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COMPARISON THEOREMS FOR SOLUTIONS OF NEUTRAL HYPERBOLIC DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we investigate Sturmian comparison theorems for second order neutral hyperbolic differential equations with two kinds of boundary conditions. Certain Sturm type of oscillation comparison theorems are obtained.

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1. Introduction

In this paper, we consider second order neutral hyperbolic differential equations of the form

$$\frac{\partial}{\partial t}\left\{p(t)\frac{\partial}{\partial t}\left[u(y,t)+\sum_{i=1}^{l}\lambda_i(t)u(y,\tau_i(t))\right]\right\}+\sum_{j=1}^{m}q_j(y,t)f_j(u(y,\sigma_j(t)))$$
$$=\sum_{k=1}^{s}a_k(t)\triangle u(y,\rho_k(t)),\quad (y,t)\in\Omega\times R_+\equiv G \tag{1}$$

where $\triangle$ is the Laplacian in Eucliden n-space $R^n, R_+=[0,\infty)$, and Ω is a

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bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$.

In what follows, we denote $I_n = \{1, 2, \ldots, n\}$ and assume that the following hold.

(C_1) $p \in C(R_+, R_+)$ such that $\int_{t_0}^{\infty} \frac{ds}{p(s)} = \infty$, $q_j \in C(\overline{G}, R_+)$ for $j \in I_m$, and $a_k \in C(R_+, R_+)$ for $k \in I_s$;

(C_2) $f_j \in C(R, R)$ is convex in $(0, \infty)$ and satisfied $f_j(-u) = -f_j(u)$ and $f_j(u)/u \geq K_j > 0$ for $u \neq 0$ and $j \in I_m$;

(C_3) $\lambda_i \in C(R_+, R_+)$, such that $0 \leq \sum_{i=1}^{l} \lambda_i \leq 1$ for $i \in I_l$;

(C_4) for $i \in I_l$, $j \in I_m$, and $k \in I_s$, τ_i, σ_j, $\rho_k \in C(R_+, R)$ such that σ_j and ρ_k are nondecreasing, and

$$\max_{i,j,k}\{\tau_i(t), \sigma_j(t), \rho_k(t)\} \leq t \quad \text{and} \quad \lim_{t\to\infty} \min_{i,j,k}\{\tau_i(t), \sigma_j(t), \rho_k(t)\} = \infty.$$

We denote $q_j(t) = \min_{y\in\bar{\Omega}} q_j(y, t)$, $j \in I_m$.

Together with equation (1) we shall consider two types of boundary conditions

$$\frac{\partial u(y,t)}{\partial N} + \mu(y,t)u(y,t) = 0, \qquad (y,t) \in \partial\Omega \times R_+, \tag{2}$$

where N is the unit exterior normal vector to $\partial\Omega$ and $\mu(y, t)$ is a nonnegative continuous function on $\partial\Omega \times R_+$, and

$$u(y,t) = 0, \quad (y,t) \in \partial\Omega \times R_+. \tag{3}$$

The theory of partial differential equations with deviating arguments has received much attention in recent years, see [1] and the references therein. In particular, the oscillatory behavior of solutions of some parabolic and hyperbolic equations with deviating arguments have been studied widely, see for example, [1-6]. Considering that the Sturmian theory for linear and half-linear differential equations plays an important role in the study of qualitative behavior of solutions of both linear and nonlinear equations (see [7]), in this paper, we will establish certain type of Sturm comparison theorems between the nonlinear problem (1), (2) (or (1), (3)) and a related linear differential equation.

The following definitions are needed.

Definition 1. A function $u(y, t) \in C(\overline{G})$ is said to be a solution of the problems (1), (2) if for a fixed $y \in \Omega$, both

$$u(y,t) + \sum_{i=1}^{l} \lambda_i(t)u(y, \tau_i(t)) \quad \text{and} \quad p(t)\frac{\partial}{\partial t}\left[u(y,t) + \sum_{i=1}^{l} \lambda_i(t)u(y, \tau_i(t))\right]$$

are differentiable with respect to t such that it satisfies (1) in the domain G; moreover, for a fixed $t \in R_+$, $u(y,t)$ is differentiable in y and satisfies the boundary condition (2). A solution of the problems (1), (3) is similarly defined.

Definition 2. A solution $u(y,t)$ of (1) is said to be oscillatory in the domain G if for each positive number T, there exists a point $(y_T, t_T) \in \Omega \times [T, \infty)$ such that $u(y_T, t_T) = 0$.

2. Main Results

We begin with the following lemma obtained in [8, Lemma 2.1].

Lemma A. Assume that $z(t) \in C^2[t_0, \infty)$ with $t_0 \geq 0$ satisfies

$$z(t) > 0,\ z'(t) > 0,\ z''(t) \leq 0,\ \ t \geq t_0.$$

Then for each $0 < \gamma_j < 1$, there exists a $t_1 \geq t_0$ such that

$$z(\sigma_j(t)) \geq \gamma_j z(t)\frac{\sigma_j(t)}{t},\ \ t \geq t_1,\ \ i = 1, 2, ..., n.$$

Theorem 2.1. *Assume that $u(y,t)$ is a nonoscillatory solution of problem (1), (2). Let*

$$V(t) = \frac{1}{|\Omega|}\int_\Omega u(y,t)dy, \quad |\Omega| = \int_\Omega dx, \quad Z(t) = V(t) + \sum_{i=1}^{l} \lambda_i(t) V(\tau_i(t)),$$
$$t \geq t_1 \geq t_0.$$

Then the following inequality

$$[p(t)Z'(t)]' + \sum_{j=1}^{m} K_j q_j(t)\left[1 - \sum_{i=1}^{l}\lambda_i(\sigma_j(t))\right] Z(\sigma_j(t)) \leq 0, \tag{4}$$

has an eventually positive solution.

Proof. Without loss of generality, we may assume that $u(y,t) > 0$, $u(y,\tau_i(t)) > 0$, $u(y,\sigma_j(t)) > 0$, and $u(y,\rho_k(t)) > 0$ in $\Omega \times [t_1, \infty), t_1 > t_0, i \in I_l, j \in I_m, k \in I_s$. Integrating (1) with respect to y over the domain Ω, we have

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u(y,t)dy + \sum_{i=1}^{l}\lambda_i(t)\int_\Omega u(y,\tau_i(t))dy\right)\right]$$

$$+\sum_{j=1}^{m}\int_{\Omega} q_j(y,t)f_j(u(y,\sigma_j(t)))dy = \sum_{k=1}^{s} a_k(t)\int_{\Omega}\triangle u(y,\rho_k(t))dy,\ \ t\geq t_1. \quad (5)$$

Using Green's formula and boundary condition (2), it follows that

$$\int_{\Omega}\triangle u(y,\rho_k(t))dy = \int_{\partial\Omega}\frac{\partial u(y,\rho_k(t))}{\partial N}d\omega = -\int_{\partial\Omega}\mu(y,\rho_k(t))u(y,\rho_k(t))d\omega \leq 0, \quad k\in I_s, \quad (6)$$

where $d\omega$ is the surface integral element on $\partial\Omega$. Moreover, from $(C_1),(C_3)$ and Jensen's inequality it follows that

$$\int_{\Omega} q_j(y,t)f_j(u(y,\sigma_j(t)))dy \geq q_j(t)\int_{\Omega} f_j(u(y,\sigma_j(t)))dy$$
$$\geq |\Omega| q_j(t) f_j\left(\frac{1}{|\Omega|}\int_{\Omega} u(y,\sigma_j(t))dy\right) = |\Omega| q_j(t)f_j(V(\sigma_j(t))),\ t\geq t_1. \quad (7)$$

Applying (6)-(7) to (5) we have

$$\frac{d}{dt}\left\{p(t)\frac{d}{dt}\left[V(t)+\sum_{i=1}^{l}\lambda_i(t)V(\tau_i(t))\right]\right\}+\sum_{j=1}^{m} q_j(t)f_j(V(\sigma_j(t))\leq 0,\ t\geq t_1. \quad (8)$$

Let $Z(t)$ be defined as in the theorem. Then (8) yields that

$$[p(t)Z'(t)]'+\sum_{j=1}^{m} q_j(t)f_j(V(\sigma_j(t))\leq 0, \quad t\geq t_2\geq t_1. \quad (9)$$

From (C_4), it follows that

$$[p(t)Z'(t)]'+\sum_{j=1}^{m} K_j q_j(t)V(\sigma_j(t))\leq 0, \quad t\geq t_2,$$

or

$$[p(t)Z'(t)]'+\sum_{j=1}^{m} K_j q_j(t)\left[Z(\sigma_j(t))-\sum_{i=1}^{l}\lambda_i(\sigma_j(t))V(\tau_i(\sigma_j(t)))\right]\leq 0, \quad t\geq t_2.$$

Obviously, $Z(t)\geq V(t)>0, [p(t)Z'(t)]'<0$, and $Z'(t)>0$ for $t\geq t_2$. It follows that

$$[p(t)Z'(t)]'+\sum_{j=1}^{m} K_j q_j(t)\left[1-\sum_{i=1}^{l}\lambda_i(\sigma_j(t))\right]Z(\sigma_j(t))\leq 0, \quad t\geq t_2,$$

i.e., $Z(t)$ is a positive solution of (4).

For the case when $u(y,t) < 0$ for $(y,t) \in \Omega \times [t_1, \infty)$, the proof follows from the fact that $-u(y,t)$ is a positive solution of the problem (1), (2). This completes the proof of Theorem 2.1.

In order to study oscillation of the solutions of the problem (1), (3), the following result from [2] will be used: The smallest eigenvalue η_0 of the Dirichlet problem

$$\begin{cases} \triangle u + \eta u = 0, & (y,t) \in G, \\ u = 0, & (y,t) \in \partial\Omega \times R_+, \end{cases} \tag{10}$$

is positive and the corresponding eigenfunction $\varphi(y)$ is positive in Ω.

Theorem 2.2. *Assume that $u(y,t)$ is a nonoscillatory solution of problem (1),(3). Let*

$$V(t) = \frac{\int_\Omega u(y,t)\varphi(y)dy}{\int_\Omega \varphi(y)dy}, \quad Z(t) = V(t) + \sum_{i=1}^{l} \lambda_i(t) V(\tau_i(t)), \quad t \geq t_1 \geq t_0.$$

Then inequality (4) has an eventually positive solutions.

Proof. Without loss of generality, we may assume that $u(y,t) > 0, u(y,\tau_i(t)) > 0, u(y,\sigma_j(t)) > 0$, and $u(y,\rho_k(t)) > 0$ in $\Omega \times [t_1,\infty), t_1 > t_0, i \in I_l, j \in I_m, k \in I_s$. Multiplying both side of equation (1) by the eigenfunction $\varphi(y)$ of (10) associated with η_0 and then integrating with respect to y over the domain Ω, we have

$$\begin{aligned} \frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u(y,t)\varphi(y)dy + \sum_{i=1}^{l} \lambda_i(t) \int_\Omega u(y,\tau_i(t))\varphi(y)dy\right)\right] \\ + \sum_{j=1}^{m} \int_\Omega q_j(y,t) f_j(u(y,\sigma_j(t)))\varphi(y)dy \\ = \sum_{k=1}^{s} a_k(t) \int_\Omega \triangle u(y,\rho_k(t))\varphi(y)dy, \; t \geq t_1. \end{aligned} \tag{11}$$

Green's formula and boundary condition (3) yield that

$$\int_\Omega \triangle u(y,\rho_k(t))\varphi(y)dy = -\eta_0 \int_\Omega u(y,\rho_k(t))\varphi(y)dy, \;\; t \geq t_1. \tag{12}$$

From $(C_1), (C_3)$ and Jensen' inequality, we have

$$\int_\Omega q_j(y,t) f_j(u(y,\sigma_j(t)))\varphi(y)dy \geq q_j(t) \int_\Omega f_j(u(y,\sigma_j(t)))\varphi(y)dy$$

$$\geq \quad q_j(t)\int_\Omega \varphi(y)dy \cdot f_j\left(\frac{\int_\Omega u(y,\sigma_j(t))\varphi(y)dy}{\int_\Omega \varphi(y)dy}\right). \tag{13}$$

Applying (12)–(13) to (11) we obtain

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(V(t)+\sum_{i=1}^{l}\lambda_i(t)V(\tau_i(t))\right)\right]+\sum_{k=1}^{s}\eta_0 a_k(t)V(\rho_k(t))$$
$$+\sum_{j=1}^{m} q_j(t)f_j(V(\sigma_j(t)) \leq 0,\ t\geq t_1. \tag{14}$$

Let $Z(t)$ be defined as in theorem. Then (14) yields that

$$[p(t)Z'(t)]'+\sum_{j=1}^{m} q_j(t)f_j(V(\sigma_j(t)))\leq 0, \quad t\geq t_1. \tag{15}$$

The remainder of the proof is similar to that of Theorem 2.1, we omit it here.

Theorem 2.3. *Assume that $u(y,t)$ is a nonoscillatory solution of problem (1),(3). Let*

$$V(t)=\frac{\int_\Omega u(y,t)\varphi(y)dy}{\int_\Omega \varphi(y)dy}, \qquad Z(t)=V(t)+\sum_{i=1}^{l}\lambda_i(t)V(\tau_i(t)), \quad t\geq t_1\geq t_0.$$

Then there exists $k_0\in I_s$ such that the following inequality

$$[p(t)Z'(t)]'+\eta_0\sum_{k=1}^{s} a_k(t)\left[1-\sum_{i=1}^{l}\lambda_i(\rho_k(t))\right]Z(\rho_k(t))\leq 0, \tag{16}$$

has an eventually positive solutions.

Proof. As in the proof of Theorem 2.2, we obtain (14). Then (14) yields that

$$[p(t)Z'(t)]'+\eta_0\sum_{k=1}^{s} a_k(t)V(\rho_k(t))\leq 0, \quad t\geq t_1. \tag{17}$$

The remainder of the proof is similar to that of Theorem 2.1, we omit it here.

In order to establish some Sturm type comparison theorems, we also need to consider the following ordinary differential equation

$$\big[P(t)x'(t)\big]'+Q(t)x(t)=0, \tag{18}$$

where $P \in C^2(R_+, (0, +\infty)), Q \in C(R_+, R_+)$. A solution x of (18) is said to be oscillatory if for each positive number T, there exists a point $t_T \in (T, \infty)$ such that $x(t_T) = 0$.

Lemma 2.1. *Let $x(t)$ and $u(y,t)$ be nontrivial solutions of equation (18) and Problem (1),(2) [or Problem (1),(3)], respectively; and $Z(t)$ be defined as in Theorem 2.1 (or 2.2). If $u(y,t)$ is nonoscillatory, then the following inequality holds eventually:*

$$\left\{\frac{x(t)}{Z(t)}\left[P(t)x'(t)Z(t) - x(t)p(t)Z'(t)\right]\right\}'$$
$$\geq \; p(t)\left[\frac{x(t)Z'(t) - x'(t)Z(t)}{Z(t)}\right]^2 + [P(t) - p(t)]x'^2(t)$$
$$+\left\{\sum_{j=1}^{m} K_j\gamma_j q_j(t)\left[1 - \sum_{i=1}^{l}\lambda_i(\sigma_j(t))\right]\frac{\sigma_j(t)}{t} - Q(t)\right\}x^2, \qquad (19)$$

where $\gamma_j \in (0,1)$.

Proof. Let $u(y,t)$ be a nonoscillatory solution of problem (1),(2) [or (1),(3)]. From Theorem 2.1 or 2.2, there exists $Z(t) > 0$ such that (4) hold for $t > t_1$. Then we have

$$\left\{\frac{x(t)}{Z(t)}\left[P(t)x'(t)Z(t) - x(t)p(t)Z'(t)\right]\right\}'$$
$$= P(t)x'^2(t) + [P(t)x'(t)]'x(t) - 2x(t)x'(t)\frac{p(t)Z'(t)}{Z(t)} - x^2(t)\frac{[p(t)Z'(t)]'}{Z(t)}$$
$$+x^2(t)\frac{p(t)Z'^2(t)}{Z^2(t)}$$
$$\geq P(t)x'^2(t) - Q(t)x^2(t) - 2x(t)x'(t)\frac{p(t)Z'(t)}{Z(t)}$$
$$+\sum_{k=1}^{m} K_j q_j(t)\left[1 - \sum_{i=1}^{l}\lambda_i(\sigma_j(t))\right]\frac{Z(\sigma_j(t))}{Z(t)}x^2(t) + p(t)\left[\frac{x(t)Z'(t)}{Z(t)}\right]^2.$$

In view of Lemma A, we have that for $t \geq t_1$

$$\left\{\frac{x(t)}{Z(t)}\left[P(t)x'(t)Z(t) - x(t)p(t)Z'(t)\right]\right\}'$$
$$\geq \; p(t)\left[\frac{x(t)Z'(t) - x'(t)Z(t)}{Z(t)}\right]^2 + [P(t) - p(t)]x'^2(t)$$

$$+\left\{\sum_{k=1}^{m} K_j\gamma_j q_j(t)\left[1-\sum_{i=1}^{l}\lambda_i(\sigma_j(t))\right]\frac{\sigma_j(t)}{t}-Q(t)\right\}x^2(t).$$

This completes the proof.

Lemma 2.2. *Let $x(t)$, $u(y,t)$ be nontrivial solutions of equation (18) and Problem (1),(2)[or Problem (1),(3)], respectively; and $Z(t)$ be defined as in Theorem 2.1 (or 2.2). If $u(y,t)$ is nonoscillatory, and there exists $j_0 \in I_m$ such that $\sigma'_{j_0}(t) > 0$, then the following inequality holds eventually:*

$$\begin{aligned}
&\left\{\frac{x(t)}{Z(\sigma_{j_0}(t))}[P(t)x'(t)Z(\sigma_{j_0}(t)) - x(t)p(t)Z'(t)]\right\}' \\
\geq\ & p(t)\left[\frac{x(t)Z'(t)}{Z(\sigma_{j_0}(t))}\sqrt{\frac{p(t)\sigma'_{j_0}(t)}{p(\sigma_{j_0}(t))}} - x'\sqrt{\frac{p(\sigma_{j_0}(t))}{p(t)\sigma'_{j_0}(t)}}\right]^2 \\
&+\left[P(t)-\frac{p(\sigma_{j_0}(t))}{\sigma'_{j_0}(t)}\right]x'^2 \\
&+\left[K_{j_0}q_{j_0}(t)\left(1-\sum_{i=1}^{l}\lambda_i(\sigma_{j_0}(t))\right)-Q(t)\right]x^2(t). \qquad (20)
\end{aligned}$$

Proof. Let $u(y,t)$ be a nonoscillatory solution of problem (1),(2) [or (1),(3)]. From Theorem 2.1 or 2.2, we have $Z(t) > 0, [p(t)Z'(t)]' < 0$ and for some $j_0 \in I_m$

$$[p(t)Z'(t)]' + K_{j_0}q_{j_0}(t)\left[1-\sum_{i=1}^{l}\lambda_i(\sigma_{j_0}(t))\right]Z(\sigma_{j_0}(t)) \leq 0, \quad t \geq t_1,$$

holds, hence we obtain

$$p(\sigma_{j_0}(t))Z'(\sigma_{j_0}(t)) \geq p(t)Z'(t),\ t \geq t_1. \qquad (21)$$

Then we have

$$\begin{aligned}
&\left\{\frac{x(t)}{Z(\sigma_{j_0}(t))}[P(t)x'(t)Z(\sigma_{j_0}(t)) - x(t)p(t)Z'(t)]\right\}' \\
&= P(t)x'^2(t) + (P(t)x'(t))'x(t) - 2x(t)x'(t)\frac{p(t)Z'(t)}{Z(\sigma_{j_0}(t))} - x^2(t)\frac{(p(t)Z'(t))'}{Z(\sigma_{j_0}(t))} \\
&\quad +x^2(t)\frac{p(t)Z'(t)Z'(\sigma_{j_0}(t))\sigma'_{j_0}(t)}{Z^2(\sigma_{j_0}(t))}
\end{aligned}$$

$$
\begin{aligned}
&\geq P(t)x'^2(t) - Q(t)x^2(t) - 2x(t)x'(t)\frac{p(t)Z'(t)}{Z(\sigma_{j_0}(t))} \\
&+K_{j_0}q_{j_0}(t)\left[1-\sum_{i=1}^{l}\lambda_i(\sigma_{j_0}(t))\right]x^2(t) \\
&+p(t)\left(\frac{x(t)Z'(t)}{Z(\sigma_{j_0}(t))}\right)^2\frac{p(t)\sigma'_{j_0}(t)}{p(\sigma_{j_0}(t))}
\end{aligned}
$$

$$
\begin{aligned}
&\geq p(t)\left[\left(\frac{x(t)Z'(t)}{Z(\sigma_{j_0}(t))}\right)^2\frac{p(t)\sigma'_{j_0}(t)}{p(\sigma_{j_0}(t))} - 2x(t)x'(t)\frac{Z'(t)}{Z(\sigma_{j_0}(t))}\right] \\
&+P(t)x'^2(t) + \left[K_{j_0}q_{j_0}(t)\left(1-\sum_{i=1}^{l}\lambda_i(\sigma_{j_0}(t))\right) - Q(t)\right]x^2(t) \\
&= p(t)\left[\frac{x(t)Z'(t)}{Z(\sigma_{j_0}(t))}\sqrt{\frac{p(t)\sigma'_{j_0}(t)}{p(\sigma_{j_0}(t))}} - x'\sqrt{\frac{p(\sigma_{j_0}(t))}{p(t)\sigma'_{j_0}(t)}}\right]^2 \\
&+\left[K_{j_0}q_{j_0}(t)\left(1-\sum_{i=1}^{l}\lambda_i(\sigma_{j_0}(t))\right) - Q(t)\right]x^2(t) \\
&+\left[P(t) - \frac{p(\sigma_{j_0}(t))}{\sigma'_{j_0}(t)}\right]x'^2(t),
\end{aligned}
$$

that is

$$
\begin{aligned}
&\left\{\frac{x(t)}{Z(\sigma_{j_0}(t))}[P(t)x'(t)Z(\sigma_{j_0}(t)) - x(t)p(t)Z'(t)]\right\}' \\
\geq\ & p(t)\left[\frac{x(t)Z'(t)}{Z(\sigma_{j_0}(t))}\sqrt{\frac{p(t)\sigma'_{j_0}(t)}{p(\sigma_{j_0}(t))}} - x'(t)\sqrt{\frac{p(\sigma_{j_0}(t))}{p(t)\sigma'_{j_0}(t)}}\right]^2 \\
&+\left[K_{j_0}q_{j_0}(t)\left(1-\sum_{i=1}^{l}\lambda_i(\sigma_{j_0}(t))\right) - Q\right]x^2(t) \\
&+\left[P(t) - \frac{p(\sigma_{j_0}(t))}{\sigma'_{j_0}(t)}\right]x'^2(t). \qquad (22)
\end{aligned}
$$

This completes the proof.

Lemma 2.3. *Let $x(t)$, $u(y,t)$ be nontrivial solutions of equation (18) and Problem (1),(2)[or Problem (1),(3)], respectively; and $Z(t)$ be defined as in*

Theorem 2.1 (or 2.2). If $u(y,t)$ is nonoscillatory, then the following inequality holds eventually:

$$\begin{aligned}
&\left[\left(p(t)x'(t) - \frac{x(t)p(t)Z'(t)}{Z(t)}\right)x(t) - \frac{1}{2}x^2(t)\left[p'(t) - p(t)P^{-1}(t)P'(t)\right]\right]' \\
\geq\ & p(t)\left[\frac{x(t)Z'(t)}{Z(t)} - x'(t)\right]^2 + \left\{\sum_{j=1}^{m} K_j\gamma_j q_j(t)\left[1 - \sum_{i=1}^{l}\lambda_i(\sigma_j(t))\right]\frac{\sigma_j(t)}{t}\right. \\
& \left. -p(t)P^{-1}(t)Q(t) - \frac{1}{2}[p'(t) - p(t)P^{-1}(t)P'(t)]'\right\}x^2(t), \qquad (23)
\end{aligned}$$

where $\gamma_j \in (0,1)$.

The proof is similar to that of Lemma 2.1 and we omit it here.

From Theorem 2.3, It is not difficult to see that the following results which related to the smallest eigenvalue η_0 of the Dirichlet problem (10) are true.

Lemma 2.4. *Let $x(t)$, $u(y,t)$ be nontrivial solutions of equation (18) and Problem (1),(3), respectively; and $Z(t)$ be defined as in Theorem 2.3. If $u(y,t)$ is nonoscillatory, then the following inequality holds eventually:*

$$\begin{aligned}
&\left\{\frac{x(t)}{Z(t)}\left[P(t)x'(t)Z(t) - x(t)p(t)Z'(t)\right]\right\}' \\
\geq\ & p(t)\left[\frac{x(t)Z'(t) - x'(t)Z(t)}{Z(t)}\right]^2 + [P(t) - p(t)]x'^2(t) \\
& + \left\{\eta_0\sum_{k=1}^{s}\gamma_k a_k(t)\left[1 - \sum_{i=1}^{l}\lambda_i(\rho_k(t))\right]\frac{\rho_k(t)}{t} - Q(t)\right\}x^2, \qquad (24)
\end{aligned}$$

where $\gamma_k \in (0,1)$.

Lemma 2.5. *Let $x(t)$, $u(y,t)$ be nontrivial solutions of equation (18) and Problem (1),(3), respectively; and $Z(t)$ be defined as in Theorem 2.3. If $u(y,t)$ is nonoscillatory, and there exists $k_0 \in I_s$ such that $\rho'_{k_0}(t) > 0$, then the following inequality holds eventually:*

$$\begin{aligned}
&\left\{\frac{x(t)}{Z(\rho_{k_0}(t))}[P(t)x'(t)Z(\rho_{k_0}(t)) - x(t)p(t)Z'(t)]\right\}' \\
\geq\ & p(t)\left[\frac{xZ'}{Z(\rho_{k_0}(t))}\sqrt{\frac{p(t)\rho'_{k_0}(t)}{p(\rho_{k_0}(t))}} - x'\sqrt{\frac{p(\rho_{k_0}(t))}{p(t)\rho'_{k_0}(t)}}\right]^2
\end{aligned}$$

$$+\left[P(t)-\frac{p(\rho_{k_0}(t))}{\rho'_{k_0}(t)}\right]x'^2$$
$$+\left\{\eta_0 a_{k_0}[1-\sum_{i=1}^{l}\lambda_i(\rho_{k_0}(t))]-Q(t)\right\}x^2(t). \tag{25}$$

Lemma 2.6. *Let $x(t)$, $u(y,t)$ be nontrivial solutions of equation (18) and Problem (1),(3), respectively; and $Z(t)$ be defined as in Theorem 2.3. If $u(y,t)$ is nonoscillatory, then the following inequality holds eventually:*

$$\left[\left(p(t)x'(t)-\frac{x(t)p(t)Z'(t)}{Z(t)}\right)x(t)-\frac{1}{2}x^2(t)\left[p'(t)-p(t)P^{-1}(t)P'(t)\right]\right]'$$
$$\geq p(t)\left[\frac{x(t)Z'(t)}{Z(t)}-x'(t)\right]^2+\left\{\eta_0\sum_{k=1}^{s}\gamma_k a_k(t)\left[1-\sum_{i=1}^{l}\lambda_i(\rho_k(t))\right]\frac{\rho_k(t)}{t}\right.$$
$$\left.-p(t)P^{-1}(t)Q(t)-\frac{1}{2}[p'(t)-p(t)P^{-1}(t)P'(t)]'\right\}x^2(t), \tag{26}$$

where $\gamma_k\in(0,1)$.

The following theorems are immediate results from Lemmas 2.1-2.6.

Theorem 2.4. *Let $x(t)$ be a oscillatory solution of equation (18) and $\{a_n\}$ is a real sequence such that $x(a_n)=0$, where $a_n\to\infty$ as $n\to\infty$. Suppose that for each sufficiently large $T>0$, there exists a $a_n>T$ such that*

$$\int_{a_n}^{a_{n+1}}\left\{\left[\sum_{k=1}^{m}K_j\gamma_j q_j(t)[1-\sum_{i=1}^{l}\lambda_i(\sigma_j(t))]\frac{\sigma_j(t)}{t}-Q(t)\right]x^2(t)\right.$$
$$\left.+[P(t)-p(t)]x'^2(t)\right\}dt>0, \tag{27}$$

where $\gamma_j\in(0,1)$. Then:

(I) every solution $u(y,t)$ of the problem (1), (2) is oscillatory in G; and

(II) every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Proof. (I) Suppose to the contrary, then without loss of generality, we may assume that there exist a solution $u(y,t)$ of problem (1), (2) and sufficiently large $T>0$ such that $u(y,t)>0, u(y,\tau_i(t))>0, u(y,\sigma_j(t))>0$ and $u(y,\rho_k(t))>0$ in $\Omega\times[T,\infty), T>0, i\in I_l, j\in I_m, k\in I_s$. Choose $a_n\geq T$. Then $u(y,t)>0, u(y,\tau_i(t))>0, u(y,\sigma_j(t))>0$ and $u(y,\rho_k(t))>0$

in $\Omega \times [a_n, a_{n+1}], i \in I_l, j \in I_m, k \in I_s$, and (19) holds. Now integrating (19) from a_n to a_{n+1}, we have

$$\int_{a_n}^{a_{n+1}} \left\{ \frac{x(t)}{Z(t)} \left[P(t)x'(t)Z(t) - x(t)p(t)Z'(t)\right] \right\}' dt$$
$$\geq \int_{a_n}^{a_{n+1}} p(t) \left[\frac{x(t)Z'(t) - x'(t)Z(t)}{Z(t)}\right]^2 dt + \int_{a_n}^{a_{n+1}} \left\{[P(t) - p(t)]x'^2(t)\right.$$
$$\left. + \left[\sum_{k=1}^{m} K_j\gamma_j q_j(t)[1 - \sum_{i=1}^{l} \lambda_i(\sigma_j(t))]\frac{\sigma_j(t)}{t} - Q(t)\right] x^2(t) \right\} dt, \quad (28)$$

By the assumption, the right side of (28) is reduced to

$$\int_{a_n}^{a_{n+1}} p(t) \left(\frac{x(t)Z'(t) - x'(t)Z(t)}{Z(t)}\right)^2 dt + \int_{a_n}^{a_{n+1}} \left\{[P(t) - p(t)]x'^2(t)\right.$$
$$\left. + \left[\sum_{k=1}^{m} K_j\gamma_j q_j(t)[1 - \sum_{i=1}^{l} \lambda_i(\sigma_j(t))]\frac{\sigma_j(t)}{t} - Q\right] x^2(t) \right\} dt > 0, \quad (29)$$

but the left side of (28) is change to

$$\int_{a_n}^{a_{n+1}} \left\{ \frac{x(t)}{Z(t)} \left[P(t)x'(t)Z(t) - x(t)p(t)Z'(t)\right] \right\}' dt$$
$$= \left\{ \frac{x(t)}{Z(t)} \left[P(t)x'(t)Z(t) - x(t)p(t)Z'(t)\right] \right\}_{a_n}^{a_{n+1}} = 0.$$

This contradiction proves that every nontrivial solution $u(y,t)$ has at least one $t_n \in (a_n, a_{n+1})$ such that $u(y, t_n) = 0$. Noting that $a_{n+1} > t_n > a_n \geq T, n \in N$, we see that every nontrivial solution $u(y,t)$ is oscillatory.

(II) According to Theorem 2.2, and Lemma 2.1, the remainder of the proof is similar to that of the proof of part (I), so we omit the details. The proof of Theorem 2.4 is complete.

Corollary 2.1. *Assume that equation (18) is oscillatory. If*

$$P(t) \geq p(t) \text{ and } \sum_{k=1}^{m} K_j\gamma_j q_j(t) \left[1 - \sum_{i=1}^{l} \lambda_i(\sigma_j(t))\right] \frac{\sigma_j(t)}{t} \geq Q(t)$$

hold eventually with $\gamma_j \in (0,1)$, and are not identical in any subinterval of $[t_0, \infty)$. Then:

(I) every solution $u(y,t)$ of the problem (1), (2) is oscillatory in G; and

(II) every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Theorem 2.5. *Let $x(t)$ be a oscillatory solution of equation (18) and $\{a_n\}$ is a real sequence such that $x(a_n) = 0$, where $a_n \to \infty$ as $n \to \infty$. Suppose that there exists $\sigma'_{j_0}(t) > 0, j_0 \in I_m$, and for each sufficiently large $T > 0$, there exists a $a_n > T$ such that*

$$\int_{a_n}^{a_{n+1}} \left\{ \left[K_{j_0} q_{j_0} \left(1 - \sum_{i=1}^{l} \lambda_i(\sigma_{j_0}(t)) \right) - Q(t) \right] x^2 + \left[P - \frac{p(\sigma_{j_0}(t))}{\sigma'_{j_0}(t)} \right] {x'}^2 \right\} dt > 0. \quad (30)$$

Then:

(I) every solution $u(y,t)$ of the problem (1), (2) is oscillatory in G; and

(II) every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

The proof is similar to that of the proof of Theorem 2.4, we omit it.

Corollary 2.2. *Assume that equation (18) is oscillatory. If there exists $\sigma'_{j_0}(t) > 0, j_0 \in I_m$, such that*

$$P(t) \geq \frac{p(\sigma_{j_0}(t))}{\sigma'_{j_0}(t)} \quad \text{and} \quad K_{j_0} q_{j_0}(t)[1 - \sum_{i=1}^{l} \lambda_i(\sigma_{j_0}(t))] \geq Q(t)$$

hold eventually, and are not identic in any subinterval of $[t_0, \infty)$. Then:

(I) every solution $u(y,t)$ of the problem (1), (2) is oscillatory in G; and

(II) every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Theorem 2.6. *Let $x(t)$ be a oscillatory solution of equation (18) and $\{a_n\}$ is a real sequence such that $x(a_n) = 0$, where $a_n \to \infty$ as $n \to \infty$. Suppose that for each sufficiently large $T > 0$, there exists a $a_n > T$ such that*

$$\int_{a_n}^{a_{n+1}} \left\{ \sum_{j=1}^{m} K_j \gamma_j q_j(t) \left[1 - \sum_{i=1}^{l} \lambda_i(\sigma_j(t)) \right] \frac{\sigma_j(t)}{t} - p(t) P^{-1}(t) Q(t) - \frac{1}{2} [p'(t) - p(t) P^{-1}(t) P'(t)]' \right\} x^2(t) dt > 0, \quad (31)$$

where $\gamma_j \in (0,1)$. Then:

(I) every solution $u(y,t)$ of the problem (1), (2) is oscillatory in G; and

(II) every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Corollary 2.3. *Assume that equation (18) is oscillatory. If*

$$\sum_{j=1}^{m} K_{j_0}\gamma_j q_j(t)\left[1-\sum_{i=1}^{l}\lambda_i(\sigma_j(t))\right]\frac{\sigma_j(t)}{t}-p(t)P^{-1}(t)Q(t)$$
$$\geq \frac{1}{2}[p'(t)-p(t)P^{-1}(t)P'(t)]'$$

holds eventually with $\gamma_j \in (0,1)$, and is not identic in any subinterval of $[t_0,\infty)$. Then:

(I) every solution $u(y,t)$ of the problem (1), (2) is oscillatory in G; and

(II) every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Theorem 2.7. *Let $x(t)$ be a oscillatory solution of equation (18) and $\{a_n\}$ is a real sequence such that $x(a_n)=0$, where $a_n \to \infty$ as $n \to \infty$. If for each sufficiently large $T>0$, there exists a $a_n > T$ such that*

$$\int_{a_n}^{a_{n+1}}\left\{\left[\eta_0\sum_{k=1}^{s}\gamma_k a_k[1-\sum_{i=1}^{l}\lambda_i(\rho_k(t))]\frac{\rho_k(t)}{t}-Q(t)\right]x^2+[P(t)-p(t)]x'^2\right\}dt$$
$$>0, \quad (32)$$

where $\gamma_k \in (0,1)$. Then every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Corollary 2.4. *Assume that equation (18) is oscillatory. If*

$$P(t)\geq p(t)\ \text{ and }\ \eta_0\sum_{k=1}^{s}\gamma_k a_k(t)\left[1-\sum_{i=1}^{l}\lambda_i(\rho_k(t))\right]\frac{\rho_k(t)}{t}\geq Q(t)$$

eventually hold with $\gamma_k \in (0,1)$, and are not identic in any subinterval of $[t_0,\infty)$. Then every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Theorem 2.8. *Let $x(t)$ be a oscillatory solution of equation (18), and $\{a_n\}$ is a real sequence such that $x(a_n)=0$, where $a_n \to \infty$ as $n \to \infty$. Suppose that there exists $\rho'_{k_0}(t)>0, k_0 \in I_s$, and for each sufficiently large $T>0$, there exists a $a_n > T$ such that*

$$\int_{a_n}^{a_{n+1}}\left\{\left[\eta_0 a_{k_0}\left(1-\sum_{i=1}^{l}\lambda_i(\rho_{k_0}(t))\right)-Q(t)\right]x^2+\left[P-\frac{p(\rho_{k_0}(t))}{\rho'_{k_0}(t)}\right]x'^2\right\}dt$$

$$> 0. \quad (33)$$

Then every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Corollary 2.5. *Assume that equation (18) is oscillatory. If there exists $\rho'_{k_0}(t) > 0, k_0 \in I_s$, such that*

$$P(t) \geq \frac{p(\rho_{k_0}(t))}{\rho'_{k_0}(t)} \text{ and } \eta_0 a_{k_0}(t)\left[1 - \sum_{i=1}^{l} \lambda_i(\rho_{k_0}(t))\right] \geq Q(t)$$

hold eventually, and are not identic in any subinterval of $[t_0, \infty)$. Then every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Theorem 2.9. *Let $x(t)$ be a oscillatory solution of equation (18) and $\{a_n\}$ is a real sequence such that $x(a_n) = 0$, where $a_n \to \infty$ as $n \to \infty$. Suppose that for each sufficiently large $T > 0$, there exists a $a_n > T$ such that*

$$\int_{a_n}^{a_{n+1}} \Bigg\{ \eta_0 \sum_{k=1}^{s} \gamma_k a_k(t)\left[1 - \sum_{i=1}^{l} \lambda_i(\rho_k(t))\right] \frac{\rho_k(t)}{t} \\ - p(t)P^{-1}(t)Q(t) - \frac{1}{2}[p'(t) - p(t)P^{-1}(t)P'(t)]' \Bigg\} x^2(t)dt > 0, \quad (34)$$

where $\gamma_k \in (0,1)$. Then every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Corollary 2.6. *Assume that equation (18) is oscillatory. If*

$$\eta_0 \sum_{k=1}^{s} \gamma_k a_k(t)\left[1 - \sum_{i=1}^{l} \lambda_i(\rho_k(t))\right] \frac{\rho_k(t)}{t} \\ \geq p(t)P^{-1}(t)Q(t) - \frac{1}{2}[p'(t) - p(t)P^{-1}(t)P'(t)]'$$

holds eventually with $\gamma_k \in (0,1)$, and is not identic in any subinterval of $[t_0, \infty)$. Then every solution $u(y,t)$ of the problem (1), (3) is oscillatory in G.

Example 1. Let constants $c > 0$ and $\mu > 0$. Consider the partial differential

$$\frac{\partial}{\partial t}\left\{ \frac{1}{t+\pi+1} \frac{\partial}{\partial t}\left[u(x,t) + \frac{3}{t+2\pi} u(x, t-2\pi)\right]\right\} \\ = \frac{1}{t+\pi+1} \triangle u(x,t) + \Big[\frac{1}{(t+\pi+1)^2} + \frac{6}{(t+\pi+1)(t+2\pi)^2}$$

$$
\begin{aligned}
&+\frac{3}{(t+\pi+1)^2(t+2\pi)}\Big]\triangle u(x,t-\frac{3\pi}{2}) \\
&+\Big[\frac{6}{(t+\pi+1)(t+2\pi)^3}+\frac{3}{(t+\pi+1)^2(t+2\pi)^2}\Big]\triangle u(x,t-\pi) \\
&-\Big[\frac{3}{(t+\pi+1)(t+2\pi)}+\frac{t+\pi}{t+\pi-3}\frac{\mu}{\sqrt{t+1}}\Big]u(x,t)\Big[1+\frac{c}{1+u^2(x,t)}\Big] \\
&-\frac{t+\pi}{t+\pi-3}\frac{\mu}{\sqrt{t+1}}u(x,t-\pi), \quad (x,t)\in(0,\pi)\times R^+\equiv G, \qquad (35)
\end{aligned}
$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0, \quad t\ge 1, \tag{36}$$

and

$$y''+\frac{\mu}{t}y=0, \quad t\ge 1. \tag{37}$$

A straightforward verification shows that the functions

$$P(t)=1,\ Q(t)=\frac{\mu}{t},\ q_1(t)=\frac{t+\pi}{t+\pi-3}\frac{\mu}{\sqrt{t+1}},\ \lambda_1(\sigma_1)=\lambda_1(t-\pi)=\frac{3}{t+\pi}.$$

By simple computation, for constant $\mu>0$ and for each $t\ge 1$, we have

$$\int_1^\infty \frac{1}{P(t)}dt=\int_1^\infty dt=\infty, \quad \int_1^\infty Q(t)=\int_1^\infty \frac{\mu}{t}dt=\infty.$$

From Wintner-Lighton type theorem [9], we know that equation (37) is oscillatory. Moreover,

$$K_{j_0}q_{j_0}\left[1-\sum_{i=1}^{l}\lambda_i(\sigma_{j_0}(t))\right]=\frac{\mu}{\sqrt{t+1}}>Q(t)=\frac{\mu}{t}, \quad P(t)>\frac{p(\sigma_{j_0}(t))}{\sigma'_{j_0}(t)}=\frac{1}{t+1}.$$

Hence, by Corollary 2.2, equation (35) is oscillatory if $\mu>0$. For example, if $c=0, u(x,t)=sinxcost$ is such a solution.

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References

[1] J.H. Wu, *Theory and Application of Partial Differential Equations*, Springer-Verlag, New York (1996).

[2] D.P. Mishev, D.D. Bainov, Oscillation of the solutions of parabolic differential equations of neutral type, *Appl. Math. Comput.*, **28**, No. 12 (1988), 97-111.

[3] L.H. Erbe, Q. Kong, B.G. Zhang, Oscillation of partial differential equations with functional argument, *Util. Math.*, **43** (1993), 129-139.

[4] W.N. Li, B.T. Cui, Oscillation of solutions of neutral partial functional differential equations, *J. Math. Anal. Appl.*, **234** (1999), 123-146.

[5] D.D. Bainov, B.T. Cui, D.P. Mishev, Forced oscillation of solutions of certain parabolic differential equations of neutral type, *J. Comput. 16 Appl. Math.*, **72** (1996), 309-318.

[6] Q.G. Yang, Oscillation of solutions of a class of nonlinear neutral partial differential equations, *Indian J. Pure Appl. Math.*, **35**, No. 1 (2004), 3-22.

[7] W.T. Reid, *Sturmin Theory for Ordinary Differential Equations*, Springer-verlag, New York (1980).

[8] L.H. Erbe, Oscillation criteria for second order nonlinear delay equation, *Canad. Math. Bull.*, **16** (1972), 49-59.

[9] W. Leighton, On self-adjonit differential equations of second order, *J. London Math, Soc.*, **27** (1952), 37-47.