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SIMPSON-LIKE AND HERMITE-HADAMARD-LIKE TYPE INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE PREINVEX FUNCTIONS

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Abstract: In this article, we extend some estimates of Simpson-like and Hermite-Hadamard-like type integral inequalities for functions whose second twice derivatives in absolute value at certain powers are preinvex and prequasi-invex.

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1. Introduction

For an interval I on the real line R, let $f : I \to R$ be a convex function and let $a, b \in I$ with $a < b$. We consider the well-known Hadamard's inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality [1].

In recent years, several extensions and generalizations have been considered for classical convexity [9, 11, 12, 13]. A significant generalization of convex func-

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tions is that of invex functions introduced by Hanson in [7]. Weir and Mond [14] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. Pini [10] introduced the concept of preinvex function as a generalization of invex functions. Later Mohan and Neogy [8] obtained some properties of generalized preinvex functions.

Yang et al. in [15] studied prequasiinvex function, and semistrictly prequasiinvex functions. Noor [2, 3, 4, 5] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers, Noor and Barani et al. in [1, 2, 3, 6] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

Definition 1. A set $S \subseteq R$ is said to be *invex* with respect to the map $\eta : S \times S \to R$, if for any $x, y \in S$ and $t \in [0,1]$, $x + t\eta(y,x) \in S$.

It is obvious that every convex set is invex with respect to the map $\eta(x,y) = y - x$, but there exist invex sets which are not convex [8].

Definition 2. Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \to R$. Then the function $f : S \to R$ is said to be *preinvex* with respect to η, if for any $x, y \in S$ and $t \in [0,1]$,

$$f(y + t\eta(x,y)) \le tf(x) + (1-t)f(y).$$

In [6], Barani et al. introduced some generalizations of Hermite-Hadamard type inequality for functions whose second derivatives absolute values are preinvex:

Theorem 1.1. *Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \to R$ and $\eta(b,a) \neq 0$ for all $a \neq b$. Suppose that $f : S \to R$ is a twice differentiable function on S. If $\left|f''\right|$ is preinvex on S and $f'' \in L[a, a+\eta(b,a)]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\le \frac{\eta^2(b,a)}{24}\Big(\left|f''(a)\right| + \left|f''(b)\right|\Big). \end{aligned} \tag{1}$$

Theorem 1.2. *Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \to R$ and $\eta(b,a) \neq 0$ for all $a \neq b$. Suppose that $f : S \to R$ is a twice* *differentiable function on* S*. If* $\left|f''\right|^{\frac{p}{p-1}}$ *is preinvex on* S *and* $f'' \in L[a, a+\eta(b,a)]$*, for* $p > 1$*, then the following inequality holds:*

$$\Big|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\Big|$$
$$\leq \frac{\eta^2(b,a)}{16}\beta^{\frac{1}{p}}(\frac{1}{2}, 1+p)\Big(\left|f''(a)\right|^q + \left|f''(b)\right|^q\Big)^{\frac{1}{q}}, \tag{2}$$

where β *is a beta function defined by* $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$*.*

Theorem 1.3. *Let* $S \subseteq R$ *be an invex set with respect to the map* $\eta : S \times S \to R$ *and* $\eta(b,a) \neq 0$ *for all* $a \neq b$*. Suppose that* $f : S \to R$ *is a twice differentiable function on* S*. If* $\left|f''\right|^q$ *is preinvex on* S *and* $f'' \in L[a, a+\eta(b,a)]$*, for* $q \geq 1$*. then the following inequality holds:*

$$\Big|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\Big|$$
$$\leq \frac{\eta^2(b,a)}{12}(\frac{1}{2})^{\frac{1}{q}}\Big(\left|f''(a)\right|^q + \left|f''(b)\right|^q\Big)^{\frac{1}{q}}. \tag{3}$$

Definition 3. Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \to R$. Then the function $f : S \to R$ is said to be *prequasiinvex* with respect to η, if, for any $x, y \in S$ and $t \in [0,1]$,

$$f(y + t\eta(x,y)) \leq \max\{f(x), f(y)\}.$$

Mohan and Neogy [8] introduced Condition C defined as follows:

Definition 4. Let $S \subseteq R$ be an open invex subset with respect to the map $\eta : S \times S \to R$. We say that the function η satisfies the *Condition C* if, for any $x, y \in S$ and any $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y), \tag{4}$$
$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y). \tag{5}$$

Note that, from the Condition C, we have

$$\eta(y + t_2\eta(x,y)y, y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y)$$

for any $x, y \in S$ and any $t_1, t_2 \in [0,1]$.

In this article, using a general integral identity for twice differentiable functions, we establish generalized Simpson-like and Hermite-Hadamard-like type integral inequalities for mappings whose twice derivatives in absolute value at certain powers are preinvex and prequasiinvex.

2. Main Results

To prove our main results, we need the following lemma [12]:

Lemma 1. Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \to R^n$ and, $\eta(b,a) \neq 0$ and $0 \leq a < a + \eta(b,a) < \infty$ for all $a \neq b$. Suppose that $f : S \to R$ is a twice differentiable function on S such that $f'' \in L[a, a+\eta(b,a)]$. Then the following identity holds:

$$\begin{aligned} &I(f : \eta : a, b : \lambda) \\ &\equiv^{\text{let}} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \\ &\quad - (1-\lambda) f(a + \frac{\eta(b,a)}{2}) - \lambda \{ \frac{f(a) + f(a+\eta(b,a))}{2} \} \\ &= \frac{\eta^2(b,a)}{2} \int_0^1 k(t) f''(a + t\eta(b,a)))dt \end{aligned} \tag{6}$$

for all $x \in [a, a+\eta(b,a)]$, where

$$k(t) = \begin{cases} t(t-\lambda) & 0 \leq t < \frac{1}{2} \\ (1-t)(1-\lambda-t) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Proof. Suppose that $a, b \in S$. Since S is an invex set with respect to $\eta : S \times S \to R$, for any $t \in [0,1]$, we have $a + t\eta(b,a) \in S$. Integrating by parts this is proved.

Theorem 2.1. *Let $S \subseteq R$ be an invex set with respect to the map $\eta : S \times S \to R$ and, $\eta(b,a) \neq 0$ and $0 \leq a < a + \eta(b,a) < \infty$ for all $a \neq b$. Suppose that $f : S \to R$ is a twice differentiable function on S such that $f'' \in L[a, a+\eta(b,a)]$. If $|f''|$ is preinvex with respect to η on S, then the following inequalities hold:*

(a) For $0 \leq \lambda \leq \frac{1}{2}$:

$$\Big| I(f : \eta : a, b : \lambda) \Big| \leq \frac{\eta^2(b,a)}{48} (8\lambda^3 - 3\lambda + 1) \Big(|f''(a)| + |f''(b)| \Big).$$

(b) For $\frac{1}{2} \leq \lambda \leq 1$:

$$\Big| I(f : \eta : a, b : \lambda) \Big| \leq \frac{\eta^2(b,a)}{48} (3\lambda - 1) \Big(|f''(a)| + |f''(b)| \Big).$$

Proof. From Lemma 1 and by the definition of $k(t)$, we get

$$\begin{aligned}\Big|I(f:\eta:a,b:\lambda)\Big| &\leq \frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|dt \\ &\quad+\int_{\frac{1}{2}}^{1}\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|dt\Big] \\ &= \frac{\eta^2(b,a)}{2}\Big\{J_1+J_2\Big\}. \qquad \text{(say)}\end{aligned} \tag{7}$$

(a) Assume that $0 \leq \lambda \leq \frac{1}{2}$.
By using the preinvexity of $\big|f''\big|$ with repect to η on S, we get

$$\begin{aligned}J_1 &= \int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|dt \\ &\leq \int_0^{\lambda} t(\lambda-t)\big\{t\big|f''(b)\big|+(1-t)\big|f''(a)\big|\big\}dt \\ &\quad+\int_{\lambda}^{\frac{1}{2}} t(t-\lambda)\big\{t\big|f''(b)\big|+(1-t)\big|f''(a)\big|\big\}dt \\ &= \left\{\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\right\}|f''(b)| \\ &\quad+\left\{\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\right\}|f''(a)|,\end{aligned} \tag{8}$$

and

$$\begin{aligned}J_2 &= \int_{\frac{1}{2}}^{1}\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|dt \\ &\leq \left\{\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}+\frac{\lambda^4}{6}\right\}\big|f''(b)\big|+\left\{\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\right\}\big|f''(a)\big|.\end{aligned} \tag{9}$$

By using (8) and (9) in (7), the part (a) is proved.
(b) Assume that $\frac{1}{2} \leq \lambda \leq 1$.
By using the preinvexity of $\big|f''\big|$ with repect to η on S, we get

$$\begin{aligned}J_1 &= \int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|dt \\ &\leq \left\{\frac{\lambda}{24}-\frac{1}{64}\right\}\big|f''(b)\big|+\left\{\frac{\lambda}{12}-\frac{5}{192}\right\}\big|f''(a)\big|,\end{aligned} \tag{10}$$

and

$$J_2 = \int_{\frac{1}{2}}^{1} \left|(1-t)(1-\lambda-t)\right|\left|f''(a+t\eta(b,a))\right|dt$$
$$\leq \left\{\frac{\lambda}{12} - \frac{5}{192}\right\}\left|f''(b)\right| + \left\{\frac{\lambda}{24} - \frac{1}{64}\right\}\left|f''(a)\right|. \quad (11)$$

By using (10) and (11) in (7), the part (b) is proved.

Corollary 1. *In Theorem 2.1,*
(a)(Midpoint inequality) if we choose $\lambda = 0$*, then we get*

$$I(f:\eta:a,b:0) \leq \frac{\eta^2(b,a)}{48}\Big(\left|f''(a)\right| + \left|f''(b)\right|\Big).$$

(b)(Trapezoid inequality) if we choose $\lambda = 1$*, then we have*

$$I(f:\eta:a,b:1) \leq \frac{\eta^2(b,a)}{8}\Big(\left|f''(a)\right| + \left|f''(b)\right|\Big).$$

(c)(Simpson inequality) if we choose $\lambda = \frac{1}{3}$*, then we get*

$$I(f:\eta:a,b:\frac{1}{3}) \leq \frac{\eta^2(b,a)}{162}\Big(\left|f''(a)\right| + \left|f''(b)\right|\Big).$$

(d) if we choose $\lambda = \frac{1}{2}$*, then we have*

$$I(f:\eta:a,b:\frac{1}{2}) \leq \frac{\eta^2(b,a)}{96}\Big(\left|f''(a)\right| + \left|f''(b)\right|\Big).$$

Theorem 2.2. *Let* $S \subseteq R$ *be an invex set with respect to the map* $\eta : S \times S \to R$ *and,* $\eta(b,a) \neq 0$ *and* $0 \leq a < a+\eta(b,a) < \infty$ *for all* $a \neq b$*. Suppose that* $f : S \to R$ *is a twice differentiable function on* S *such that* $f'' \in L[a, a+\eta(b,a)]$ *and* η *satisfies Condition* C*. If* $\left|f''\right|$ *is preinvex with respect to* η *on* S*, then the following inequalities hold:*
(a) For $0 \leq \lambda \leq \frac{1}{2}$ *:*

$$\Big|I(f:\eta:a,b:\lambda)\Big|$$
$$\leq \frac{\eta^2(b,a)}{48}(8\lambda^3 - 3\lambda + 1)\Big(\left|f''(a+\eta(b,a))\right| + \left|f''(a)\right|\Big).$$

(b) For $\frac{1}{2} \leq \lambda \leq 1$*:*

$$\Big|I(f:\eta:a,b:\lambda)\Big|$$
$$\leq \frac{\eta^2(b,a)}{48}(3\lambda - 1)\Big(\left|f''(a+\eta(b,a))\right| + \left|f''(a)\right|\Big).$$

Proof. By Condition C and the preinvexity of $\left|f''\right|$ with repect to η on S, we have

$$\begin{aligned} &\left|f''(a+t\eta(b,a))\right| \\ &= \left|f''(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a)))\right| \\ &\leq t\left|f''(a+\eta(b,a))\right|+(1-t)\left|f''(a)\right|. \end{aligned} \tag{12}$$

(a) Assume that $0 \leq \lambda \leq \frac{1}{2}$. By (12), we get

$$\begin{aligned} J_1 &= \int_0^{\frac{1}{2}} \left|t(t-\lambda)\right|\left|f''(a+t\eta(b,a))\right|dt \\ &\leq \left\{\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\right\}\left|f''(a+\eta(b,a))\right| \\ &\qquad + \left\{\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\right\}\left|f''(a)\right|, \end{aligned} \tag{13}$$

and

$$\begin{aligned} J_2 &= \int_{\frac{1}{2}}^1 \left|(1-t)(1-\lambda-t)\right|\left|f''(a+t\eta(b,a))\right|dt \\ &\leq \left\{\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}+\frac{\lambda^4}{6}\right\}\left|f''(a+\eta(b,a))\right| \\ &\qquad + \left\{\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\right\}\left|f''(a)\right|. \end{aligned} \tag{14}$$

By using (13) and (14) in (7), the part (a) is proved.
(b) Assume that $\frac{1}{2} \leq \lambda \leq 1$.
By using the preinvexity of $\left|f''\right|$ with repect to η on S, we get

$$\begin{aligned} J_1 &= \int_0^{\frac{1}{2}} \left|t(t-\lambda)\right|\left|f''(a+t\eta(b,a))\right|dt \\ &\leq \int_0^{\frac{1}{2}} \left|t(t-\lambda)\right|\left\{t\left|f''(a+\eta(b,a))\right|+(1-t)\left|f''(a)\right|\right\}dt \\ &\leq \left\{\frac{\lambda}{24}-\frac{1}{64}\right\}\left|f''(a+\eta(b,a))\right|+\left\{\frac{\lambda}{12}-\frac{5}{192}\right\}\left|f''(a)\right|, \end{aligned} \tag{15}$$

and

$$J_2 = \int_{\frac{1}{2}}^1 \left|(1-t)(1-\lambda-t)\right|\left|f''(a+t\eta(b,a))\right|dt$$

$$\leq \int_{\frac{1}{2}}^{1} (1-t)(t+\lambda-1)tdt\big|f''(a+\eta(b,a))\big|$$
$$+\int_{\frac{1}{2}}^{1} (1-t)(t+\lambda-1)(1-t)dt\big|f''(a)\big|$$
$$=\Big\{\frac{\lambda}{12}-\frac{5}{192}\Big\}\big|f''(a+\eta(b,a))\big|+\Big\{\frac{\lambda}{24}-\frac{1}{64}\Big\}\big|f''(a)\big|. \quad (16)$$

By using (15) and (16) in (7), the part (b) is proved.

Theorem 2.3. *Let $S \subseteq R$ be an open invex subset with respect to the map $\eta : S \times S \to R$ and, $\eta(b,a) \neq 0$ and $0 \leq a < a+\eta(b,a) < \infty$ for all $a \neq b$. Suppose that $f : S \to R$ is a twice differentiable function on S such that $f'' \in L[a, a+\eta(b,a)]$. If $\big|f''\big|^q$ is preinvex with respect to η on S for $q \geq 1$ and $0 \leq \lambda \leq 1$, then the following inequalities hold:*

(a) For $0 \leq \lambda \leq \frac{1}{2}$:

$$\Big|I(f:\eta:a,b:\lambda)\Big|$$
$$\leq \frac{\eta^2(b,a)}{2}\Big(\frac{8\lambda^3-3\lambda+1}{24}\Big)^{1-\frac{1}{q}}\Big[\Big\{\big(\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\big)\big|f''(b)\big|^q$$
$$+\big(\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\big)\big|f''(a)\big|^q\Big\}^{\frac{1}{q}}$$

$$+\Big\{\big(\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\big)\big|f''(b)\big|^q$$
$$+\big(\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\big)\big|f''(a)\big|^q\Big\}^{\frac{1}{q}}\Big].$$

(b) For $\frac{1}{2} \leq \lambda \leq 1$:

$$\Big|I(f:\eta:a,b:\lambda)\Big|$$
$$\leq \frac{\eta^2(b,a)}{2}\Big(\frac{3\lambda-1}{24}\Big)^{1-\frac{1}{q}}$$
$$\times\Big[\Big\{\big(\frac{\lambda}{24}-\frac{1}{64}\big)\big|f''(b)\big|^q+\big(\frac{\lambda}{12}-\frac{5}{192}\big)\big|f''(a)\big|^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\big(\frac{\lambda}{12}-\frac{5}{192}\big)\big|f''(b)\big|^q+\big(\frac{\lambda}{24}-\frac{1}{64}\big)\big|f''(a)\big|^q\Big\}^{\frac{1}{q}}\Big].$$

Proof. From Lemma 1 and by the definition of $k(t)$, we get

$$\begin{aligned}
&\Big|I(f:\eta:a,b:\lambda)\Big| \\
&\leq \frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|dt \\
&\qquad + \int_{\frac{1}{2}}^1\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|dt\Big] \\
&\leq \frac{\eta^2(b,a)}{2}\Big[\Big\{\int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|dt\Big\}^{1-\frac{1}{q}} \\
&\qquad \times\Big\{\int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|^q dt\Big\}^{\frac{1}{q}} \\
&\quad +\Big\{\int_{\frac{1}{2}}^1\big|(1-t)(1-\lambda-t)\big|dt\Big\}^{1-\frac{1}{q}} \\
&\qquad \times\Big\{\int_{\frac{1}{2}}^1\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|^q dt\Big\}^{\frac{1}{q}}\Big]. \qquad (17)
\end{aligned}$$

(a) Assume that $0 \leq \lambda \leq \frac{1}{2}$.
Since $\big|f''\big|^q$ is preinvex with repect to η on S, we know that for $t \in [0,1]$

$$\big|f''(a+t\eta(b,a))\big|^q \leq t\big|f''(b)\big|^q + (1-t)\big|f''(a)\big|^q. \qquad (18)$$

Here by our assumption, we have that

$$\begin{aligned}
&(i)\int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|^q dt \\
&\leq \Big\{\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\Big\}\big|f''(b)\big|^q + \Big\{\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\Big\}\big|f''(a)\big|^q, \qquad (19)
\end{aligned}$$

$$\begin{aligned}
&(ii)\int_{\frac{1}{2}}^1\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|^q dt \\
&\leq \Big\{\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\Big\}\big|f''(b)\big|^q + \Big\{\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\Big\}\big|f''(a)\big|^q. \qquad (20)
\end{aligned}$$

By (18)-(20) in (17), the part (a) is proved.
(b) Assume that $\frac{1}{2} \leq \lambda \leq 1$.

By using the preinvexity of $|f''|$ with repect to η on S, we get that

$$(i) \int_0^{\frac{1}{2}} \left|t(t-\lambda)\right|\left|f''(a+t\eta(b,a))\right|^q dt \\ \leq \left\{\frac{\lambda}{24}-\frac{1}{64}\right\}\left|f''(b)\right|^q + \left\{\frac{\lambda}{12}-\frac{5}{192}\right\}\left|f''(a)\right|^q, \tag{21}$$

$$(ii) \int_{\frac{1}{2}}^{1} \left|(1-t)(1-\lambda-t)\right|\left|f''(a+t\eta(b,a))\right|^q dt \\ \leq \left\{\frac{\lambda}{12}-\frac{5}{192}\right\}\left|f''(b)\right|^q + \left\{\frac{\lambda}{24}-\frac{1}{64}\right\}\left|f''(a)\right|^q. \tag{22}$$

By using (18) and (21)-(22) in (17), the part (b) is proved.

Corollary 2. *In Theorem 2.3,*
(a)(Midpoint inequality) if we choose $\lambda = 0$*, then we get*

$$\left|I(f:\eta:a,b:0)\right| \leq \frac{\eta^2(b,a)}{48}\Big[\Big\{\frac{3\left|f''(b)\right|^q+5\left|f''(a)\right|^q}{8}\Big\}^{\frac{1}{q}} \\ + \Big\{\frac{5\left|f''(b)\right|^q+3\left|f''(a)\right|^q}{8}\Big\}^{\frac{1}{q}}\Big].$$

(b)(Trapezoid inequality) if we choose $\lambda = 1$*, then we have*

$$\left|I(f:\eta:a,b:1)\right| \leq \frac{\eta^2(b,a)}{24}\Big[\Big\{\frac{5\left|f''(b)\right|^q+11\left|f''(a)\right|^q}{16}\Big\}^{\frac{1}{q}} \\ + \Big\{\frac{11\left|f''(b)\right|^q+5\left|f''(a)\right|^q}{16}\Big\}^{\frac{1}{q}}\Big].$$

(c)(Simpson inequality) if we choose $\lambda = \frac{1}{3}$*, then we get*

$$\left|I(f:\eta:a,b:\frac{1}{3})\right| \leq \frac{\eta^2(b,a)}{162}\Big[\Big\{\frac{59\left|f''(b)\right|^q+133\left|f''(a)\right|^q}{192}\Big\}^{\frac{1}{q}} \\ + \Big\{\frac{133\left|f''(b)\right|^q+59\left|f''(a)\right|^q}{192}\Big\}^{\frac{1}{q}}\Big].$$

(d) if we choose $\lambda = \frac{1}{2}$*, then we have*

$$\left|I(f:\eta:a,b:\frac{1}{2})\right| \leq \frac{\eta^2(b,a)}{96}\Big[\Big\{\frac{\left|f''(b)\right|^q+3\left|f''(a)\right|^q}{4}\Big\}^{\frac{1}{q}} \\ + \Big\{\frac{3\left|f''(b)\right|^q+\left|f''(a)\right|^q}{4}\Big\}^{\frac{1}{q}}\Big].$$

Theorem 2.4. *Let $S \subseteq R$ be an open invex subset with respect to the map $\eta : S \times S \to R$ and, $\eta(b,a) \neq 0$ and $0 \leq a < a+\eta(b,a) < \infty$ for all $a \neq b$. Suppose that $f : S \to R$ is a twice differentiable function on S such that $f'' \in L[a, a+\eta(b,a)]$ and η satisfies Condition C. If $\left|f''\right|^q$ is preinvex with respect to η on S for $q \geq 1$ and $0 \leq \lambda \leq 1$, then the following inequalities hold:*

(a) For $0 \leq \lambda \leq \frac{1}{2}$:

$$
\begin{aligned}
&\left|I(f:\eta:a,b:\lambda)\right| \\
&\leq \frac{\eta^2(b,a)}{2}\Big(\frac{\lambda^3}{3}+\frac{1-3\lambda}{24}\Big)^{1-\frac{1}{q}} \\
&\quad \times \Big[\Big\{\Big(\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\Big)\left|f''(a+\eta(b,a))\right|^q \\
&\qquad + \Big(\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\Big)\left|f''(a)\right|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\Big(\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\Big)\left|f''(a+\eta(b,a))\right|^q \\
&\qquad + \Big(\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\Big)\left|f''(a)\right|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}
$$

(b) For $\frac{1}{2} \leq \lambda \leq 1$:

$$
\begin{aligned}
&\left|I(f:\eta:a,b:\lambda)\right| \\
&\leq \frac{\eta^2(b,a)}{2}\Big(\frac{3\lambda-1}{24}\Big)^{1-\frac{1}{q}} \\
&\times \Big[\Big\{\Big(\frac{\lambda}{24}-\frac{1}{64}\Big)\left|f''(a+\eta(b,a))\right|^q + \Big(\frac{\lambda}{12}-\frac{5}{192}\Big)\left|f''(a)\right|^q\Big\}^{\frac{1}{q}} \\
&+ \Big\{\Big(\frac{\lambda}{12}-\frac{5}{192}\Big)\left|f''(a+\eta(b,a))\right|^q + \Big(\frac{\lambda}{24}-\frac{1}{64}\Big)\left|f''(a)\right|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}
$$

Proof. (a) Assume that $0 \leq \lambda \leq \frac{1}{2}$.
Since $\left|f''\right|^q$ is preinvex with repect to η on S, we know that for $t \in [0,1]$

$$
\left|f''(a+t\eta(b,a)\right|^q \leq t\left|f''(a+\eta(b,a))\right|^q + (1-t)\left|f''(a)\right|^q. \tag{23}
$$

Here by our assumption, we have that

$$
(i) \int_0^{\frac{1}{2}} \left|t(t-\lambda)\right|\left|f''(a+t\eta(b,a))\right|^q dt
$$

$$\leq \left\{\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\right\}\left|f''(a+\eta(b,a))\right|^q$$
$$+\left\{\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\right\}\left|f''(a)\right|^q, \quad (24)$$

$$(ii)\int_{\frac{1}{2}}^{1}\left|(1-t)(1-\lambda-t)\right|\left|f''(a+t\eta(b,a))\right|^q dt$$
$$\leq \left\{\frac{5}{192}-\frac{\lambda}{12}+\frac{\lambda^3}{3}-\frac{\lambda^4}{6}\right\}\left|f''(a+\eta(b,a))\right|^q$$
$$+\left\{\frac{1}{64}-\frac{\lambda}{24}+\frac{\lambda^4}{6}\right\}\left|f''(a)\right|^q. \quad (25)$$

By (23)-(25) in (17), the part (a) is proved.
(b) Assume that $\frac{1}{2}\leq\lambda\leq 1$.
By using the preinvexity of $\left|f''\right|$ with repect to η on S, we get

$$(i)\int_{0}^{\frac{1}{2}}\left|t(t-\lambda)\right|\left|f''(a+t\eta(b,a))\right|^q dt$$
$$\leq \left\{\frac{\lambda}{24}-\frac{1}{64}\right\}\left|f''(a+\eta(b,a))\right|^q+\left\{\frac{\lambda}{12}-\frac{5}{192}\right\}\left|f''(a)\right|^q, \quad (26)$$
$$(ii)\int_{\frac{1}{2}}^{1}\left|(1-t)(1-\lambda-t)\right|\left|f''(a+t\eta(b,a))\right|^q dt$$
$$\leq \left\{\frac{\lambda}{12}-\frac{5}{192}\right\}\left|f''(a+\eta(b,a))\right|^q+\left\{\frac{\lambda}{24}-\frac{1}{64}\right\}\left|f''(a)\right|^q. \quad (27)$$

By using (23) and (26)-(27) in (17), the part (b) is proved.

Corollary 3. *In Theorem 2.4,*
(a)(Midpoint inequality) if we choose $\lambda=0$, then we get

$$\left|I(f:\eta:a,b:0)\right|$$
$$\leq \frac{\eta^2(b,a)}{48}\left[\left\{\frac{3\left|f''(a+\eta(b,a))\right|^q+5\left|f''(a)\right|^q}{8}\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{5\left|f''(a+\eta(b,a))\right|^q+3\left|f''(a)\right|^q}{8}\right\}^{\frac{1}{q}}\right].$$

(b)(Trapezoid inequality) if we choose $\lambda=1$, then we have

$$\left|I(f:\eta:a,b:1)\right|$$

$$\leq \frac{\eta^2(b,a)}{24}\Big[\Big\{\frac{5\big|f''(a+\eta(b,a))\big|^q + 11\big|f''(a)\big|^q}{16}\Big\}^{\frac{1}{q}} + \Big\{\frac{11\big|f''(a+\eta(b,a))\big|^q + 5\big|f''(a)\big|^q}{16}\Big\}^{\frac{1}{q}}\Big].$$

(c)(Simpson inequality) if we choose $\lambda = \frac{1}{3}$*, then we get*

$$\Big|I(f:\eta:a,b:\frac{1}{3})\Big| \leq \frac{\eta^2(b,a)}{162}\Big[\Big\{\frac{59\big|f''(a+\eta(b,a))\big|^q + 133\big|f''(a)\big|^q}{192}\Big\}^{\frac{1}{q}} + \Big\{\frac{133\big|f''(a+\eta(b,a))\big|^q + 59\big|f''(a)\big|^q}{192}\Big\}^{\frac{1}{q}}\Big].$$

(d) if we choose $\lambda = \frac{1}{2}$*, then we have*

$$\Big|I(f:\eta:a,b:\frac{1}{2})\Big| \leq \frac{\eta^2(b,a)}{96}\Big[\Big\{\frac{\big|f''(a+\eta(b,a))\big|^q + 3\big|f''(a)\big|^q}{4}\Big\}^{\frac{1}{q}} + \Big\{\frac{3\big|f''(a+\eta(b,a))\big|^q + \big|f''(a)\big|^q}{4}\Big\}^{\frac{1}{q}}\Big].$$

Theorem 2.5. *Let* $S \subseteq R$ *be an open invex subset with respect to the map* $\eta : S \times S \to R$ *and,* $\eta(b,a) \neq 0$ *and* $0 \leq a < a + \eta(b,a) < \infty$ *for all* $a \neq b$*. Suppose that* $f : S \to R$ *is a twice differentiable function on* S *such that* $f'' \in L[a, a+\eta(b,a)]$*. If* $\big|f''\big|$ *is prequasiinvex with respect to* η *on* S *and* $0 \leq \lambda \leq 1$*, then the following inequalities hold:*

(a) For $0 \leq \lambda \leq \frac{1}{2}$*:*

$$\Big|I(f:\eta:a,b:\lambda)\Big| \leq \frac{\eta^2(b,a)}{24}\Big(8\lambda^3 - 3\lambda + 1\Big)\max\Big\{\big|f''(a)\big|, \big|f''(b)\big|\Big\}.$$

(b) For $\frac{1}{2} \leq \lambda \leq 1$*:*

$$\Big|I(f:\eta:a,b:\lambda)\Big| \leq \frac{\eta^2(b,a)}{24}\Big(3\lambda - 1\Big)\max\Big\{\big|f''(a)\big|, \big|f''(b)\big|\Big\}.$$

Proof. From Lemma 1 and by the definition of $k(t)$, we get

$$\begin{aligned}
&\Big|I(f:\eta:a,b:\lambda)\Big|\\
&\le \frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|dt\\
&\quad+\int_{\frac{1}{2}}^{1}\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|dt\Big]\\
&\le \frac{\eta^2(b,a)}{2}\Big\{\int_0^{\frac{1}{2}}\big|t(t-\lambda)\big|dt+\int_{\frac{1}{2}}^{1}\big|(1-t)(1-\lambda-t)\big|dt\Big\}\\
&\quad\times\max\Big\{\big|f''(a)\big|,\big|f''(b)\big|\Big\}. \qquad (28)
\end{aligned}$$

(a) Assume that $0\le\lambda\le\frac{1}{2}$.
Since $\big|f''\big|$ is prequasiinvex with repect to η on S, we know that for $t\in[0,1]$

$$\big|f''(a+t\eta(b,a)\big|\le\max\Big\{\big|f''(a)\big|,\big|f''(b)\big|\Big\}. \qquad (29)$$

By (29) in (28), we get

$$\begin{aligned}
&\Big|I(f:\eta:a,b:\lambda)\Big|\\
&\le\frac{\eta^2(b,a)}{24}\Big(8\lambda^3-3\lambda+1\Big)\max\Big\{\big|f''(a)\big|,\big|f''(b)\big|\Big\}.
\end{aligned}$$

(b) Assume that $\frac{1}{2}\le\lambda\le 1$.
By using the prequasiinvexity of $\big|f''\big|$ with repect to η on S, we get

$$\begin{aligned}
&\Big|I(f:\eta:a,b:\lambda)\Big|\\
&\le\frac{\eta^2(b,a)}{2}\Big\{\frac{1}{24}\Big(3\lambda-1\Big)+\frac{1}{24}\Big(3\lambda-1\Big)\Big)\max\Big\{\big|f''(a)\big|,\big|f''(b)\big|\Big\}\\
&=\frac{\eta^2(a,b)}{24}\Big(3\lambda-1\Big)\max\Big\{\big|f''(a)\big|,\big|f''(b)\big|\Big\}.
\end{aligned}$$

Corollary 4. *In Theorem 2.5,*
(a)(Midpoint inequality) if we choose $\lambda=0$, then we get

$$\Big|I(f:\eta:a,b:0)\Big|\le\frac{\eta^2(b,a)}{24}\Big[\max\Big\{\big|f''(a)\big|^q,\big|f''(b)\big|^q\Big\}\Big]^{\frac{1}{q}}.$$

(b)(Trapezoid inequality) if we choose $\lambda = 1$, *then we have*

$$\left|I(f:\eta:a,b:1)\right| \leq \frac{\eta^2(b,a)}{12}\Big[\max\Big\{\left|f''(a)\right|^q, \left|f''(b)\right|^q\Big\}\Big]^{\frac{1}{q}}.$$

(c)(Simpson inequality) if we choose $\lambda = \frac{1}{3}$, *then we get*

$$\left|I(f:\eta:a,b:\frac{1}{3})\right| \leq \frac{\eta^2(b,a)}{81}\Big[\max\Big\{\left|f''(a)\right|^q, \left|f''(b)\right|^q\Big\}\Big]^{\frac{1}{q}}.$$

(d) if we choose $\lambda = \frac{1}{2}$, *then we have*

$$\left|I(f:\eta:a,b:\frac{1}{2})\right| \leq \frac{\eta^2(b,a)}{48}\Big[\max\Big\{\left|f''(a)\right|^q, \left|f''(b)\right|^q\Big\}\Big]^{\frac{1}{q}}.$$

Theorem 2.6. *Let* $S \subseteq R$ *be an open invex subset with respect to the map* $\eta : S \times S \to R$ *and,* $\eta(b,a) \neq 0$ $0 \leq a < a + \eta(b,a) < \infty$ *for all* $a \neq b$. *Suppose that* $f : S \to R$ *is a twice differentiable function on* S *such that* $f'' \in L[a, a+\eta(b,a)]$. *If* $\left|f''\right|^q$ *is prequasiinvex with respect to* η *on* S *for* $q \geq 1$ *and* $0 \leq \lambda \leq 1$, *then the following inequalities hold:*

(a) For $0 \leq \lambda \leq \frac{1}{2}$*:*

$$\begin{aligned}&\left|I(f:\eta:a,b:\lambda)\right| \\ &\leq \frac{\eta^2(b,a)}{24}\Big(8\lambda^3 - 3\lambda + 1\Big)\max\Big\{\left|f''(a)\right|^q, \left|f''(b)\right|^q\Big\}^{\frac{1}{q}}.\end{aligned}$$

(b) For $\frac{1}{2} \leq \lambda \leq 1$*:*

$$\begin{aligned}&\left|I(f:\eta:a,b:\lambda)\right| \\ &\leq \frac{\eta^2(b,a)}{24}\Big(3\lambda - 1\Big)\max\Big\{\left|f''(a)\right|^q, \left|f''(b)\right|^q\Big\}^{\frac{1}{q}}.\end{aligned}$$

Proof. From Lemma 1 and by the definition of $k(t)$, we get

$$\begin{aligned}&\left|I(f:\eta:a,b:\lambda)\right| \\ &\leq \frac{\eta^2(b,a)}{2}\Big[\Big(\int_0^{\frac{1}{2}}\left|t(t-\lambda)\right|dt\Big)^{1-\frac{1}{q}} \\ &\quad\times\Big(\int_0^{\frac{1}{2}}\left|t(t-\lambda)\right|\left|f''(a+t\eta(b,a))\right|^q dt\Big)^{\frac{1}{q}}\end{aligned}$$

$$+\Big(\int_{\frac{1}{2}}^{1}\big|(1-t)(1-\lambda-t)\big|dt\Big)^{1-\frac{1}{q}}$$

$$\times\Big(\int_{\frac{1}{2}}^{1}\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|^{q}dt\Big)^{\frac{1}{q}}\Big]. \tag{30}$$

(a) Assume that $0 \le \lambda \le \frac{1}{2}$.
Since $\big|f''\big|^q$ is prequasiinvex with repect to η on S, we know that for $t \in [0,1]$

$$\big|f''(a+t\eta(b,a)\big|^{q} \le \max\Big\{\big|f''(a)\big|^{q}, \big|f''(b)\big|^{q}\Big\}. \tag{31}$$

Note that

$$(i)\int_{0}^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|^{q}dt$$
$$\le\Big\{\frac{8\lambda^3-3\lambda+1}{24}\Big\}\max\Big\{\big|f''(a)\big|^{q}, \big|f''(b)\big|^{q}\Big\}, \tag{32}$$

$$(ii)\int_{\frac{1}{2}}^{1}\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|^{q}dt$$
$$\le\Big\{\frac{8\lambda^3-3\lambda+1}{24}\Big\}\max\Big\{\big|f''(a)\big|^{q}, \big|f''(b)\big|^{q}\Big\}. \tag{33}$$

By using (31)-(33) in (30), the part (a) is proved.
(b) Assume that $\frac{1}{2} \le \lambda \le 1$.
By using the prequasiinvexity of $\big|f''\big|$ with repect to η on S, we get that

$$(i)\int_{0}^{\frac{1}{2}}\big|t(t-\lambda)\big|\big|f''(a+t\eta(b,a))\big|^{q}dt$$
$$\le\frac{3\lambda-1}{24}\max\Big\{\big|f''(a)\big|^{q}, \big|f''(b)\big|^{q}\Big\}, \tag{34}$$
$$(ii)\int_{\frac{1}{2}}^{1}\big|(1-t)(1-\lambda-t)\big|\big|f''(a+t\eta(b,a))\big|^{q}dt$$
$$\le\frac{3\lambda-1}{24}\max\Big\{\big|f''(a)\big|^{q}, \big|f''(b)\big|^{q}\Big\}. \tag{35}$$

By using (34) and (35) in (30), we get

$$\begin{aligned}&\Big|I(f:\eta:a,b:\lambda)\Big|\\&\le \frac{\eta^2(b,a)}{2}\Big\{\frac{1}{24}\Big(3\lambda-1\Big)+\frac{1}{24}\Big(3\lambda-1\Big)\Big)\Big[\max\Big\{\big|f''(a)\big|,\big|f''(b)\big|\Big\}\Big]^{\frac{1}{q}}\\&=\frac{\eta^2(b,a)}{24}\Big(3\lambda-1\Big)\Big[\max\Big\{\big|f''(a)\big|^q,\big|f''(b)\big|^q\Big\}\Big]^{\frac{1}{q}}.\end{aligned}$$

Corollary 5. *In Theorem 2.6,*
(a)(Midpoint inequality) if we choose $\lambda = 0$, then we get

$$\Big|I(f:\eta:a,b:0)\Big|\le\frac{\eta^2(b,a)}{24}\Big[\max\Big\{\big|f''(a)\big|^q,\big|f''(b)\big|^q\Big\}\Big]^{\frac{1}{q}}.$$

(b)(Trapezoid inequality) if we choose $\lambda = 1$, then we have

$$\Big|I(f:\eta:a,b:1)\Big|\le\frac{\eta^2(b,a)}{12}\Big[\max\Big\{\big|f''(a)\big|^q,\big|f''(b)\big|^q\Big\}\Big]^{\frac{1}{q}}.$$

(c)(Simpson inequality) if we choose $\lambda = \frac{1}{3}$, then we get

$$\Big|I(f:\eta:a,b:\frac{1}{3})\Big|\le\frac{\eta^2(b,a)}{81}\Big[\max\Big\{\big|f''(a)\big|^q,\big|f''(b)\big|^q\Big\}\Big]^{\frac{1}{q}}.$$

(d) if we choose $\lambda = \frac{1}{2}$, then we have

$$\Big|I(f:\eta:a,b:\frac{1}{2})\Big|\le\frac{\eta^2(b,a)}{48}\Big[\max\Big\{\big|f''(a)\big|^q,\big|f''(b)\big|^q\Big\}\Big]^{\frac{1}{q}}.$$

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