemma 10, in $\mathrm{Cay}(S, A)$, $\overrightarrow{d}_{**}(x) = |Y_\alpha^1||A \cap G_\alpha| = 0$ for all $x \in G_\alpha, \alpha < \pi$ and $\overrightarrow{d}_{**}(x) \leq |Y_\pi^1||A \cap G_\pi| + \sum_{\gamma \in Y_\pi^0} |G_\gamma| = 0$ for all $x \in G_\pi$. Thus $\overrightarrow{d}_{**}(x) = 0$ for all $x \in \mathrm{Cay}(S, A)$. We must have $\overrightarrow{d}_{**}(x) = 0$ for all $x \in \mathrm{Cay}(S, B)$. Suppose that $B \nsubseteq G_\pi$, so there exists $b \in B \cap G_\beta$ for some $\beta < \pi$. We have $f_{\pi,\beta}(e_\pi)b = e_\beta b = b$. By Lemma 3(2), (e_π, b) is an arc in $\mathrm{Cay}(S, B)$, that is, $\overrightarrow{d}_{**}(b) \neq 0$, this contradicts our assumption that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Thus $B \subseteq G_\pi$.

Now we want to show that $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\pi, B)$. If $A = \{e_\pi\}$, then $A_\alpha = \{e_\alpha\}$ for all $\alpha \in Y$. Thus the arc set $E(\mathrm{Cay}(S, A))$ $= \{(x, x)|x \in S\}$. Of course, $E(\mathrm{Cay}(S, B)) = \{(x, x)|x \in S\}$. Hence $B = \{e_\pi\}$. Therefore, $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\pi, B)$. If $A \setminus \{e_\pi\} \neq \emptyset$, then there exists $a \in A \setminus \{e_\pi\}$. Let $x \in G_\pi$, then $(x, xa, xa^2, ..., xa^{p_\pi} = x)$ is a directed cycle of order $p_\pi = |G_\pi|$ in $\mathrm{Cay}(S, A)$. Obviously, $g(x) \notin G_\alpha$ where $f_{\pi,\alpha}$ is a zero-mapping. We have $(g(x), g(xa), g(xa^2), ..., g(x))$ is also directed cycle of order p_π in $\mathrm{Cay}(S, B)$. By Lemma 4, $g(x), g(xa), g(xa^2), ..., g(xa^{(p_\pi - 1)}) \in G_\gamma$ for some $\gamma \in Y$ such that $|G_\gamma| = p_\gamma \geq p_\pi$. If $p_\gamma \neq p_\pi$, then $f_{\pi,\gamma}$ is a zero-mapping. Thus $g(G_\pi) \subseteq G_\rho$ for some G_ρ such that $|G_\rho| = p_\rho = p_\pi$ and $f_{\pi,\rho}$ is an isomorphism. Hence $g(G_\pi) = G_\rho$ for some G_ρ such that $f_{\pi,\rho}$ is an isomorphism, that is, $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\rho, B_\rho) \cong \mathrm{Cay}(G_\pi, B)$. □

Lemma 12. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup where Y is a chain, all groups G_α are cyclic groups of order prime p_α. If $A \subseteq G_\rho$ for some $\rho \in Y$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, then $B \subseteq G_\rho$ and $\mathrm{Cay}(G_\rho, A) \cong \mathrm{Cay}G_\rho, B)$.*

Proof. Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. If $A = \emptyset$, then $\mathrm{Cay}(S, A)$ is an empty graph. Obviously, $\mathrm{Cay}(S, B)$ is also an empty graph and $B = \emptyset \subseteq G_\rho$. If ρ is the maximum element of Y, then, by Lemma 11, $B \subseteq G_\rho$ and $\mathrm{Cay}(G_\rho, A) \cong \mathrm{Cay}G_\rho, B)$. Let $A \neq \emptyset$ and ρ is not the maximum element of Y. We claim that

$A \cap G_\alpha = \emptyset$ for all $\alpha \neq \rho$, $A_\alpha \neq \emptyset$ for all $\alpha \leq \rho$ and $A_\alpha = \emptyset$ for all $\alpha > \rho$. By Lemma 5, in Cay(S, A), $\overrightarrow{d}(x) \geq |A_\alpha| > 0$ for all $x \in G_\alpha, \alpha \leq \rho$ and $\overrightarrow{d}(y) = |A_\gamma| + |Y_\gamma^1||A \cap G_\gamma| = 0$ for all $y \in G_\gamma, \gamma > \rho$. Thus $N_0^{\text{Cay}(S,A)} = \sum_{\gamma \in Y_\rho} |G_\gamma|$. By Lemma 10, in Cay$(S, A)$, $\overrightarrow{d}_{**}(y) = |Y_\gamma^1||A \cap G_\gamma| = 0$ for all $y \in G_\gamma, \gamma \neq \rho$, that is, $N_{\neq 0}^{**\text{Cay}(S,A)} \leq |G_\rho|$. Moreover, $\overrightarrow{d}_{**}(x) \leq |Y_\rho^1||A| + \sum_{\gamma \in Y_\rho^0} |G_\gamma|$ for all $x \in G_\rho$. Thus

$$\text{in Cay}(S, A), \overrightarrow{d}_{**}(s) \leq |Y_\rho^1||A| + \sum_{\gamma \in Y_\rho^0} |G_\gamma| \text{ for all } s \in S. \tag{1}$$

If $B \nsubseteq G_\rho$, then we need to consider 4 cases:

Case1. There exists $b \in B \cap G_\beta$ for some $\beta > \rho$. then $B_\alpha \neq \emptyset$ for all $\alpha \leq \beta$. By Lemma 5, in Cay(S, B), $\overrightarrow{d}(x) \geq |B_\alpha| > 0$ for all $x \in G_\alpha, \alpha \leq \beta$. Thus $N_0^{\text{Cay}(S,B)} \leq \sum_{\alpha > \beta} |G_\alpha| < \sum_{\alpha > \beta} |G_\alpha| + |G_\beta| \leq \sum_{\alpha \in Y_\rho} |G_\alpha| = N_0^{\text{Cay}(S,A)}$, this contradicts our assumption that Cay$(S, A) \cong$ Cay(S, B).

Case2. $B \cap G_\rho = \emptyset$. By case1, $B \cap G_\gamma = \emptyset$ for all $\gamma > \rho$. Let $x \in G_\gamma$ for some $\gamma \geq \rho$ and $b \in B \cap G_\alpha$, then $\alpha < \rho$ and $xb \in G_\alpha$. Thus, in Cay(S, B), $\overrightarrow{d}(x) = 0$ for all $x \in \bigcup_{\gamma \geq \rho} G\gamma$, that is, $N_0^{\text{Cay}(S,B)} \geq \sum_{\gamma \geq \rho} |G_\gamma| = \sum_{\gamma \in Y_\rho} |G_\gamma| + |G_\rho| > \sum_{\gamma \in Y_\rho} |G_\gamma| = N_0^{\text{Cay}(S,A)}$, this contradicts our assumption that Cay$(S, A) \cong$ Cay(S, B).

Case3. There exists $b \in B \cap G_\beta$ for some $\beta < \rho$ where $f_{\rho,\beta}$ is a zero-mapping. Then $f_{\alpha,\beta}$ is s zero-mapping for all $\alpha \in Y_\rho$, that is, $(Y_\rho \cup \{\rho\}) \subseteq Y_\beta^0$. By Lemma 10, in Cay$(S, B)$, $\overrightarrow{d}_{**}(b) = |Y_\beta^1||B \cap G_\beta| + \sum_{\gamma \in Y_\beta^0} |G_\gamma| \geq \sum_{\gamma \in Y_\beta^0} |G_\gamma| \geq \sum_{\gamma \in Y_\rho} |G_\gamma| + |G_\rho| > \sum_{\gamma \in Y_\rho} |G_\gamma| = \sum_{\gamma \in Y_\rho^1} |G_\gamma| + \sum_{\gamma \in Y_\rho^0} |G_\gamma| = |Y_\rho^1||G_\rho| + \sum_{\gamma \in Y_\rho^0} |G_\gamma| \geq |Y_\rho^1||A| + \sum_{\gamma \in Y_\rho^0} |G_\gamma|$. By (1), we have $\overrightarrow{d}_{**}(b) \neq \overrightarrow{d}_{**}(s)$ for all s in Cay(S, A), this contradicts our assumption that Cay$(S, A) \cong$ Cay(S, B).

Case4. There exists $b \in B \cap G_\beta$ for some $\beta < \rho$ where $f_{\rho,\beta}$ is an isomorphism. Obviously, $B \cap G_\beta \neq \emptyset, Y_\beta^1 \neq \emptyset$ and $|G_\beta| = |G_\rho|$. We have $Y_\rho^0 \subseteq Y_\beta^0, Y_\rho^1 \subseteq Y_\beta^1$ and $f_{\alpha,\beta}$ is an isomorphism for all $\beta < \alpha \leq \rho$. Thus $Y_\rho^0 = Y_\beta^0$ and $(Y_\rho^1 \cup \{\rho\}) \subseteq Y_\beta^1$ By Lemma 10, in Cay(S, B), $\overrightarrow{d}_{**}(x) \geq |Y_\beta^1||B \cap G_\beta| > 0$ for all $x \in G_\beta$. By case2, there exists $b_1 \in B \cap G_\rho$. There exists $e_\gamma \in G_\gamma, \gamma > \rho$ such that $(e_\gamma, e_\gamma b_1 = f_{\gamma,\rho}(e_\gamma) b_1 = e_\rho b_1 = b_1)$ is an arc in Cay(S, B), so $\overrightarrow{d}_{**}(b_1) \neq 0$. Hence $N_{\neq 0}^{**\text{Cay}(S,B)} \geq |G_\beta| + 1 > |G_\beta| = |G_\rho| \geq N_{\neq 0}^{**\text{Cay}(S,A)}$, this contradicts our assumption that Cay$(S, A) \cong$ Cay(S, B).

These 4 cases give $B \subseteq G_\rho$. As in $\mathrm{Cay}(S, A)$, Lemma 5 gives, in $\mathrm{Cay}(S, B)$, $\overrightarrow{d}(x) \geq |B_\alpha| > 0$ for all $x \in G_\alpha, \alpha \leq \rho$ and $\overrightarrow{d}(y) = |B_\gamma| + |Y_\gamma^1||B \cap G_\gamma| = 0$ for all $y \in G_\gamma, \gamma > \rho$. Thus $g(\bigcup_{\alpha \leq \rho} G_\alpha) = \bigcup_{\alpha \leq \rho} G_\alpha$, that is, $\mathrm{Cay}(\bigcup_{\alpha \leq \rho} G_\alpha, A) \cong \mathrm{Cay}(\bigcup_{\alpha \leq \rho} G_\alpha, B)$. Since ρ is a maximum element of $\{\alpha | \alpha \leq \rho\}$, analysis similar to that in the proof of Theorem 11 shows that $\mathrm{Cay}(G_\rho, A) \cong \mathrm{Cay} G_\rho, B)$. □

3. Main Results

We first give an example of a Cayley graph of a Clifford semigroup which is not a CI-graph and that Clifford semigroup is also not a CI-semigroup.

Example 13. Let Y be a semilattice $\{\alpha, \beta, \gamma\}$ such that $\alpha \wedge \beta = \alpha \wedge \gamma = \beta \wedge \gamma = \gamma$. Let $G_\alpha = \mathbb{Z}_2 = \{\overline{0}_\alpha, \overline{1}_\alpha\}$, $G_\beta = \mathbb{Z}_3 = \{\overline{0}_\beta, \overline{1}_\beta, \overline{2}_\beta\}$, $G_\gamma = \mathbb{Z}_5 = \{\overline{0}_\gamma, \overline{1}_\gamma, \overline{2}_\gamma, \overline{3}_\gamma, \overline{4}_\gamma\}$ and let $f_{\alpha,\gamma}$, $f_{\beta,\gamma}$ be zero-mappings, i.e. $f_{\alpha,\gamma}(G_\alpha) = f_{\beta,\gamma}(G_\beta) = \{\overline{0}_\gamma\}$. Then $S = [Y; G_\alpha, f_{\alpha,\beta}]$ is a Clifford semigroup (see Figure 1).

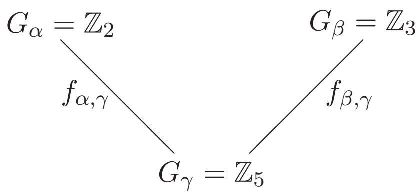

Figure 1: $S = [Y; G_\alpha, f_{\alpha,\beta}]$

Consider two subsets $A = \{\overline{1}_\alpha, \overline{1}_\beta\}$ and $B = \{\overline{1}_\alpha, \overline{1}_\beta, \overline{0}_\gamma\}$ of S. Then $\mathrm{Cay}(S, A) = \mathrm{Cay}(S, B)$ (see Figure 2).

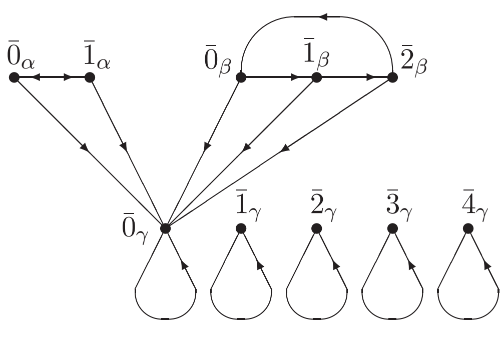

Figure 2: $\mathrm{Cay}(S, A) = \mathrm{Cay}(S, B)$

Since $|A| \neq |B|$, there is no $\sigma \in \mathrm{Aut}(S)$ such that $\sigma(A) = B$. Therefore, $\mathrm{Cay}(S, A)$ and $\mathrm{Cay}(S, B)$ are not CI-graphs, and S is not a CI-semigroup.

Here we investigate the conditions for Clifford semigroups enjoy the property of being CI-semigroups.

Theorem 14. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup. If Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings, then S is a CI-semigroup.*

Proof. Let $A \subseteq S$. Suppose that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 8, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$ for all $\alpha \in Y$ such that $A \cap G_\alpha \neq \emptyset$. Since $G_\alpha = \mathbb{Z}_{p_\alpha}$ is a CI-group for all $\alpha \in Y$, there exists $g_\alpha \in \mathrm{Aut}(G_\alpha)$ such that $g_\alpha(A \cap G_\alpha) = B \cap G_\alpha$ for all $\alpha \in Y$ such that $A \cap G_\alpha \neq \emptyset$. Now we will construct an automorphism on S as Proposition 1. Let $\eta : Y \to Y$ be an identity mapping id_Y and for each $\alpha \in Y$, let

$$\chi_\alpha = \begin{cases} g_\alpha, & \text{if } A \cap G_\alpha \neq \emptyset \\ id_{G_\alpha}, & \text{otherwise.} \end{cases}$$

Define a mapping χ on S by $\chi(x) = \chi_\alpha(x)$ if $x \in G_\alpha$. Clearly, $\chi(A) = B$. To show that $\chi \in \mathrm{Aut}(S)$, it is sufficient to show that $f_{\alpha,\beta}\chi_\alpha = \chi_\beta f_{\alpha,\beta}$. Let $x \in G_\alpha$. Hence $f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta}(\chi_\alpha(x)) = e_\beta$ and $\chi_\beta f_{\alpha,\beta}(x) = \chi_\beta(e_\beta) = e_\beta$. □

Corollary 15. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup. If Y is a chain, all groups G_α are different cyclic groups of order prime p_α, then S is a CI-semigroup.*

Proof. Since all groups G_α are different cyclic groups of order prime p_α, all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings. By Theorem 14, S is a CI-semigroup. □

Now we investigate the conditions for Cayley graphs of Clifford semigroups enjoy the property of being CI-graphs.

Theorem 16. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup. If Y is a chain, all groups G_α are cyclic groups of order prime p_α and $A \subseteq G_\rho$ for some $\rho \in Y$, then $\mathrm{Cay}(S, A)$ is a CI-graph.*

Proof. Let $A \subseteq G_\rho \subseteq S$. Suppose that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 12, $B \subseteq G_\rho$ and $\mathrm{Cay}(G_\rho, A) \cong \mathrm{Cay}(G_\rho, B)$. Since $G_\rho = \mathbb{Z}_{p_\rho}$ is a CI-group, there exists $g_\rho \in \mathrm{Aut}(G_\rho)$ s.t. $g_\rho(A) = B$. Now we will construct an automorphism on S as Proposition 1. Let $\eta : Y \to Y$ be the identity mapping id_Y and for each $\alpha \in Y$, let

$$\chi_\alpha(x) = \begin{cases} f_{\rho,\alpha} g_\rho f_{\rho,\alpha}^{-1}(x), & \text{if } \alpha \leq \rho \text{ and } f_{\rho,\alpha} \text{ is an isomorphism} \\ f_{\alpha,\rho}^{-1} g_\rho f_{\alpha,\rho}(x), & \text{if } \rho < \alpha \text{ and } f_{\alpha,\rho} \text{ is an isomorphism} \\ x, & \text{otherwise.} \end{cases}$$

Define a mapping χ on S by $\chi(x) = \chi_\alpha(x)$ if $x \in G_\alpha$. To show that $\chi \in \mathrm{Aut}(S)$, it is sufficient to show that $f_{\alpha,\beta}\chi_\alpha = \chi_\beta f_{\alpha,\beta}$. Let $x \in G_\alpha$. If $\alpha = \beta$, then it is easily seen that $f_{\alpha,\beta}\chi_\alpha(x) = \chi_\beta f_{\alpha,\beta}(x)$. If $\beta < \alpha$ and $f_{\alpha,\beta}$ is a zero-mapping, then $f_{\alpha,\beta}\chi_\alpha(x) = e_\beta$ and $\chi_\beta f_{\alpha,\beta}(x) = \chi_\beta(e_\beta) = e_\beta$. For $\beta < \alpha$ and $f_{\alpha,\beta}$ is an isomorphism, we need to prove 5 cases:

Case1. $\beta < \alpha \leq \rho$ and $f_{\rho,\alpha}$ is an isomorphism. Then $f_{\rho,\beta}$ is an isomorphism. Thus $f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta} f_{\rho,\alpha} g_\rho f_{\rho,\alpha}^{-1}(x) = f_{\rho,\beta} g_\rho f_{\rho,\alpha}^{-1}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\rho,\beta} g_\rho f_{\rho,\beta}^{-1} f_{\alpha,\beta}(x) = f_{\rho,\beta} g_\rho f_{\rho,\alpha}^{-1}(x)$.

Case2. $\beta < \alpha \leq \rho$ and $f_{\rho,\alpha}$ is a zero-mapping. Then $f_{\rho,\beta}$ is a zero-mapping. Thus$f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\alpha,\beta}(x)$.

Case3. $\rho < \beta < \alpha$ and $f_{\beta,\rho}$ is an isomorphism. Then $f_{\alpha,\rho}$ is an isomorphism. Thus$f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta} f_{\alpha,\rho}^{-1} g_\rho f_{\alpha,\rho}(x) = f_{\beta,\rho}^{-1} g_\rho f_{\alpha,\rho}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\beta,\rho}^{-1} g_\rho f_{\beta,\rho} f_{\alpha,\beta}(x) = f_{\beta,\rho}^{-1} g_\rho f_{\alpha,\rho}(x)$.

Case4. $\rho < \beta < \alpha$ and $f_{\beta,\rho}$ is a zero-mapping. Then $f_{\alpha,\rho}$ is a zero-mapping. Thus $f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\alpha,\beta}(x)$.

Case5. $\beta \leq \rho \leq \alpha$. Then $f_{\alpha,\rho}$ and $f_{\rho,\beta}$ are isomorphisms. Thus$f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta} f_{\alpha,\rho}^{-1} g_\rho f_{\alpha,\rho}(x) = f_{\rho,\beta} g_\rho f_{\alpha,\rho}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\rho,\beta} g_\rho f_{\rho,\beta}^{-1} f_{\alpha,\beta}(x) = f_{\rho,\beta} g_\rho f_{\alpha,\rho}(x)$.

□

Theorem 17. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, ω the minimum element of Y. If all groups G_α are cyclic groups of order prime p_α and $A \subseteq G_\omega$, then* $\mathrm{Cay}(S, A)$ *is a CI-graph.*

Proof. Let $A \subseteq G_\omega$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 9, $B \subseteq G_\omega$ and $\mathrm{Cay}(G_\omega, A) \cong \mathrm{Cay}(G_\omega, B)$. Since $G_\omega = \mathbb{Z}_{p_\omega}$ is a CI-group, then there exists $g_\omega \in \mathrm{Aut}(G_\omega)$ such that $g_\omega(A) = B$. Now we will construct an automorphism on S as Proposition 1. Let $\eta : Y \to Y$ be an identity mapping id_Y and for each $\alpha \in Y$, let

$$\chi_\alpha(x) = \begin{cases} f_{\alpha,\omega}^{-1} g_\omega f_{\alpha,\omega}(x), & \text{if } f_{\alpha,\omega} \text{ is an isomorphism} \\ x, & \text{otherwise.} \end{cases}$$

Define a mapping χ on S by $\chi(x) = \chi_\alpha(x)$ if $x \in G_\omega$. To show that $\chi \in \mathrm{Aut}(S)$, it is sufficient to show that $f_{\alpha,\beta}\chi_\alpha = \chi_\beta f_{\alpha,\beta}$. Let $x \in G_\alpha$. If $\alpha = \beta$, then it is easily seen that $f_{\alpha,\beta}\chi_\alpha(x) = \chi_\beta f_{\alpha,\beta}(x)$. If $\beta < \alpha$ and $f_{\alpha,\beta}$ is a zero-mapping, then $f_{\alpha,\beta}\chi_\alpha(x) = e_\beta$ and $\chi_\beta f_{\alpha,\beta}(x) = \chi_\beta(e_\beta) = e_\beta$. For the other cases that $\beta < \alpha$ and $f_{\alpha,\beta}$ is an isomorphism see case3 and case4 in Theorem 16, with ρ replaced by ω. □

Theorem 18. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, π the maximum element of Y. If all group G_α are cyclic groups of order prime p_α and $A \subseteq G_\pi$, then* $\mathrm{Cay}(S, A)$ *is a CI-graph.*

Proof. Let $A \subseteq G_\pi$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 11, $B \subseteq G_\pi$ and $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\pi, B)$. Since $G_\pi = \mathbb{Z}_{p_\pi}$ is a CI-group, there exists $g_\pi \in \mathrm{Aut}(G_\pi)$ such that $g_\pi(A) = B$. Now we will construct an automorphism on S as Proposition 1. Let $\eta : Y \to Y$ be an identity mapping id_Y and for each $\alpha \in Y$, let

$$\chi_\alpha(x) = \begin{cases} f_{\pi,\alpha} g_\pi f^{-1}_{\pi,\alpha}(x), & \text{if } f_{\pi,\alpha} \text{ is an isomorphism} \\ x, & \text{otherwise.} \end{cases}$$

Define a mapping χ on S by $\chi(x) = \chi_\alpha(x)$ if $x \in G_\alpha$. To show that $\chi \in \mathrm{Aut}(S)$, it is sufficient to show that $f_{\alpha,\beta}\chi_\alpha = \chi_\beta f_{\alpha,\beta}$. Let $x \in G_\alpha$. If $\alpha = \beta$, then it is easily seen that $f_{\alpha,\beta}\chi_\alpha(x) = \chi_\beta f_{\alpha,\beta}(x)$. If $\beta < \alpha$ and $f_{\alpha,\beta}$ is a zero-mapping, then $f_{\alpha,\beta}\chi_\alpha(x) = e_\beta$ and $\chi_\beta f_{\alpha,\beta}(x) = \chi_\beta(e_\beta) = e_\beta$. For the other cases that $\beta < \alpha$ and $f_{\alpha,\beta}$ is an isomorphism see case1 and case2 in Theorem 16, with ρ replaced by π. □

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