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FINITE ABELIAN GROUPS BASED ON JR-3CN

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Abstract: A new type of finite Abelian groups, whose elements are the integer numbers that can be represented as a sum of three signed cubes, is constructed in this paper. These integers are the elements of three signed cubes sum family that has been introduced in JR-3CN family. Following the definition of this family, the representation of each element in the infinite Abelian groups is as an ordered pair, whose components and its subscribed parameter are the three signed cubes. The finite Abelian groups are constructed here, considered as a continuum to the study that has been done on $2JR_n$. Each finite Abelian group is denoted by $3JR_n$, where, 3 refers to the three signed cubes and n refers to the modulo and determines the order of the group. An addition binary operation has been defined on $3JR_n$ based on the addition binary operation associated with JR-3CN. This addition binary operation is operated under modulo n. Since the structure of the integers in JR-3CN is represented as ordered pairs, therefore, it is decisive to apply the modulo on each component for each ordered pair whenever it is needed. In order to support the claim, theorems and propositions on constructing the finite sets and the finite Abelian groups $3JR_n$ are stated and proved. Where they are mainly concern with determining the type of the elements and the order of each finite set.

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1. Introduction

In [1], we introduced the finite Abelian group $2\mathrm{JR}_n$. Analogy, $3\mathrm{JR}_n$ is introduced in this paper. The main stations are the same between $2\mathrm{JR}_n$ and $3\mathrm{JR}_n$, nevertheless; the details are different. The study in this paper has been done based on the Abelian group JR-3CN, which is the family and the group of all integers that can be represented as a sum of three signed cubes. The definition of JR-3CN has been given in [3], we are recalling it here,

$$\begin{aligned} JR-3CN &= \{(j,r)_\alpha : j^3+r^3-\alpha^3 = 3\alpha r^2 - 3\alpha^2 r, \\ &\quad \text{where}, j=\alpha-r,\ \forall \alpha, r \in Z\} \end{aligned} \tag{1}$$

We have to start, first, by collecting the ordered pairs of the finite sets before we construct the groups. Based on [1], it is already known that the constructing of $3\mathrm{JR}_n$ needs eliminating process. Therefore, we will start directly with the theorems and the propositions of collecting the ordered pairs. The proof that these finite sets are, actually, finite Abelian groups is followed. We will denote each finite set as $3\mathrm{JR}_n$, which is related to JR-3CN. We are using the same addition binary operation that has been defined on JR-3CN, but under modulo n. We recall here the definition of the addition binary operation on JR-3CN, $\oplus_{JR3CN}$ from [2]. For $i=1,2$, let $j_i = \alpha_i - r_i, \forall (j_1,r_1)_{\alpha_1}, (j_2,r_2)_{\alpha_2} \in JR-3CN$, we have defined the addition binary operation as the following,

$$\begin{aligned} (j_1,r_1)_{\alpha_1} \oplus_{JR3CN} (j_2,r_2)_{\alpha_2} &= (j_1+j_2, r_1+r_2)_{\alpha_1+\alpha_2} \text{ iff} \\ (j_1+j_2)^3 + (r_1+r_2)^3 - (\alpha_1+\alpha_2)^3 &= 3(\alpha_1+\alpha_2)(r_1+r_2)^2 \\ &\quad -3(\alpha_1+\alpha_2)^2(r_1+r_2), \end{aligned} \tag{2}$$

The addition binary operation of the finite set $3\mathrm{JR}_n$ is symbolized as $\oplus_{3_n}$, where 3 refers to the sum of three signed cubic numbers and n refers to the modulo of the set, and later, of the finite Abelian group.

As we said above, the method of collecting the elements is basically depending on modulo n, however, it is still a bit different. In this, and in the previous study, the modulo has not been applied on integer as it is usual, but it has been applied on ordered pairs. Therefore, to collect the ordered pairs of $3\mathrm{JR}_n$ out from JR-3CN, the modulo must be applied on both components and the parameter of each ordered pair in both directions, the negative and the positive. It means; each ordered pair is bound to the set $\{0, \pm 1, \pm 2, \cdots, \pm(n-1)\}$. The constructing of the first five finite Abelian groups $3\mathrm{JR}_n$ will be given in the appendix as examples.

We have faced some difficulties with finding references to this work, because there is no previous work on applying the arithmetic modulo on an ordered pair. Therefore, the references we depended on can be organized as two types, the first type is the authors' previous work on this topic. The second type contains the references that are related to the theoretical background of constructing the finite set $2\mathrm{JR}_n$; we referred to [7] and [4]. Moreover, in regard to constructing the finite sets and calculate the orders, we referred to [5] and [6]. Finally, in constructing the finite Abelian groups, we referred to and [8].

The paper is organized as follows. Beside the introduction, we allocate Section 2 for constructing and collecting the elements of the finite sets $3\mathrm{JR}_n$. Where we presented and proved the supporting theorems. In addition, the section also contains the proofs that these finite sets are actually finite Abelian groups. Finally, the appendix contains implementation examples for some specific n.

2. The Constructing of $3\mathrm{JR}_n$

Since $3\mathrm{JR}_n$ that we are about to construct in this section is based on JR-3CN, therefore, its elements must follow the same definition. We rely on [7] and [4] regarding some background information.

Now, the elements of JR-3CN are symbolized as $(j,r)_\alpha$, where $j=\alpha-r$ and each order pair is an integer represented as $j^3+r^3-\alpha^3$, which is the sum of three signed cubes. Before we start, we need to give a general definition for $3\mathrm{JR}_n$.

Definition 2.1. $3\mathrm{JR}_n$ is a subset of JR-3CN, where its elements are the ordered pairs, whose components and the parameter are ranged from $-(n-1)$ to $n-1$. In other words,

$$\begin{aligned} 3JR_n \quad = \quad & \{(j,r)_\alpha \ mod\ n : (j,r)_\alpha \in JR-2CN, \text{ where} \\ & |j| \le n-1, |r| \le n-1, |\alpha| \le n-1\} \end{aligned} \tag{3}$$

Definition 2.2. let $\oplus_{3_n}$ be a relation defined on $3JR_n \times 3JR_n$ into $3JR_n$, such that, $\forall (j,r)_\alpha, (k,s)_\beta \in 3JR_n$, we have

$$\begin{aligned} (j,r)_\alpha \oplus_{3_n} (k,s)_\beta \quad & \equiv \quad (j+k, r+s)_{(\alpha+\beta)} mod\ n \\ & \equiv \quad ((j+k) mod\ n, (r+s) mod\ n)_{(\alpha+\beta) mod\ n}. \end{aligned}$$

Proposition 2.1. *The relation $\oplus_{3_n}$ is well defined binary operation, closed, associative and commutative on $3JR_n$.*

The proposition can be proved directly by using the concept of the arithmetic modulo n, definition (2.2) and ([2]-definition 3.1, propositions 3.1-3.3).

These finite set $3\mathrm{JR}_n$ cannot be considered as a subgroup of JR-3CN, because the binary operation $\oplus_{3_n}$ that is associated with $3\mathrm{JR}_n$ is not the same binary operation $\oplus_{JR-3CN}$ that is defined on JR-3CN, in fact, $\oplus_{3_n}$ is the modulo n of $\oplus_{JR-3CN}$. In this section, the theorems of constructing the finite sets $3\mathrm{JR}_n$ are introduced. Moreover, the question of how we can generate the ordered pairs under the arithmetic modulo n is answered. In the end of the section, we will prove that these finite sets are actually finite Abelian groups.

2.1. Constructing the Finite Set $3\mathrm{JR}_n$

In [3] the family JR-3CN is introduced. We have seen this family has infinitely many integers which can be represented as a sum of three signed cubes, where, these integers have been represented as ordered pairs. Looking at the sample table of these ordered pairs in ([3]–table 2) will be neither enough nor practical to collect the ordered pairs of each $3\mathrm{JR}_n$. One needs to be careful with the collecting process to avoid missing any ordered pair that might belong to $3\mathrm{JR}_n$. The safest way is by following the last condition in definition (2.1), gathering all the ordered pairs that fall in the range $\pm(n-1)$ would be the first step, then, eliminates those which are not in JR-3CN.

The theorem below gives the method of constructing $3\mathrm{JR}_n$, it is actually, a method for eliminating, or in other words, a method of setting up the finite sets $3\mathrm{JR}_n$. The eliminating process must pass through two steps. First step is eliminating the ordered pairs that are not coincided with the definition JR-3CN. Second step is eliminating the congruent ordered pairs. What we aim to achieve in this study is formulating finite Abelian groups out of $3\mathrm{JR}_n$, thus, for $3\mathrm{JR}_n$ to be Abelian group must not have two congruent ordered pairs.

Theorem 2.1. *The ordered pair $(j,r)_\alpha$ in the finite set $3JR_n$ under the arithmetic modulo n is satisfied the following conditions:*

1. *The components j,r and the parameter α satisfy the solutions of the congruence equation $j+r \equiv \alpha \mod n$, where the integers j,r and α are strictly belong to the set $\{0,\pm1,\cdots\pm(n-1)\}$.*
2. *Each ordered pair whose either or all components and parameter are negative then it is congruent to an ordered pair whose components and parameter are positive.*

Proof. For the first condition, recalling the definition of JR-3CN, [3]-Def.(1), $(j,r)_\alpha \in$ JR-3CN with $j+r=\alpha$. Based on definition (2.1), we have, $\{|j|,|r|,|\alpha|\}$

$\leq n-1$. Keeping in mind that the ordered pairs in $3\mathrm{JR}_n$ are compelled to be under modulo n, so that the components j and r and the parameter α must satisfy the congruence equation,

$$j + r \equiv \alpha \bmod n \tag{4}$$

In the range from $-(n-1)$ to $n-1$.

As for the second condition, and as how the statement suggests, we have no way but to split it into two cases. The first one is when either component or the parameter is negative. And the second case is when both components and parameter are negative. Below we will discuss both of these two cases in details.

Case 1. When at least one integer in $(j, r)_\alpha$ is negative: let j be a negative integer, r and α are positive, and let $j + r \equiv \alpha \bmod n$. Then it is also true that $(n + j) + r \equiv \alpha \bmod n$ and $n + j$ here is positive.

Case 2. When all the integers in $(j, r)_\alpha$ are negative: let j, r and α be negative integers, and $j + r \equiv \alpha \bmod n$ then it is obviously true that $(n + j) + (n + r) \equiv (n + \alpha) \bmod n$ where $(n + j), (n + r)$ and $(n + \alpha)$ are all positive and belong to the complete residue system of the solutions of the congruence equation (4).

To summarize the two cases above, if $j + r \equiv \alpha \bmod n$, then $kn + (j + r)$ is also congruent to $\alpha \bmod n$. As long as the ranges of j, r, α bounded by $\mp(n-1)$ then if any or all of them were negative, then, applying modulo n on them will turn them to be positive. Therefore, if $(j, r)_\alpha \in 3JR_n$ and either j, r, α or all of them are negative, then it is congruent to an ordered pair whose both components and parameter are positive. □

A very important note we conclude from theorem (2.1) above, which is, there are no ordered pairs with negative integers could be found in any set of $3\mathrm{JR}_n$. Appendix (A, the sets (A.2), (A.3), (A.4), and (A.5)) provide very good examples on this theorem.

Theorem 2.2. *No two different ordered pairs are congruent in $3\mathrm{JR}_n$.*

Proof. Assume that, there exist $(j_k, r_k)_{\alpha_k}, (j_l, r_l)_{\alpha_l} \in 3JR_n$ such that $(j_k, r_k)_{\alpha_k} \neq (j_l, r_l)_{\alpha_l}$ and $(j_k, r_k)_{\alpha_k} \equiv (j_l, r_l)_{\alpha_l} \bmod n$. Now, if j_l (respectively r_l, α_l) is zero, then j_k (respectively r_k, α_k) is zero too. Otherwise, j_l, r_l, α_l are positive and that means j_k, r_k, α_k are negative numbers because, the ranges of j_l, r_l, α_l are from $-(n-1)$ to $(n-1)$. This is definitely cannot happen because condition (2) of theorem (2.1) states that there are no ordered pairs in $3\mathrm{JR}_n$

whose components and parameters are negative. Thus, if $(j_k, r_k)_{\alpha_k} \neq (j_l, r_l)_{\alpha_l}$, then $(j_k, r_k)_{\alpha_k} \not\equiv (j_l, r_l)_{\alpha_l} \mod n$

□

Theorem 2.3. *The order of 3JR$_n$ is n^2*

Proof. Originally, the total number of the ordered pairs in 3JR$_n$ must be $(2n-1)^3$ following to the permutations $P_3^{(2n-1)}$, where the set of choices is $\{0, \pm 1, \cdots, \pm(n-1)\}$, and the places of choices are 3. However, based on theorem (2.1, condition 2), 3JR$_n$ contains only the ordered pairs whose components and parameters are positive so that the set of choices is $\{0, 1, \cdots, (n-1)\}$ and therefore, the permutation is P_3^n and so that,

$$|3JR_n|_{\text{Number of ordered pairs before eliminating}} = n^3. \tag{5}$$

Now, we need to eliminate all the ordered pairs that are not in JR-3CN; it means; we need to eliminate all the ordered pairs that do not satisfy theorem (2.1- condition 1), which they are the ordered pairs that do not satisfy the congruence equation $j + r \equiv \alpha \mod n$. It is easier to break it down into the two cases below,

Case 1. When $j = r$: we need to find the total number of ordered pairs $(j, j)_\alpha$ that do not satisfy the congruent equation

$$2j \equiv \alpha \mod n \tag{6}$$

In the residue system $\{0, 1, 2, \cdots, n-1\}$ there is only one solution for each j to satisfy the congruent equation (6). It means, there are $(n-1)$ incongruent results. Therefore, there are $(n-1)$ choice does not satisfy the congruent equation for each j.
Now, j can take n different values in the set $\{0, 1, 2, \cdots, n-1\}$, therefore,

$$|(j, j)_\alpha|_{\text{not satisfying the congruent equation } 2j\equiv\alpha \text{ mod } n} = n(n-1) \tag{7}$$

Case 2. When $j \neq r$: for the ordered pair $(j, r)_\alpha$, the first component j takes its values from the set $\{0, 1, \cdots, n-1\}$, then the second components r depends on the choice of j to avoid being equaled and to avoid the repeating. The number of these ordered pairs follows the summation of the natural series from 1 to $n-1$, where the number of this kind of the ordered pairs is $\frac{n(n-1)}{2}$. However, if $(j, r)_\alpha$ exists, then $(r, j)_\alpha$ exists too. Therefore,

$$|(j,r)_\alpha|_{j\neq r} = n(n-1) \tag{8}$$

Now, with regards to α, we need to check for which values of α, these ordered pairs satisfied theorem (2.1-condition 1). For each choice of j and r, in the residue system $\{0, 1, \cdots, n-1\}$, there are $(n-1)$ choices for α that do not satisfy the congruent equation $j + r \equiv \alpha \bmod n$. Therefore, the total number of ordered pairs for this case is

$$|(j,r)_\alpha|_{\text{not satisfying the congruent equation } j+r\equiv\alpha \bmod n} = n(n-1)^2 \tag{9}$$

From these two cases and the results in equations (7) and (9), the total number of ordered pairs that do not satisfy theorem (2.1-condition 1) is

$$n(n-1) + n(n-1)^2 = n^3 - n^2 \tag{10}$$

Subtracting this result from the total number of the ordered pairs in equation (5) gives n^2, which is the order of 3JR_n.

□

2.2. $(3JR_n, \oplus_{3_n})$ are Finite Abelian Groups

We also will introduce the binary operation and general refereing to the identity and the inverse.

Definition 2.3. Under the addition binary operation $\oplus_{3_n}$, the identity element is $(0,0)_0$ and the inverse element of $(j,r)_\alpha$ is $(j,r)_\alpha^{-1}$, where, $(j,r)_\alpha \oplus_{3_n} (j,r)_\alpha^{-1} \equiv (0,0)_0 \, mod \; n$

Section (A) will assist to see how we can construct the finite abelian groups. Few examples are given, where we explain in details of how the theorems (2.1, 2.2 and 2.3) are actually working. In this subsection, we will give the theoretical background to prove that each $(3JR_n, \oplus_{3_n})$ is a finite Abelian group for any n.

We defined the binary operation in definition (2.2), and we proved in Proposition (2.1) that the binary operation $\oplus_{3_n}$ is actually well defined, closed, associative, and, commutative over 3JR_n. The main issue that has left to prove is the existence of the unique inverse ordered pair for each ordered pair in 3JR_n for each n.

Proposition 2.2. *$(3JR_n, \oplus_{3_n})$ is finite Abelian group with identity $(0,0)_0$ and a unique inverse ordered pair $(n-j, n-r)_{n-\alpha} \bmod n$ for every ordered pair $(j,r)_\alpha$.*

Proof. It is easy to show that $(0,0)_0$ is the identity ordered pair in the set $3\mathrm{JR}_n$. To prove that $(n-j, n-r)_{n-\alpha}$ mod n is the inverse element of $(j,r)_\alpha$. We have $(j,r)_\alpha \in 3JR_n$ then $j+r \equiv \alpha$ mod n and both of the components and the parameter are positive according to theorem (2.1-condition 1), therefore, $(n-j)+(n-r)$ is congruent $(n-\alpha)$ mod n and both of $n-j, n-r$ and the parameter $n-\alpha$ are positive and so that $(n-j, n-r)_{n-\alpha} \in 3JR_n$. Moreover, $(n-j, n-r)_{n-\alpha} \oplus_{3_n} (j,r)_\alpha \equiv (0,0)_0$ mod n.

As for the uniqueness part, the only other possible ordered pairs that could be placed as inverses for the ordered pair $(j,r)_\alpha$ are the ordered pairs $(j,r)_\alpha$ were either components and parameter or all are negative. However, this scenario cannot be happen, because, according to theorem (2.1-condition 2) there are no ordered pairs whose components and parameter are negative in $3\mathrm{JR}_n$.

□

3. Conclusion

We have introduced in this paper a new type of finite Abelian groups that derived from the family JR-3CN which is the family of all integer numbers that can be represented as a sum of three signed cubs, and we have denoted them as $3\mathrm{JR}_n$. We have started by giving the method of how to collect the ordered pairs of each finite set $3\mathrm{JR}_n$. Embedded in this method a way to eliminate the ordered pairs that cannot be in JR-3CN and the congruent ordered pairs, therefore we can also called this method as a set up of $3\mathrm{JR}_n$. We have given the theorems of the setting up, the order of each finite set, and the assurance of there are no congruent ordered pairs in one finite set. We have ended the paper by proving that each finite set $3\mathrm{JR}_n$ is actually an Abelian group. The Appendix contains examples of to how construct the first five finite abelian groups $3\mathrm{JR}_n$.

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A. Ordered Pairs Eliminating of $3\mathrm{JR}_n$, $n = 1, 2, 3, 4, 5$

Few examples is given to be as practical implementation to the theoretical part where we construct the finite sets and the finite Abelian groups $3\mathrm{JR}_n$. The theoretical part has been already explained and proved in section (2.1). We depend partially on [5] and [6] to assist with the theoretical background of calculating the permutations

1. **Modulo 1, $3\mathrm{JR}_1$.**

   The only available ordered pair in the set $3\mathrm{JR}_1$ is the one whose components and the parameter are 0

$$3JR_1 = \{(0, 0)_0\} \tag{A.1}$$

$(3JR_1, \oplus_{3_1})$ represents finite abelian group.

2. **Modulo 2, $\mathbf{3JR_2}$.**

The easiest way is to illustrate all the possible ordered pairs under modulo 2. Modulo 2 will be taken over the components and the subscript parameter.

The possible choices under modulo 2 are taken from the set $\{0, \pm 1\}$. Therefore, the original set should be,

$$\begin{aligned} 3JR_2 &= \{(-1,-1)_{-1}, (-1,-1)_0, (-1,-1)_1, (-1,0)_{-1}, (-1,0)_0, \\ &\quad (-1,0)_1, (-1,1)_{-1}, (-1,1)_0, (-1,1)_1, (0,-1)_{-1}, (0,-1)_0, \\ &\quad (0,-1)_1, (0,0)_{-1}, (0,0)_0, (0,0)_1, (0,1)_{-1}, (0,1)_0, (0,1)_1, \\ &\quad (1,-1)_{-1}, (1,-1)_0, (1,-1)_1, (1,0)_{-1}, (1,0)_0, (1,0)_1, \\ &\quad (1,1)_{-1}, (1,1)_0, (1,1)_1\}. \end{aligned}$$

First step of determining the final shape of this set is by eliminating the order pairs that do not belong to JR-3CN. Which means, eliminate the ordered pairs whose components and the parameter do not coincide with the definition of JR-3CN, such as, $(-1,-1)_{-1}, (-1,-1)_1, (-1,0)_0$ and so on.

$$\begin{aligned} 3JR_2 &= \{(-1,-1)_0, (-1,0)_{-1}, (-1,0)_1, (-1,1)_0, (0,-1)_{-1}, (0,-1)_1, \\ &\quad (0,0)_0, (0,1)_{-1}, (0,1)_1, (1,-1)_0, (1,0)_{-1}, (1,0)_1, (1,1)_0\} \end{aligned}$$

Now, we must eliminate the ordered pairs that are congruent under modulo 2 to other ordered pairs in the set. for instance, $(-1,0)_{-1} \equiv (1,0)_1 \bmod 2$. Obviously, it is preferable to keep the ordered pairs whose components and parameters are the most positive and small and eliminate those whose components and parameters are negative and large. All the equivalent ordered pairs are listed below,

$$\begin{aligned} \{(-1,-1)_0, (1,-1)_0, (-1,1)_0\} &\equiv (1,1)_0 \bmod 2 \\ \{(-1,0)_{-1}, (1,0)_{-1}, (-1,0)_1\} &\equiv (1,0)_1 \bmod 2 \\ \{(0,-1)_{-1}, (0,1)_{-1}, (0,-1)_1\} &\equiv (0,1)_1 \bmod 2 \end{aligned}$$

Therefore, by eliminating these ordered pairs, the final shape of $3JR_2$ is,

$$3JR_2 = \{(0,0)_0, (0,1)_1, (1,0)_1, (1,1)_0\} \tag{A.2}$$

$3JR_2$ is a finite Abelian group. The binary operation $\oplus_{3_2}$ is well defined, closed, associative and commutative over $3JR_2$, the identity element is $(0,0)_0$ and there is inverse element for each element in the set as follows,

$$(0,1)_1^{-1} = (0,1)_1, \quad (1,0)_1^{-1} = (1,0)_1, \quad (1,1)_0^{-1} = (1,1)_0.$$

3. **Modulo 3, $\mathbf{3JR_3}$.**
All the possible ordered pairs under modulo 3 take their components from the set $\{0, \pm 1, \pm 2\}$. Therefore, the set should be,

$$\begin{aligned}3JR_3 =\{&(-2,-2)_{-2},(-2,-2)_{-1},(-2,-2)_0,(-2,-2)_1,(-2,-2)_2,(-2,-1)_{-2},\\ &(-2,-1)_{-1},(-2,-1)_0,(-2,-1)_1,(-2,-1)_2,(-2,0)_{-2},(-2,0)_{-1},(-2,0)_0,\\ &(-2,0)_1,(-2,0)_2,(-2,1)_{-2},(-2,1)_{-1},(-2,1)_0,(-2,1)_1,(-2,1)_2,(-2,2)_{-2},\\ &(-2,2)_{-1},(-2,2)_0,(-2,2)_1,(-2,2)_2,(-1,-2)_{-2},(-1,-2)_{-1},(-1,-2)_0,\\ &(-1,-2)_1,(-1,-2)_2,(-1,-1)_{-2},(-1,-1)_{-1},(-1,-1)_0,(-1,-1)_1,(-1,-1)_2,\\ &(-1,0)_{-2},(-1,0)_{-1},(-1,0)_0,(-1,0)_1,(-1,0)_2,(-1,1)_{-2},(-1,1)_{-1},(-1,1)_0,\\ &(-1,1)_1,(-1,1)_2,(-1,2)_{-2},(-1,2)_{-1},(-1,2)_0,(-1,2)_1,(-1,2)_2,(0,-2)_{-2},\\ &(0,-2)_{-1},(0,-2)_0,(0,-2)_1,(0,-2)_2,(0,-1)_{-2},(0,-1)_{-1},(0,-1)_0,(0,-1)_1,\\ &(0,-1)_2,(0,0)_{-2},(0,0)_{-1},(0,0)_0,(0,0)_1,(0,0)_2,(0,1)_{-2},(0,1)_{-1},(0,1)_0,(0,1)_1,\\ &(0,1)_2,(0,2)_{-2},(0,2)_{-1},(0,2)_0,(0,2)_1,(0,2)_2,(1,-2)_{-2},(1,-2)_{-1},(1,-2)_0,\\ &(1,-2)_1,(1,-2)_2,(1,-1)_{-2},(1,-1)_{-1},(1,-1)_0,(1,-1)_1,(1,-1)_2,(1,0)_{-2},\\ &(1,0)_{-1},(1,0)_0,(1,0)_1,(1,0)_2,(1,1)_{-2},(1,1)_{-1},(1,1)_0,(1,1)_1,(1,1)_2,(1,2)_{-2},\\ &(1,2)_{-1},(1,2)_0,(1,2)_1,(1,2)_2,(2,-2)_{-2},(2,-2)_{-1},(2,-2)_0,(2,-2)_1,(2,-2)_2,\\ &(2,-1)_{-2},(2,-1)_{-1},(2,-1)_0,(2,-1)_1,(2,-1)_2,(2,0)_{-2},(2,0)_{-1},(2,0)_0,(2,0)_1,\\ &(2,0)_2,(2,1)_{-2},(2,1)_{-1},(2,1)_0,(2,1)_1,(2,1)_2,(2,2)_{-2},(2,2)_{-1},(2,2)_0,\\ &(2,2)_1,(2,2)_2\}.\end{aligned}$$

Now we have to eliminate all the ordered pairs whose components summation is not equal to the parameter under modulo 3. Therefore, the set will be,

$$\begin{aligned}3JR_3 = \{&(-2,-2)_{-1},(-2,-2)_2,(-2,-1)_0,(-2,0)_{-2},(-2,0)_1,(-2,1)_{-1},(-2,1)_2,\\ &(-2,2)_0,(-1,-2)_0,(-1,-1)_{-2},(-1,-1)_1,(-1,0)_{-1},(-1,0)_2,(-1,1)_0,\\ &(-1,2)_{-2},(-1,2)_1,(0,-2)_{-2},(0,-2)_1,(0,-1)_{-1},(0,-1)_2,(0,0)_0,(0,1)_{-2},\\ &(0,1)_1,(0,2)_{-1},(0,2)_2,(1,-2)_{-1},(1,-2)_2,(1,-1)_0,(1,0)_{-2},(1,0)_1,(1,1)_{-1},\\ &(1,1)_2,(1,2)_0,(2,-2)_0,(2,-1)_{-2},(2,-1)_1,(2,0)_{-1},(2,0)_2,(2,1)_0,(2,2)_{-2},\\ &(2,2)_1\}.\end{aligned}$$

The set above can also be reduced by eliminating the equivalent ordered pairs. The list of the congruent ordered pairs is,

$$\begin{aligned}\{&(-2,-2)_{-1},(-2,-2)_2,(-2,1)_{-1},(-2,1)_2,\\ &(1,-2)_{-1},(1,-2)_2,(1,1)_{-1}\} \equiv (1,1)_2 \bmod 2\end{aligned}$$

$$\begin{aligned}
\{(-1,-1)_{-2},(-1,-1)_1,(-1,2)_{-2},(-1,2)_1,&\\
(2,-1)_{-2},(2,-1)_1,(2,2)_{-2}\} &\equiv (2,2)_1 \bmod 2\\
\{(-2,-1)_0,(-2,2)_0,(1,-1)_0\} &\equiv (1,2)_0 \bmod 2\\
\{(-1,-2)_0,(2,-2)_0,(-1,1)_0\} &\equiv (2,1)_0 \bmod 2\\
\{(-1,0)_{-1},(-1,0)_2,(2,0)_{-1}\} &\equiv (2,0)_2 \bmod 2\\
\{(0,-1)_{-1},(0,-1)_2,(0,2)_{-1}\} &\equiv (0,2)_2 \bmod 2\\
\{(-2,0)_{-2},(-2,0)_1,(1,0)_{-2}\} &\equiv (1,0)_1 \bmod 2\\
\{(0,-2)_{-2},(0,-2)_1,(0,1)_{-2}\} &\equiv (0,1)_1 \bmod 2
\end{aligned}$$

Therefore, by eliminating these ordered pairs, the final shape of $3JR_3$ is,

$$\begin{aligned}
3JR_3 = \{&(0,0)_0,(0,1)_1,(0,2)_2,(1,0)_1,(1,1)_2,\\
&(1,2)_0,(2,0)_2,(2,1)_0,(2,2)_1\}
\end{aligned} \tag{A.3}$$

$3JR_3$ is a finite Abelian group. The binary operation $\oplus_{3_3}$ is well defined, closed, associative and commutative over $3JR_3$, the identity element is $(0,0)_0$ and there is inverse element for each element in the set as follows,

$$\begin{aligned}
(0,1)_1^{-1} &= (0,2)_2, & (1,0)_1^{-1} &= (2,0)_2,\\
(1,1)_2^{-1} &= (2,2)_1, & (1,2)_0^{-1} &= (2,1)_0.
\end{aligned}$$

For the two examples above, the groups $3JR_2$ and $3JR_3$ do not contain ordered pairs whose components and parameter are negative. This situation will be repeated in the next groups $3JR_4$ and $3JR_5$.

4. **Modulo 4, $3JR_4$.**

All the possible ordered pairs under modulo 4 take their components from the set $\{0,\pm1,\pm2,\pm3\}$. Therefore, the set should contain 343 ordered pairs. After eliminating those which do not belong to JR-3CN, remains,

$$\begin{aligned}
3JR_4 = \{&(-3,-3)_{-2},(-3,-3)_2,(-3,-2)_{-1},(-3,-2)_3,(-3,-1)_0,(-3,0)_{-3},\\
&(-3,0)_1,(-3,1)_{-2},(-3,1)_2,(-3,2)_{-1},(-3,2)_3,(-3,3)_0,(-2,-3)_{-1},\\
&(-2,-3)_3,(-2,-2)_0,(-2,-1)_{-3},(-2,-1)_1,(-2,0)_{-2},(-2,0)_2,\\
&(-2,1)_{-1},(-2,1)_3,(-2,2)_0,(-2,3)_{-3},(-2,-2)_0,(-1,-3)_0,\\
&(-1,-2)_{-3},(-1,-2)_1,(-1,-1)_{-2},(-1,-1)_2,(-1,0)_{-1},(-1,0)_3,\\
&(-1,1)_0,(-1,2)_{-3},(-1,2)_1,(-1,3)_{-2},(-1,3)_2,(0,-3)_{-3},(0,-3)_1,\\
&(0,-2)_{-2},(0,-2)_2,(0,-1)_{-1},(0,-1)_3,(0,0)_0,(0,1)_{-3},(0,1)_1,(0,2)_{-2},\\
&(0,2)_2,(0,3)_{-1},(0,3)_3,(1,-3)_{-2},(1,-3)_2,(1,-2)_{-1},(1,-2)_3,(1,-1)_0,\\
&(1,0)_{-3},(1,0)_1,(1,2)_3,(1,3)_0,(2,-3)_{-1},(2,-3)_3,(2,-2)_0,(2,-1)_{-3},\\
&(2,-1)_1,(2,0)_{-2},(2,0)_2,(2,1)_{-1},(1,1)_{-2},(1,1)_2,(1,2)_{-1},(2,1)_3,
\end{aligned}$$

$(2,2)_0,(2,3)_{-3},(2,3)_1,(3,-3)_0,(3,-2)_{-3},(3,-2)_1,(3,-1)_{-2},(3,-1)_2,$
$(3,0)_{-1},(3,0)_3,(3,1)_0,(3,2)_{-3},(3,2)_1,(3,3)_{-2},(3,3)_2\}.$

The set above can also be reduced by eliminating the equivalent ordered pairs. Therefore, by eliminating these ordered pairs, the final shape of $3\mathrm{JR}_4$ is,

$$3JR_4 = \{(0,0)_0,(0,1)_1,(0,2)_2,(0,3)_3,(1,0)_1,(1,2)_3,(1,3)_0,(2,0)_2,$$
$$(1,1)_2,(2,1)_3,(2,2)_0,(2,3)_1,(3,0)_3,(3,1)_0,(3,2)_1,(3,3)_2\}. \quad \text{(A.4)}$$

$3\mathrm{JR}_4$ is a finite Abelian group. The binary operation $\oplus_{3_4}$ is well defined, closed, associative and commutative over $3\mathrm{JR}_4$, the identity element is $(0,0)_0$ and there is inverse element for each element in the set as follows,

$$(0,1)_1^{-1}=(0,3)_3,\quad (0,2)_2^{-1}=(0,2)_2,\quad (1,0)_1^{-1}=(3,0)_3,$$
$$(1,2)_3^{-1}=(3,2)_1,\quad (1,3)_0^{-1}=(3,1)_0,\quad (2,0)_2^{-1}=(2,0)_2,$$
$$(1,1)_2^{-1}=(3,3)_2,\quad (2,1)_3^{-1}=(2,3)_1,\quad (2,2)_0^{-1}=(2,2)_0.$$

5. **Modulo 5, $3\mathrm{JR}_5$.**
All the possible ordered pairs under modulo 5 take their components from the set $\{0,\pm1,\pm2,\pm3,\pm4\}$. Below, we will put the set of the ordered pairs except those ordered pairs that do not belong to JR-3CN.

$$3JR_5 = \{(-4,-4)_{-3},(-4,-4)_2,(-4,-3)_{-2},(-4,-3)_3,(-4,-2)_{-1},$$
$$(-4,-2)_4,(-4,-1)_0,(-4,0)_{-4},(-4,0)_1,(-4,1)_{-3},(-4,1)_2,$$
$$(-4,2)_{-2},(-4,2)_3,(-4,3)_{-1},(-4,3)_4,(-4,4)_0,(-3,-4)_{-2},$$
$$(-3,-4)_3,(-3,-3)_{-1},(-3,-3)_4,(-3,-2)_0,(-3,-1)_{-4},(-3,-1)_1,$$
$$(-3,0)_{-3},(-3,0)_2,(-3,1)_{-2},(-3,1)_3,(-3,2)_{-1},(-3,2)_4,$$
$$(-3,3)_0,(-3,4)_{-4},(-3,4)_1,(-2,-4)_{-1},(-2,-4)_4,(-2,-3)_0,$$
$$(-2,-2)_{-4},(-2,-2)_1,(-2,-1)_{-3},(-2,-1)_2,(-2,0)_{-2},(-2,0)_3,$$
$$(-2,1)_{-1},(-2,1)_4,(-2,2)_0,(-2,3)_{-4},(-2,3)_1,(-2,4)_{-3},$$
$$(-2,4)_2,(-1,-4)_0,(-1,-3)_{-4},(-1,-3)_1,(-1,-2)_{-3},(-1,-2)_2,$$
$$(-1,-1)_{-2},(-1,-1)_3,(-1,0)_{-1},(-1,0)_4,(-1,1)_0,(-1,2)_{-4},$$
$$(-1,2)_1,(-1,3)_{-3},(-1,3)_2,(-1,4)_{-2},(-1,4)_3,(0,-4)_{-4},$$
$$(0,-4)_1,(0,-3)_{-3},(0,-3)_2,(0,-2)_{-2},(0,-2)_3,(0,-1)_{-1},$$
$$(0,-1)_4,(0,0)_0,(0,1)_{-4},(0,1)_1,(0,2)_{-3},(0,2)_2,(0,3)_{-2},$$
$$(0,3)_3,(0,4)_{-1},(0,4)_4,(1,-4)_{-3},(1,-4)_2,(1,-3)_{-2},$$
$$(1,-3)_3,(1,-2)_{-1},(1,-2)_4,(1,-1)_0,(1,0)_{-4},(1,0)_1,$$
$$(1,1)_{-3},(1,1)_2,(1,2)_{-2},(1,2)_3,(1,3)_{-1},(1,3)_4,(1,4)_0,$$
$$(2,-4)_{-2},(2,-4)_3,(2,-3)_{-1},(2,-3)_4,(2,-2)_0,(2,-1)_{-4},$$

$(2,-1)_1, (2,0)_{-3}, (2,0)_2, (2,1)_{-2}, (2,1)_3, (2,2)_{-1}, (2,2)_0,$
$(2,2)_1, (2,2)_4, (2,3)_0, (2,4)_{-4}, (2,4)_1, (3,-4)_{-1}, (3,-4)_4,$
$(3,-3)_0, (3,-2)_{-4}, (3,-2)_1, (3,-1)_{-3}, (3,-1)_2, (3,0)_{-2},$
$(3,0)_3, (3,1)_{-1}, (3,1)_4, (3,2)_0, (3,3)_{-4}, (3,3)_1, (3,4)_{-3},$
$(3,4)_2, (4,-4)_0, (4,-3)_{-4}, (4,-3)_1, (4,-2)_{-3}, (4,-2)_2,$
$(4,-1)_{-2}, (4,-1)_3, (4,0)_{-1}, (4,0)_4, (4,1)_0, (4,2)_{-4}, (4,2)_1,$
$(4,3)_{-3}, (4,3)_2, (4,4)_{-2}, (4,4)_3.\}$

The set above can also be reduced by eliminating the equivalent ordered pairs. Therefore, by eliminating these ordered pairs, the final shape of $3\mathrm{JR}_5$ is,

$$\begin{aligned} 3JR_5 \;=\; & \{(0,0)_0, (0,1)_1, (0,2)_2, (0,3)_3, (0,4)_4, (1,0)_1, (1,1)_2, (1,2)_3, \\ & (1,3)_4, (1,4)_0, (2,0)_2, (2,1)_3, (2,2)_4, (2,3)_0, (2,4)_1, (3,0)_3, \\ & (3,1)_4, (3,2)_0, (3,3)_1, (3,4)_2, (4,0)_4, (4,1)_0, (4,2)_1, (4,3)_2, (4,4)_3.\} \end{aligned} \tag{A.5}$$

$3\mathrm{JR}_5$ is a finite Abelian group. The binary operation $\oplus_{3_5}$ is well defined, closed, associative and commutative over $3\mathrm{JR}_5$, the identity element is $(0,0)_0$ and there is inverse element for each element in the set as follows,

$$\begin{array}{lll} (0,1)_1^{-1} = (0,4)_4, & (0,2)_2^{-1} = (0,3)_3, & (1,0)_1^{-1} = (4,0)_4, \\ (1,1)_2^{-1} = (4,4)_3, & (1,2)_3^{-1} = (4,3)_2, & (1,3)_4^{-1} = (4,2)_1, \\ (1,4)_0^{-1} = (4,1)_0, & (2,0)_2^{-1} = (3,0)_3, & (2,1)_3^{-1} = (3,4)_2, \\ (2,2)_4^{-1} = (3,3)_1, & (2,3)_0^{-1} = (3,2)_0, & (2,4)_1^{-1} = (3,1)_4. \end{array}$$

The examples above for the finite Abelian groups $3\mathrm{JR}_n$ when $n = 1,2,3,4,5$ do not contain ordered pairs whose components and parameter are negative. Which is what we have shown Section 2, where we proved $3\mathrm{JR}_n$, for any n, does not contain ordered pairs whose components and parameters are negative.