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# INJECTIVITY AND SURJECTIVITY EQUIVALENCY OF LINEAR MAPPING: A REVERSAL OF THE TRADITIONAL APPROACH

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**Abstract:** Equivalency of injectivity and surjectivity of linear mapping on $R^n$ is determined by an explicit formulation for pullback of any vector by Gauss-Jordan row (GJR) operations, not by matrix inversion that is the traditional approach.

It is shown that the necessary and sufficient conditions for the equivalency are iff the complete GJR operations can be performed successfully on matrix transformation.

The inverse matrix that is needed for the proof of the equivalency in the standard approach falls out as a by-product.

The nice property of this symbolic approach is considerable columns reduction with savings in both computer time and storage. Finite computational complexities are given.

This reversal to the equivalency enables us to use the system of equations module rather than matrix inversion module in using symbolic computation systems, such as MAPLE.

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**Key Words:** injectivity and surjectivity equivalency, symbolic matrix inversion, Gauss-Jordan matrix inversion is not operation, symbolic, Cramer rule, Matrix inversion by solving system of equations, symbolic computation systems

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## 1. Introduction

It is sometimes necessary or desirable to consider a linear transformation such as rotation, reflection, dilation, magnifications and projections. Let U and V be vector space with finite and same dimension. It is well known that every linear transformation T such as rotation, reflection, dilation, magnifications and projections that maps U into V has matrix representation [12].

The set of transformations of all elements of U is called the range of the transformation. If all elements of V are TU, than the transformation T is called a transformation of U *onto* V (or is surjective transformation or a surjection). If for every element of V that is a transformation of only one element of U, than the transformation is called one-to-one (or an injective transformation, or an injective). For a recent application, see [11]

In linear algebra textbooks one sees formula $X = A^{-1}b$ that suggests one must invert a matrix to compute $X$. This is very inefficient approach. One almost never needed to invert matrices to solve a system of linear equations In fact, inverting a matrix is a "computational waste' to solve system of equations. It requires at least three times as much computation compared with LU-approach [3]. There are, however, certain problem in statistics and structural mechanic when element $A^{-1}$ have an "influence coefficient" meaning, see, e.g., [5].

The matrix inverse is an important tool with many other useful applications. In addition to the classical applications, one needs only the simplex method of solving general linear programs.

Much work has been done on the problem of computing the inverse of a matrix with numeric (real and/or complex) entries. Golub [6] describes the background of early method such as: Gaussian Row Operations, Gaussian Row- and Column Operations, Algebraic Equivalence approach, Adjoint method, Method of partitions, Using Characteristic Equation, Circular Method, Frame's Method, Newton's Formula, and Method of Expansion [6]. Some recent developments are reported in [7] and [14]. The inversion of matrices with linear (i.e. parametric) entries arises in many areas of applied mathematics such as computation of Jacobians, generating functions, the perturbation problem of linear ordinary differential equations.

Gauss-Jordan elimination is similar in efficiency to Gaussian elimination for solving linear systems of equations 13]. In Gauss-Jordan elimination, the elimination above the diagonal, as well as these below the diagonal, are reduced to zero during the reduction stage. It has been recognized that Gauss was ahead of his time in choosing a method that is the most efficient in the sense of number of arithmetic operations, see e.g., [5, 8].

However, one problem with this approach is that the necessary GJR operation must be performed on an augmented matrix of the order n by 2n. Arsham et al [2] present a new approach based on the standard GJR operations principle with considerable reduction in the number of columns. A short summary of the algorithm is as follow: Place the matrix $A_{n \times n}$ which is to be inverted along side a column vector R with $r_i$ as its $i^{th}$ row element $1 \leq I \leq n$, where $r_i$ is an unspecified variable. Then perform the same (standard) on A and R in such a way that A changes to identity matrix. It is shown that the matrix coefficients of $r_i$ becomes $A^{-1}$. Clearly, this column reduction is significant for large matrices. The proposed unified approach can invert matrices with or without symbolic entries. To the best of our knowledge, this method is unique in the symbolic literature [4, 7].

## 2. Equivalency of the Injective and Surjective Mapping

We come to two important classes of linear transformations characterized by special behaviors. We shall provide an equivalency in mapping, but before that, we need to mention the following definitions:

We say that a mapping A: S $\rightarrow$ S' is injective if whatever x, y $\in$ S and x $\neq$ y then Ax $\neq$ Ay. Let A: S $\rightarrow$ S' is surjective if the image of f is all of S'. If A is both surjective and injective, then we say A is a bijective map (a one-to-one mapping).

Because of the reversible nature of the row operation, we state the following theorem:

$$A = \begin{matrix} [2 & 1 & old & a] \\ [ & & & ] \\ [1 & -1 & old & b] \end{matrix}$$

That is mapping:

New a = 2 old a + old b

New b = 1 old a - old b

Now performing GJ up to column 2, we have

$$\begin{matrix} [1 & 0 & 1/3 \text{ old } a + 1/3 \text{ old } b] \\ [ & & ] \\ [0 & 1 & 1/3 \text{ old } a - 2/3 \text{ old } b] \end{matrix}$$

The last two columns are the new point coordinates.

1/3 old a + 1/3 old b = new a

1/3 old a - 2/3 old b = new b

This one to one relation, suffice to prove the equivalency.

**Theorem 1.** *Let A: S → S' be a map on which the above GJR can be applied completely and successfully on matrix transformation A then A is both injective and surjectivity. Thus for A injectivity and surjetectivity are equivalent.*

*Proof.* Proof follows from the symbolic–based approach.

Notice that the matrix of coefficient

$$\begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}$$

is $A^{-1}$, however the proof is not straightforward.

## 3. A Column Reduction Algorithm for Matrix Inversion

The conceptual ease of the proposed approach coupled with considerable column reduction should have special appeal to instructors and students as a convenient method for manual calculations.

**Theorem 1.** *The Fundamental Consequence of Linear Symbolic GJR operations.*

If A|R is GJR operations equivalent to I|B, where B = C.R, then matrix A is invertible and C = $A^{-1}$.

Clearly, the elements of matrix B in the proposed algorithm are some linear combination of the element of column vector R with the coefficients matrix denoted by C.

The following lemmas help to proof the theorem.

The theoretical basis for the algorithm rests partially upon the following three lemmas.

**Lemma 1.** *The matrix A is invertible if A is GJR operations equivalent to the identity matrix I, see Hogben [5].*

**Lemma 2.** *If A|R is GJR operations equivalent to I|B then for every X, AX=R if and only if IX=B, see Hogben [5].*

**Lemma 3.** *If the product of two square matrices is the identity matrix the product is commutative, see Roman [6].*

*Proof of Theorm 1.* Proof of invertability follows from Lemma 1. Now, by taking

$$X = B = CR$$

in Lemma 2, we have;

$$A(CR) = R = IR.$$

By the Associative Law we have;

$$(AC)R = IR.$$

Notice that the last equality is true for all R. This implies,

$$(AC)R/IR,$$

which is an identity. Clearly, this identity holds if and only if;

$$AC = I.$$

This result combined with Lemma 3 gives

$$AC = CA = I.$$

This completes the proof.

## 4. Implementation On Symbolic Systems

The most widely used approach in symbolic computation systems, such as MAPLE, and MATHEMATICA is GJR operations. Our approach may not be as expensive, both of space and computation time. Clearly, this representation will be less expensive, by a constant factor.

Suppose we wish to inverse matrix A:

```
[2    1]
[      ]  = A
[1   -1]
```

On MAPLE V the proposed approach uses the "gaussjord" function of "linalg" module to compute the inverse matrix instead of "inverse" function which requires more computational time and storage. Detailed program is as follows.

```
>with(linalg):
```

```
>A := array ([[2, 1, r1], [1, -1, r2]]);
```

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

```
>gaussjordan (A,2); # up to column 2
```

$$A = \begin{bmatrix} 1 & 0 & 1/3r1+1/3r2 \\ 0 & 1 & 1/3r1-2/3r2 \end{bmatrix}$$

The most widely used approach in symbolic computation systems, such as MAPLE, MATHEMATICA, MACSYMA, DERIVE, and MATHCAD [9, 10] is GJR operations. The symbolic gaussjordan approach is not as expensive in terms of both memory and computation time, Arsham [2].

Although there is always only one column in the vector R, appended to the matrix A, each row in R can consist of linear function. We shall first consider the numeric case of A and then the symbolic. In the numeric case, the "inverse" function in the existing systems augments by a matrix. In the proposed approach, a column vector is augmented which could be more efficient. In the case when matrix A itself contains some linear elements, then elements in R should be treated in the same manner, i.e., in a linear ring.

## 5. Finite Computational Complexities

Since the notion of efficiency involves all the various competing resources needed for executing an algorithm, both data structure and storage requirements should be considered. Since the data structure is simpler, we believe it is space efficient with low intermediate storage. The computational complexity in terms of elementary operations for the conventional method is $n^3$, while for the proposed approach it is $(n^3+n^2)/2$.

**Inversion by Partitioning.** Another method for inverting a matrix is one that depends upon partitioning of the matrix into four sub-matrices. This technique reduces the inversion problem to finding the inverse of smaller matrices. The inverse of these smaller matrices could be found using any methods including symbolic GJ. In other words, one could apply the partitioning process repeatedly until one has to invert only small size matrices.

The price paid, of course, is an increase in the number of matrices to be inversed. Golub [6] gives an *asymptotic* improvement of Strassen's result. Combining the Strassen's portioning approach with our symbolic method the complexity in terms of the four elementary operations is $2(n+1)^3$-8n(n+1)+4n-3. One can show that beyond n=21, our complexity is superior to that of than Strassen's, and much of saving a n gets larger.

## 6. Other Consequences of the Linear-Symbolic Approach

One immediate result is:

**Theorem 2.** *Given transformation A, find $X_i$ such that its transformation AX is the base $(1,0,0\ldots0)^T$, then this X is the first column of $A^{-1}$.*

*Proof.* Obvious.

**Cramer Rule.** *Let A be an nxm non-singular matrix then the solution to AX=b is $X_n$ =b is $x_n = \{det[A(b;i)]\}/det[A]$.*

*Proof.* Let $x_n$ the $n^{th}$ variable the A.I(I;X)=A(I;b) where I is the identity matrix with the last column replaced by the column vector X. Then det[A]. $x_n$ = det[A(i;b)]. Since determinant of a lower triangular matrix is the product of its diagonal, i.e. det[I(i;X)]=$x_n$. So $x_n$ = {det[A(b;i)]}/det[A]. Clearly, one can generate other unknown by exchanging columns.

One can also extend the existing one-way connections among the following related topics: Linear System of Equations; Matrix Inversion; and the Simplex Method (and its variations) of Linear Programming. The new links empower the user to solve the associated problem to any of these two topics by having access to a computer package solver of the other topic. Clearly, the goal here is a theoretical unification as well as advancements in applications.

## 7. Conclusions

Equivalency of injectivity and surjectivityof linear mapping on $R^n$ was proven by an explicit formulation for the pullback of any vector by GJR operations, not inverse transformation that is the traditional approach. This reversal of the equivalency brings about the inverse of the matrix bas a by-product.

The argument in favor of augmenting by one column of parameter $r_i$'s rather than identity matrix which is the traditional approach in computing the inverse matrix. This saves a considerable computer time and storage [2]. For sparse matrices the argument for using this approach is even stronger due to the fact that the inverse of sparse matrices are almost invariably fully populated. Since less calculations are involved, the results are more likely to suffer less from rounding errors.

Notice also that, since $(A^T)^{-1} = (A^{-1})^T$, one can also apply the column operations instead of row operations.

Future works include generalized inverse (rectangular matrices) and vector space transformation in particular in coordinate and isomorphesis. If necessary, the process can be modified by some pivotal strategy.

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