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# FUZZY DECOMPOSITION ON THE AFFINE KAC-MOODY ALGEBRAS

A. Uma Maheswari
Department of Mathematics
Quaid-E-Millat Government College (Autonomous)
Chennai, 600002, INDIA

**Abstract:** Kac-Moody algebras is one of the modern fields of Mathematical research which has got interesting connections and applications to other fields of Mathematics and Mathematical Physics. On the other hand fuzzy theory has been finding deep rooted applications in all walks of life. In this paper, we define fuzzy sets on the Cartesian product of the simple roots of some of the affine type of Kac-Moody algebras. The fundamental fuzzy properties like normality, convexity and cardinality are studied for these affine Kac-Moody algebras. The $\alpha$-level, strong $\alpha$-level sets and $\alpha$-cut decomposition for these fuzzy sets, associated with the affine Kac-Moody algebras are computed.

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**Key Words:** generalized Cartan matrix, Kac-Moody algebra, simple roots, affine type, fuzzy set, $\alpha$-cut decomposition, $\alpha$-level sets, convexity

## 1. Introduction

### 1.1. Basic Definitions on Kac-Moody Algebras

**Definition 1.** (see [3]) An integer matrix $A = (a_{ij})_{i,j=1}^{n}$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:
(i) $a_{ii} = 2 \; \forall \; i = 1, 2, ..., n$;
(ii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \; \forall \; i, j = 1, 2, ..., n$;
(iii) $a_{ij} \leq 2 \; \forall \; i, j = 1, 2, ..., n$.

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Let us denote the index set of $A$ by $N = \{1, ..., n\}$. A GCM $A$ is said to be decomposable if there exist two non-empty subsets $I, J \subset N$ such that $I \cup J = N$ and $a_{ij} = a_{ji} = 0 \ \forall \ i \in I$ and $j \in J$. If $A$ is not decomposable, it is said to be indecomposable.

**Definition 2.** (see [2]) A realization of a matrix $A = (a_{ij})_{i,j=1}^{n}$ is a triple $(H, \Pi, \Pi^{\vee})$ where $l$ is the rank of $A$, $H$ is a $2n - l$ dimensional complex vector space, $\Pi = \{\alpha_1, ..., \alpha_n\}$ and $\Pi^{\vee} = \{\alpha_1^{\vee}, ..., \alpha_n^{\vee}\}$ are linearly independent subsets of $H^*$ and $H$ respectively, satisfying $\alpha_j(\alpha_i^{\vee}) = a_{ij}$ for $i, j = 1, ..., n$. $\Pi$ is called the root basis. Elements of $\Pi$ are called simple roots. The root lattice generated by $\Pi$ is $Q = \sum_{i=1}^{n} Z\alpha_i$.

**Definition 3.** (see [2]) The Kac-Moody algebra $g(A)$ associated with a GCM $A = (a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements $e_i$, $f_i$, $i = 1, 2, ..., n$ and $H$ with the following defining relations:
$[h, h'] \ = 0, h, h' \in H;$
$[e_i, f_j] \ = \delta_{ij}\alpha_i^{\vee};$
$[h, e_j] \ = \alpha_j(h)e_j; \ [h, f_j] \ = -\alpha_j(h)f_j, \ i, j \in N;$
$(ad \ e_i)^{1-a_{ij}} \ e_j = 0; \ (ad \ f_i)^{1-a_{ij}} \ f_j = 0 \ \forall \ i \neq j \ i, j \in N.$

The Kac-Moody algebra $g(A)$ has the root space decomposition $g(A) = \oplus_{\alpha \in Q} g_{\alpha}(A)$ where $g_{\alpha}(A) = \{x \in g(A)/[h, x] = \alpha(h)x \ \forall \ h \in H\}$. An element $\alpha$, $\alpha \neq 0$ in $Q$ is called a root if $g_{\alpha} \neq 0$ . Let $Q = \sum_{i=1}^{n} Z_{+}\alpha_i$ . Q has a partial ordering " $\leq$ " defined by $\alpha \leq \beta$ if $\beta - \alpha \in Q_{+}$, where $\alpha, \beta \in Q$.

**Definition 4.** (see [2]) Let $\triangle(= \triangle(A))$ denote the set of all roots of $g(A)$ and $\triangle_{+}$ the set of all positive roots of $g(A)$. We have $\triangle_{-} = -\triangle_{+}$ and $\triangle = \triangle_{+} \cup \triangle_{-}$.

**Definition 5.** (see [2]) To every GCM A is associated a Dynkin diagram S (A) defined as follows: S (A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|, |a_{ji}|\}$ number of lines if $a_{ij}.a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $a_{ij}.a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|, |a_{ji}|)$ of integers.

**Definition 6.** (see [2]) Let $A$ be a real n x n matrix satisfying (m1), (m2) and (m3):

(m1) $A$ is indecomposable;

(m2) $a_{ij} \leq 0$ for $i \neq j$;

(m3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Then one and only one of the following three possibilities holds for both $A$ and ${}^{t}A$:

(i) det A $\neq 0$; there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$

or $v = 0$;

(ii) co rank $A = 1$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$;

(iii) there exists $u > 0$ such that $Au < 0$; $Av \geq 0$, $v \geq 0$ imply $v = 0$.

Then $A$ is of finite, affine or indefinite type iff (i), (ii) or (iii) is satisfied.

**Definition 7.** (see [2]) A Kac- Moody algebra $g(A)$ is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.

### 1.2. Basic Definitions on Fuzzy Sets

**Definition 8.** (see [6]) A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$ that can be finite, countable or over-countable.

**Definition 9.** (see [6]) If $X$ is a collection of objects denoted generically by $x$, then a fuzzy set $\tilde{A}$ is defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))/x \in X\}$. $\mu_{\tilde{A}}(x)$ is called the membership function or "grade of membership" of $x$ in $\tilde{A}$ that maps $X$ to the membership space $M$.

**Definition 10.** (see [6]) The support of a fuzzy set $\tilde{A}$ , $S(\tilde{A})$ is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

**Definition 11.** (see [6]) The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree $\alpha$ is called the $\alpha$- level set $A_\alpha = \{x \in X/\mu_{\tilde{A}}(x) \geq \alpha\}$; $A'_\alpha = \{x \in X/\mu_{\tilde{A}}(x) > \alpha\}$ is called "Strong $\alpha$- level set" or "Strong $\alpha$- cut".

**Definition 12.** (see [6]) Let $\tilde{A}$ be a fuzzy set on $X$. Then the set $\{x \in X/\mu_{\tilde{A}}(x) = 1\}$ is called the core of the fuzzy set $\tilde{A}$. This set is denoted by core $(\tilde{A})$.

**Definition 13.** (see [6]) A fuzzy set $\tilde{A}$ is said to be normal if

$$sup_x \mu_{\tilde{A}}(x) = 1.$$

**Definition 14.** (see [6]) The membership function of the complement of a normalized fuzzy set $\tilde{A}$ is defined as $\mu_{\complement\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$, $x \in X$.

**Definition 15.** (see [6]) For a finite fuzzy set $\tilde{A}$, the cardinality $|\tilde{A}|$ is defined as $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)$. $\|\tilde{A}\| = |\tilde{A}|/|X|$ is called the relative cardinality of $\tilde{A}$.

**Definition 16.** (see [6]) A fuzzy set $\tilde{A}$ is convex if

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\},$$

$x_1, x_2 \in X$, $\lambda \in [0,1]$.

**Definition 17.** (see [1]) Let $A$ be a fuzzy set on U and $\alpha$ be a number such that $0 < \alpha \leq 1$. Then by $\alpha A$ we mean a fuzzy set on U, denoted by $\alpha A$ which is such that $(\alpha A)(x) = \alpha A(x)$ for every $x$ in $U$. This procedure of associating another fuzzy set with the given fuzzy set A is termed as restricted scalar multiplication.

**Theorem 18.** (see [1]) *Any fuzzy set A on U can be decomposed as* $A = sup\{\alpha A_\alpha / \ 0 < \alpha \leq 1\}$. *We also write* $A = \sum \alpha A_\alpha$ *or* $A = \cup \alpha A_\alpha$.

In the previous paper of [4], the new concept of fuzzy sets on the root systems of Kac-Moody algebras was introduced. The fuzzy set on $X = \Pi \times \Pi$ is defined as follows:

$$\mu_{\tilde{A}}(\alpha_i, \alpha_j) = \begin{cases} 1/\max(|a_{ij}|, |a_{ji}|), & \text{if } a_{ij} \neq 0, \\ 0, \text{if } a_{ij} = 0, \end{cases} \tag{1}$$

for $(\alpha_i, \alpha_j) \in X, \ i, j = 1, 2, ...l$.

Then $\tilde{A} = ((\alpha_i, \alpha_j), \mu_{\tilde{A}}(\alpha_i, \alpha_j))$ forms a fuzzy set on $\Pi \times \Pi$.

The following properties of fuzzy set defined by (1) on X are also given in [4].

(i) Support of $\tilde{A}$ consists of all $(\alpha_i, \alpha_j)$ such that $a_{ij} \neq 0$, for $i, j = 1, ..., l$.

(ii)Core of the fuzzy set $\tilde{A}$ is non - empty if and only if the associated Dynkin diagram contains at least one sub diagram of the form o—o.

(iii)The fuzzy set $\tilde{A}$ defined by (1) is normal iff a sub diagram of the form o—o occurs in the Dynkin diagram associated with the GCM $A$.

**Note.** For the affine type of Kac-Moody algebras the rank of the GCM $A = n - 1$. i.e., $l = n - 1$.

## 2. Some Fuzzy Properties on the Root Basis of Affine Kac-Moody Algebras

In this section we shall study about some more properties of fuzzy sets on the root basis of affine Kac- Moody algebras, belonging to Table aff(2) and Table aff(3) in [5], $A_l^{(2)}, A_{2l-2}^{(2)}(l \geq 3)$, $A_{2l-3}^{(2)}(l \geq 4), D_l^{(2)}(l \geq 3), E_6^{(2)}, D_4^{(3)}$. Using table 1,2 and 3 which shows all possible membership grades for the elements of X, the convexity is verified.

| $\mu_{\tilde{A}}(\alpha_i, \alpha_j)$ | $\mu_{\tilde{A}}(\alpha_k, \alpha_l)$ | $\mu_{\tilde{A}}(\lambda(\alpha_i, \alpha_j)+$ $(1-\lambda)(\alpha_k, \alpha_l))$ | $min\{\mu_{\tilde{A}}(\alpha_i, \alpha_j),$ $\mu_{\tilde{A}}(\alpha_k, \alpha_l)\}$ |
|---|---|---|---|
| 1/2 | 1/4 | $(1+\lambda)/4$ | 1/4 |
| 1/4 | 1/2 | $(2-\lambda)/4$ | 1/4 |
| 1/2 | 1/2 | 1/2 | 1/2 |
| 1/4 | 1/4 | 1/4 | 1/4 |

Table 1: Membership grade attained by every element in X.

**Lemma 19.** *A fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebras $A_l^{(2)}$ defined by equation (1) is convex.*

*Proof.* Consider the affine type of Kac-Moody algebras $A_l^{(2)}$. Table 1 shows all possible membership grades for the elements of X and the conditions for checking convexity is listed. For every element in $X = \Pi \times \Pi$, we see that the following inequality is satisfied:

$$\mu_{\tilde{A}}(\lambda(\alpha_i, \alpha_j) + (1-\lambda)(\alpha_k, \alpha_l)) \geq \min\{\mu_{\tilde{A}}(\alpha_i, \alpha_j), \mu_{\tilde{A}}(\alpha_k, \alpha_l)\}.$$

Hence the fuzzy set $\tilde{A}$ corresponding to the affine type of Kac-Moody algebra $A_l^{(2)}$ is convex. □

**Lemma 20.** *A fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebras $A_{2l-2}^{(2)}(l \geq 3), A_{2l-3}^{(2)}(l \geq 4), D_l^{(2)}(l \geq 3), E_6^{(2)}$ defined by equation (1) is convex.*

*Proof.* Consider the affine type of Kac-Moody algebras $A_{2l-2}^{(2)}(l \geq 3)$, $A_{2l-3}^{(2)}(l \geq 4), D_l^{(2)}(l \geq 3)$, $E_6^{(2)}$. Table 2 shows all possible membership grades for the elements of X and the conditions for checking convexity is listed in the Table 2. For every element in $X = \Pi \times \Pi$, we see that the following inequality is satisfied:

$$\mu_{\tilde{A}}(\lambda(\alpha_i, \alpha_j) + (1-\lambda)(\alpha_k, \alpha_l)) \geq \min\{\mu_{\tilde{A}}(\alpha_i, \alpha_j), \mu_{\tilde{A}}(\alpha_k, \alpha_l)\}.$$

Hence the fuzzy set $\tilde{A}$ corresponding to the affine type of Kac-Moody algebra $A_{2l-2}^{(2)}(l \geq 3), A_{2l-3}^{(2)}(l \geq 4), D_l^{(2)}(l \geq 3), E_6^{(2)}$ is convex. □

**Lemma 21.** *A fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $D_4^{(3)}$ defined by equation (1) is convex.*

| $\mu_{\tilde{A}}(\alpha_i,\alpha_j)$ | $\mu_{\tilde{A}}(\alpha_k,\alpha_l)$ | $\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+$ $(1-\lambda)(\alpha_k,\alpha_l))$ | $min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j),$ $\mu_{\tilde{A}}(\alpha_k,\alpha_l)\}$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | $\lambda$ | 0 |
| 1 | 1/2 | $(\lambda+1)/2$ | 1/2 |
| 0 | 1 | $(1-\lambda)$ | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 1/2 | $(1-\lambda)/2$ | 0 |
| 1/2 | 1 | $(2-\lambda)/2$ | 1/2 |
| 1/2 | 0 | $\lambda/2$ | 0 |
| 1/2 | 1/2 | 1/2 | 1/2 |

Table 2: Membership grade attained by every element in X.

| $\mu_{\tilde{A}}(\alpha_i,\alpha_j)$ | $\mu_{\tilde{A}}(\alpha_k,\alpha_l)$ | $\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+$ $(1-\lambda)(\alpha_k,\alpha_l))$ | $min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j),$ $\mu_{\tilde{A}}(\alpha_k,\alpha_l)\}$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | $\lambda$ | 0 |
| 1 | 1/2 | $(\lambda+1)/2$ | 1/2 |
| 1 | 1/3 | $(2\lambda+1)/3$ | 1/3 |
| 0 | 1 | $(1-\lambda)$ | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 1/2 | $(1-\lambda)/2$ | 0 |
| 0 | 1/3 | $(1-\lambda)/3$ | 0 |
| 1/2 | 1 | $(2-\lambda)/2$ | 1/2 |
| 1/2 | 0 | $\lambda/2$ | 0 |
| 1/2 | 1/2 | 1/2 | 1/2 |
| 1/2 | 1/3 | $(\lambda+2)/6$ | 1/3 |
| 1/3 | 1 | $(3-2\lambda)/3$ | 1/3 |
| 1/3 | 0 | $\lambda/3$ | 0 |
| 1/3 | 1/2 | $(3-\lambda)/6$ | 1/3 |
| 1/3 | 1/3 | 1/3 | 1/3 |

Table 3: Membership grade attained by every element in X.

*Proof.* Consider the affine type of Kac-Moody algebra $D_4^{(3)}$. Table 3 shows all possible membership grades for the elements of X and the conditions for checking convexity is listed in the Table 3. For every element in $X = \Pi\times\Pi$, we see that the following inequality is satisfied: $\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l)) \geq$

$min\{\mu_{\tilde{A}}(\alpha_i, \alpha_j), \mu_{\tilde{A}}(\alpha_k, \alpha_l)\}$. Hence the fuzzy set $\tilde{A}$ corresponding to the affine type of Kac-Moody algebra $D_4^{(3)}$ is convex. □

**Theorem 22.** *For the affine Kac-Moody algebra* $A_l^{(2)}$ *associated with the indecomposable GCM* $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, *let* $\tilde{A}$ *be the fuzzy set defined on* $\Pi \times \Pi$ *given by the equation (1). Then the* $\alpha$*- level sets and strong* $\alpha$*- level sets for* $\alpha = 1, 1/2, ..., 1/k, ...$ *are given below:*

$(i) A_1 \quad = \Phi$
$(ii) A_{1/2} \quad = \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2)\} = A_{1/3}$
$(iii) A_{1/4} \quad = \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\}$
$(iv) A'_{1/2} \quad = \Phi$
$(v) A'_{1/3} \quad = A_{1/2}$
$(vi) A'_{1/4} \quad = A_{1/3}$
$(vii) A'_{1/5} \quad = A_{1/4}$
$(viii) |A_{1/2}| = 2, |A_{1/3}| = 2, |A_{1/4}| = 4, |A'_{1/5}| = 4.$
*For k = 5,6,...,* $|A_{1/k}| = |A'_{1/k}| = 4$.

*Proof.* Consider the family $A_l^{(2)}$,
$(i) A_1 \quad = \{(\alpha_i, \alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i, \alpha_j) \geq 1\} = \Phi$
$(ii) A_{1/2} \quad = \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2)\}$
$A_{1/3} = A_{1/2}$
$(iii) A_{1/4} = \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\}$
From the above relations we have,
$A_1 \subset A_{1/2} = A_{1/3} \subset A_{1/4} = ... = A_{1/k} = ...$
$(iv) A'_{1/2} \quad = \{(\alpha_i, \alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i, \alpha_j) > 1/2\} = \Phi$
$(v) A'_{1/3} \quad = \{(\alpha_i, \alpha_j) \in X / \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/3\} = A_{1/2}$
$(vi) A'_{1/4} \quad = \{(\alpha_i, \alpha_j) \in X / \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/4\} = A_{1/3}$
$(vii) A'_{1/5} = \{(\alpha_i, \alpha_j) \in X / \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/4\} = A_{1/4}$
From the above relations we see that,
$A'_1 \subset A'_{1/2} \subset A'_{1/3} = A'_{1/4} \subset A'_{1/5} = ... = A'_{1/k} = ...$
$(viii)$ By the above relations,
$A_{1/4} = ... = A_{1/l} = ... = A'_{1/5} = ... = A'_{1/k}$
$|A_{1/2}| = 2, |A_{1/3}| = 2, |A_{1/4}| = 4, |A'_{1/5}| = 7.$
For $k = 5, 6, ...,$ $|A_{1/k}| = |A'_{1/k}| = 4$. □

**Lemma 23.** *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the affine type of Kac-Moody algebra $A_l^{(2)}$ by equation (1) then $\tilde{A}$ has the following properties:*

*(i) The cardinality $|\tilde{A}| = 1.5$*

*(ii) Relative cardinality $\|\tilde{A}\| = 0.375$*

*(iii)The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A_l^{(2)}$, are listed below: For $i = 2$, $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 3/4$.*

*For $i = 1, 2$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1/2$.*

*Proof.* (i) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A_l^{(2)}$ contains 2 elements in X having membership grade 1/2 , 2 elements in X having membership grade 1/4 . By the definition of cardinality, $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = 1.5$.

(ii) $\|\tilde{A}\| = |\tilde{A}|/|X| = 0.375$.

(iii) The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A_l^2$ is normal. For $(\alpha_i, \alpha_j) \in X$ , the membership function of the complement of a normalized fuzzy set $\tilde{A}$ are listed below:

For $i = 2$, $\mu_{\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1/4 = 3/4$.

For $i = 1, 2$, $\mu_{\tilde{A}}(\alpha_i, \alpha_i) = 1 - 1/2 = 1/2$. □

**Theorem 24.** *For the affine Kac-Moody algebra $A_{2l-2}^{(2)} (l \geq 3)$ associated with the indecomposable GCM* $\begin{pmatrix} 2 & -2 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & -1 & 2 & -2 \\ 0 & 0 & ... & -1 & 2 \end{pmatrix}$, *let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ given by the equation (1). Then the $\alpha$- level sets and strong $\alpha$- level sets for $\alpha = 1, 1/2, ...,$*
*$1/k, ...$ are given below:*

*(i)$A_1 \quad = \{(\alpha_2, \alpha_3), ..., (\alpha_{l-2}, \alpha_{l-1}), (\alpha_{l-1}, \alpha_{l-2}), ..., (\alpha_3, \alpha_2)\}$*

*(ii)$A_{1/2} \ = A_1 \cup \{(\alpha_1, \alpha_1), ..., (\alpha_l, \alpha_l), (\alpha_1, \alpha_2), (\alpha_2, \alpha_1), (\alpha_{l-1}, \alpha_l), (\alpha_l, \alpha_{l-1})\}$*

*(iii)$A'_{1/2} = A_1$*

*(iv)$A'_{1/3} = A_{1/2}$*

*(v)$|A_1| \quad = 2l - 6, |A_{1/2}| = 3l - 2, |A'_{1/3}| = 3l - 2.$*

*For $k = 3, 4, ...,$ $|A_{1/k}| = |A'_{1/k}| = 3l - 2.$*

*Proof.* Consider the family $A_{2l-2}^{(2)} (l \geq 3)$.

$(i) A_1 = \{(\alpha_i, \alpha_j) \in X/ \ \ 1/max(|a_{ij}|, |a_{ji}|) \geq 1\}$
$= \{(\alpha_2, \alpha_3), ..., (\alpha_{l-2}, \alpha_{l-1}), (\alpha_{l-1}, \alpha_{l-2}), ..., (\alpha_3, \alpha_2)\}$
$(ii) A_{1/2} = A_1 \cup \{(\alpha_1, \alpha_1), ..., (\alpha_l, \alpha_l), (\alpha_1, \alpha_2), (\alpha_2, \alpha_1),$
$(\alpha_{l-1}, \alpha_l), (\alpha_l, \alpha_{l-1})\}$
$A_{1/3} \ = A_{1/2}$
We have, $A_1 \subset A_{1/2} = A_{1/3} = ... = A_{1/k} = ...$
$(iii) A'_{1/2} \ \ = \{(\alpha_i, \alpha_j) \in X/ \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/2\} = A_1$
$(iv) A'_{1/3} \ \ = \{(\alpha_i, \alpha_j) \in X/ \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/3\} = A_{1/2}$
$A'_{1/4} \ \ = \{(\alpha_i, \alpha_j) \in X/ \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/4\} = A_{1/3}$
We see that, $A'_1 \subset A'_{1/2} \subset A'_{1/3} = A'_{1/4} = ... = A'_{1/k} = ...$

$(v)$ By the above relations, $A_{1/2} = A_{1/3} = ... = A_{1/k} = ... = A'_{1/3} = ... = A'_{1/k}$ $|A_1| = 2l - 6, |A_{1/2}| = 3l - 2, |A'_{1/3}| = 3l - 2.$

For $k = 3, 4, ...,$ $|A_{1/k}| = |A'_{1/k}| = 3l - 2.$ □

**Lemma 25.** *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the affine type of Kac-Moody algebra $A^{(2)}_{2l-2} (l \geq 3)$ by equation (1) then $\tilde{A}$ has the following properties:*

*(i)The cardinality $|\tilde{A}| = (5l - 8)/2$*

*(ii)Relative cardinality $\|\tilde{A}\| = (5l - 8)/2l^2$*

*(iii)The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A^{(2)}_{2l-2} (l \geq 3)$, $\forall (\alpha_i, \alpha_j) \in X$ are listed below:*

*For $i = 3, ..., l - 1$, $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 0.$*

*For $i = 1, 2, ...l$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1/2$*

*For $i \neq j$, $i \neq j + 1$ and $i \neq j - 1$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_j) = 1$ and $\mu_{\complement\tilde{A}}(\alpha_2, \alpha_1) = \mu_{\complement\tilde{A}}(\alpha_1, \alpha_2) = \mu_{\complement\tilde{A}}(\alpha_{l-1}, \alpha_l) = \mu_{\complement\tilde{A}}(\alpha_l, \alpha_{l-1}) = 1/2$*

*Proof.* (i) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A^{(2)}_{2l-2} (l \geq 3)$ contains $(2l - 6)$ elements in X having membership grade 1 , $(l + 4)$ elements in X having membership grade 1/2 and all the other elements in X having membership grade 0. By the definition of cardinality, $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = (5l - 8)/2.$

(ii) $\|\tilde{A}\| = |\tilde{A}|/|X| = (5l - 8)/2l^2.$

(iii)The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A^{(2)}_{2l-2} (l \geq 3)$ is normal. For $(\alpha_i, \alpha_j) \in X$ , the membership function of the complement of a normalized fuzzy set $\tilde{A}$ are listed below:

For $i = 3, ..., l - 1$, $\mu_{\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1 = 0.$

For $i = 1, 2, ..., l$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1 - 1/2 = 1/2$.

For $i \neq j$, $i \neq j+1$ and $i \neq j-1$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_j) = 1-0 = 1$ and $\mu_{\complement\tilde{A}}(\alpha_2, \alpha_1) = \mu_{\complement\tilde{A}}(\alpha_1, \alpha_2) = 1-1/2 = 1/2$. $\mu_{\complement\tilde{A}}(\alpha_{l-1}, \alpha_l) = \mu_{\complement\tilde{A}}(\alpha_l, \alpha_{l-1}) = 1-1/2 = 1/2$. □

**Theorem 26.** *For the affine Kac-Moody algebra $A^{(2)}_{2l-3}(l \geq 4)$ associated with the indecomposable GCM* $\begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & -1 \\ ... & ... & ... & ... & ... \\ 0 & ... & 2 & -2 & 0 \\ 0 & ... & -1 & 2 & 0 \\ 0 & -1 & ... & 0 & 2 \end{pmatrix}$, *let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ given by the equation (1). Then the $\alpha$- level sets and strong $\alpha$- level sets for $\alpha = 1, 1/2, ...,$*
*$1/k, ...$ are given below:*

$(i) A_1 = \{(\alpha_1, \alpha_2), ..., (\alpha_{l-3}, \alpha_{l-2}), (\alpha_{l-2}, \alpha_{l-3}), ..., (\alpha_2, \alpha_1), (\alpha_2, \alpha_l), (\alpha_l, \alpha_2)\}$

$(ii) A_{1/2} = A_1 \cup \{(\alpha_1, \alpha_1), ..., (\alpha_l, \alpha_l), (\alpha_{l-2}, \alpha_{l-1}), (\alpha_{l-1}, \alpha_{l-2})\}$

$(iii) A'_{1/2} = A_1$

$(iv) A'_{1/3} = A_{1/2}$

$(v) |A_1| = 2l - 4, |A_{1/2}| = 3l - 2, |A'_{1/3}| = 3l - 2.$

*For $k = 3, 4, ...,$ $|A_{1/k}| = |A'_{1/k}| = 3l - 2$.*

*Proof.* Consider the family $A^{(2)}_{2l-3}(l \geq 4)$,

$(i) A_1 = \{(\alpha_i, \alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i, \alpha_j) \geq 1\}$
$= \{(\alpha_1, \alpha_2), ..., (\alpha_{l-3}, \alpha_{l-2}), (\alpha_{l-2}, \alpha_{l-3}), ..., (\alpha_2, \alpha_1), (\alpha_2, \alpha_l), (\alpha_l, \alpha_2)\}$

$(ii) A_{1/2} = A_1 \cup \{(\alpha_1, \alpha_1), ..., (\alpha_l, \alpha_l), (\alpha_{l-2}, \alpha_{l-1}), (\alpha_{l-1}, \alpha_{l-2})\}$

$A_{1/3} = A_{1/2}$

We have, $A_1 \subset A_{1/2} = A_{1/3} = ... = A_{1/k} = ...$

$(iii) A'_{1/2} = \{(\alpha_i, \alpha_j) \in X/ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/2\} = A_1$

$(iv) A'_{1/3} = \{(\alpha_i, \alpha_j) \in X/ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/3\} = A_{1/2}$

$A'_{1/4} = \{(\alpha_i, \alpha_j) \in X/ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/4\} = A_{1/3}$

We see that, $A'_1 \subset A'_{1/2} \subset A'_{1/3} = A'_{1/4} = ... = A'_{1/k} = ...$

$(v)$ By the above relations, $A_{1/2} = A_{1/3} = ... = A_{1/k} = ... = A'_{1/3} = ... = A'_{1/k}$ $|A_1| = 2l - 4, |A_{1/2}| = 3l - 2, |A'_{1/3}| = 3l - 2.$

For $k = 3, 4, ...,$ $|A_{1/k}| = |A'_{1/k}| = 3l - 2.$ □

**Lemma 27.** *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the affine type of Kac-Moody algebra $A_{2l-3}^{(2)}(l \geq 4)$ by equation (1) then $\tilde{A}$ has the following properties:*

*(i)The cardinality $|\tilde{A}| = (5l-6)/2$.*

*(ii)Relative cardinality $\|\tilde{A}\| = (5l-6)/2l^2$.*

*(iii)The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A_{2l-3}^{(2)}(l \geq 4)$, $\forall\ (\alpha_i, \alpha_j) \in X$ are listed below: For $i = 2, ..., l-2$, $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 0$.*

*For $i = 1, 2, ...l$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1/2$.*

*$\mu_{\complement\tilde{A}}(\alpha_{l-1}, \alpha_{l-2}) = \mu_{\complement\tilde{A}}(\alpha_{l-2}, \alpha_{l-1}) = 1/2$.*

*$\mu_{\complement\tilde{A}}(\alpha_l, \alpha_2) = \mu_{\complement\tilde{A}}(\alpha_2, \alpha_l) = 0$. For $i \neq j$, $i \neq j+1$ and $i \neq j-1$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_j) = 1$ except for $\mu_{\complement\tilde{A}}(\alpha_2, \alpha_l) = 0$.*

*Proof.* (i) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A_{2l-3}^{(2)}(l \geq 4)$ contains $(2l-4)$ elements in X having membership grade 1 , $(l+2)$ elements in X having membership grade 1/2 and all the other elements in X having membership grade 0. By the definition of cardinality, $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = (5l-6)/2$.

(ii)$\|\tilde{A}\| = |\tilde{A}|/|X| = (5l-6)/2l^2$.

(iii)The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $A_{2l-3}^{(2)}(l \geq 4)$ is normal. For $(\alpha_i, \alpha_j) \in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ are listed below:

For $i = 2, ..., l-2$, $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1 = 0$.

For $i = 1, 2, ..., l$

$$\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1 - 1/2 = 1/2,$$

$$\mu_{\complement\tilde{A}}(\alpha_{l-2}, \alpha_{l-1}) = \mu_{\tilde{A}}(\alpha_{l-1}, \alpha_{l-2}) = 1 - 1/2 = 1/2.$$

$\mu_{\complement\tilde{A}}(\alpha_2, \alpha_l) = \mu_{\complement\tilde{A}}(\alpha_l, \alpha_2) = 1 - 1 = 0$. For $i \neq j$, $i \neq j+1$ and $i \neq j-1$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_j) = 1 - 0 = 1$ except for $\mu_{\complement\tilde{A}}(\alpha_2, \alpha_l) = 0$. □

**Theorem 28.** *For the affine Kac-Moody algebra $D_l^{(2)}(l \geq 3)$ associated with the indecomposable GCM* $\begin{pmatrix} 2 & -2 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & -1 & 2 & -1 \\ 0 & 0 & ... & -2 & 2 \end{pmatrix}$, *let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ given by the equation (1). Then the $\alpha$- level sets and strong $\alpha$- level sets for $\alpha = 1, 1/2, ..., 1/k, ...$ are given below:*

*(i)$A_1 = \{(\alpha_2, \alpha_3), ..., (\alpha_{l-2}, \alpha_{l-1}), (\alpha_{l-1}, \alpha_{l-2}), ..., (\alpha_3, \alpha_2)\}$*

*(ii)*$A_{1/2} = A_1 \cup \{(\alpha_1,\alpha_1),...,(\alpha_l,\alpha_l),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_l,\alpha_{l-1}),(\alpha_{l-1},\alpha_l)\}$
*(iii)*$A'_{1/2} = A_1$
*(iv)*$A'_{1/3} = A_{1/2}$
*(v)*$|A_1| = 2l-6, |A_{1/2}| = 3l-2, |A'_{1/3}| = 3l-2.$
*For* $k = 3,4,...,$ $|A_{1/k}| = |A'_{1/k}| = 3l-2.$

*Proof.* Consider the family $D_l^{(2)}(l \geq 3)$,
(i)$A_1 = \{(\alpha_i,\alpha_j) \in X/ \ 1/max(|a_{ij}|,|a_{ji}|) \geq 1\}$
$= \{(\alpha_2,\alpha_3),...,(\alpha_{l-2},\alpha_{l-1}),(\alpha_{l-1},\alpha_{l-2}),...,(\alpha_3,\alpha_2)\}$
(ii)$A_{1/2} = A_1 \cup \{(\alpha_1,\alpha_1),...,(\alpha_l,\alpha_l),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1)\ (\alpha_l,\alpha_{l-1}),(\alpha_{l-1},\alpha_l)\}$
$A_{1/3} = A_{1/2}$.
We have, $A_1 \subset A_{1/2} = A_{1/3} = ... = A_{1/k} = ...$
(iii)$A'_{1/2} = \{(\alpha_i,\alpha_j) \in X/ \ 1/max(|a_{ij}|,|a_{ji}|) > 1/2\} = A_1$
(iv)$A'_{1/3} = \{(\alpha_i,\alpha_j) \in X/ \ 1/max(|a_{ij}|,|a_{ji}|) > 1/3\} = A_{1/2}$ $A'_{1/4} = \{(\alpha_i,\alpha_j) \in X/ \ 1/max(|a_{ij}|,|a_{ji}|) > 1/4\} = A_{1/3}$.
We see that, $A'_1 \subset A'_{1/2} \subset A'_{1/3} = A'_{1/4} = ... = A'_{1/k} = ...$
(v) By the above relations, $A_{1/2} = A_{1/3} = ... = A_{1/k} = ... = A'_{1/3} = ... = A'_{1/k}$ $|A_1| = 2l-6, |A_{1/2}| = 3l-2, |A'_{1/3}| = 3l-2.$
For $k = 3,4,...,$ $|A_{1/k}| = |A'_{1/k}| = 3l-2.$ □

**Lemma 29.** *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the affine type of Kac-Moody algebra $D_l^{(2)}(l \geq 3)$ by equation (1) then $\tilde{A}$ has the following properties:*
*(i)The cardinality $|\tilde{A}| = (5l-8)/2$.*
*(ii)Relative cardinality $\|\tilde{A}\| = (5l-8)/2l^2$.*
*(iii)The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $D_l^{(2)}(l \geq 3)$, $\forall$ $(\alpha_i,\alpha_j) \in X$ are listed below:*
*For $i = 3,...,l-1$, $\mu_{\complement\tilde{A}}(\alpha_{i-1},\alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i,\alpha_{i-1}) = 0$.*
*For $i = 1,2,...,l$ $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_i) = 1/2$, $\mu_{\complement\tilde{A}}(\alpha_1,\alpha_2) = \mu_{\complement\tilde{A}}(\alpha_2,\alpha_1) = \mu_{\complement\tilde{A}}(\alpha_{l-1},\alpha_l) = \mu_{\complement\tilde{A}}(\alpha_l,\alpha_{l-1}) = 1/2$.*
*For $i \neq j$, $i \neq j+1$ and $i \neq j-1$, $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_j) = 1$*

*Proof.* (i) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $D_l^{(2)}$ contains $(2l-6)$ elements in X having membership grade 1 , $(l+4)$ elements in X having membership grade 1/2 and all the other elements in X having membership grade 0. By the definition of cardinality, $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = (5l-8)/2$.

(ii)$\|\tilde{A}\| = |\tilde{A}|/|X| = (5l-8)/2l^2$.

(iii) The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $D_l^{(2)} (l \geq 2)$ is normal. For $(\alpha_i, \alpha_j) \in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ are listed below:

For $i = 3, ..., l-1$, $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1 = 0$.

For $i = 1, 2, ..., l$ $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1 - 1/2 = 1/2$. $\mu_{\complement\tilde{A}}(\alpha_1, \alpha_2) = \mu_{\complement\tilde{A}}(\alpha_2, \alpha_1) = \mu_{\complement\tilde{A}}(\alpha_{l-1}, \alpha_l) = \mu_{\complement\tilde{A}}(\alpha_l, \alpha_{l-1}) = 1 - 1/2 = 1/2$. For $i \neq j$, $i \neq j+1$ and $i \neq j-1$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_j) = 1$. □

**Theorem 30.** *For the affine Kac-Moody algebra $E_6^{(2)}$ associated with the indecomposable GCM* $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$, *let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ given by the equation (1). Then the $\alpha$- level sets and strong $\alpha$- level sets for $\alpha = 1, 1/2, ..., 1/k, ...$ are given below:*

*(i)*$A_1 = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_4, \alpha_5), (\alpha_5, \alpha_4), (\alpha_3, \alpha_2), (\alpha_2, \alpha_1)\}$

*(ii)*$A_{1/2} = A_1 \cup \{(\alpha_1, \alpha_1), ..., (\alpha_5, \alpha_5), (\alpha_3, \alpha_4), (\alpha_4, \alpha_3)\}$

*(iii)*$A'_{1/2} = A_1$

*(iv)*$A'_{1/3} = A_{1/2}$

*(v)*$|A_1| = 6, |A_{1/2}| = 13, |A'_{1/3}| = 13.$

*For* $k = 3, 4, ...,$ $|A_{1/k}| = |A'_{1/k}| = 13.$

*Proof.* Consider the family $E_6^{(2)}$ whose associated GCM is

*(i)*$A_1 = \{(\alpha_i, \alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i, \alpha_j) \geq 1\} = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_4, \alpha_5), (\alpha_5, \alpha_4), (\alpha_3, \alpha_2), (\alpha_2, \alpha_1)\}$

*(ii)*$A_{1/2} = A_1 \cup \{(\alpha_1, \alpha_1), ..., (\alpha_5, \alpha_5), (\alpha_3, \alpha_4), (\alpha_4, \alpha_3)\}$ $A_{1/3} = A_{1/2}$.

We have, $A_1 \subset A_{1/2} = A_{1/3} = ... = A_{1/k} = ...$

*(iii)*$A'_{1/2} = \{(\alpha_i, \alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i, \alpha_j) > 1/2\} = \{(\alpha_i, \alpha_j) \in X/\ 1/max(|a_{ij}|, |a_{ji}|) > 1/2\} = A_1$

*(iv)*$A'_{1/3} = \{(\alpha_i, \alpha_j) \in X/\ 1/max(|a_{ij}|, |a_{ji}|) > 1/3\} = A_{1/2}$ $A'_{1/4} = \{(\alpha_i, \alpha_j) \in X/\ 1/max(|a_{ij}|, |a_{ji}|) > 1/4\} = A_{1/3}$.

We see that, $A'_1 \subset A'_{1/2} \subset A'_{1/3} = A'_{1/4} = ... = A'_{1/k} = ...$

$(v)$ By the above relations, $A_{1/2} = A_{1/3} = ... = A_{1/k} = ... = A'_{1/3} = ... = A'_{1/k}$ $|A_1| = 6, |A_{1/2}| = 13, |A'_{1/3}| = 13$.

For $k = 3, 4, ...,$ $|A_{1/k}| = |A'_{1/k}| = 13$. □

**Lemma 31.** *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the affine type of Kac-Moody algebra $E_6^{(2)}$ by equation (1) then $\tilde{A}$ has the following properties:*

*(i)The cardinality $|\tilde{A}| = 9.5$.*

*(ii)Relative cardinality $\|\tilde{A}\| = 0.38$.*

*(iii)The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra*

*$E_6^{(2)}$, $\forall$ $(\alpha_i, \alpha_j) \in X$ are listed below:*

*For $i = 2, 3, 5,$ $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 0$.*

*For $i = 1, 2, ..., 5,$ $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1/2$.*

*$\mu_{\complement\tilde{A}}(\alpha_3, \alpha_4) = \mu_{\complement\tilde{A}}(\alpha_4, \alpha_3) = 1/2$.*

*For $i \neq j$, $i \neq j + 1$ and $i \neq j - 1$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_j) = 1$*

*Proof.* (i) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_6^{(2)}$ contains 6 elements in X having membership grade 1 , 7 elements in X having membership grade 1/2 and all the other elements in X having membership grade 0. By the definition of cardinality, $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = 9.5$.

(ii)$\|\tilde{A}\| = |\tilde{A}|/|X| = 0.38$.

(iii)The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_6^{(2)}$ is normal. For $(\alpha_i, \alpha_j) \in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ are listed below:

For $i = 2, 3, 5,$ $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1 = 0$.

For $i = 1, 2, ..., 5,$ $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1 - 1/2 = 1/2$. $\mu_{\complement\tilde{A}}(\alpha_3, \alpha_4) = \mu_{\complement\tilde{A}}(\alpha_4, \alpha_3) = 1 - 1/2 = 1/2$.

For $i \neq j$, $i \neq j + 1$ and $i \neq j - 1$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_j) = 1 - 0 = 1$ □

**Theorem 32.** *For the affine Kac-Moody algebra $D_4^{(3)}$ associated with the indecomposable GCM $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$, let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ given by the equation (1). Then the $\alpha$- level sets and strong $\alpha$- level sets for $\alpha = 1, 1/2, ..., 1/k, ...$ are given below:*

*(i)*$A_1 \quad = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\}$

*(ii)*$A_{1/2} \quad = A_1 \cup \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_3, \alpha_3)\}$

*(iii)*$A_{1/3} \quad = A_{1/2} \cup \{(\alpha_2, \alpha_3), (\alpha_3, \alpha_2)\}$

$(iv)A'_{1/2} \quad = A_1$
$(v)A'_{1/3} \quad = A_{1/2}$
$(vi)A'_{1/4} \quad = A_{1/3}$
$(vii)|A_1| \quad = 2, |A_{1/2}| = 5, |A_{1/3}| = 7, |A'_{1/4}| = 7.$
*For* $k =$ *5,6,...,* $|A_{1/k}| = |A'_{1/k}| = 7.$

*Proof.* Consider the family $D_4^{(3)}$,
$(i)A_1 = \{(\alpha_i, \alpha_j) \in X/\mu_{\tilde{A}}(\alpha_i, \alpha_j) \geq 1\} = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\}$
$(ii)A_{1/2} = A_1 \cup \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_3, \alpha_3)\}$
$(iii)A_{1/3} = A_{1/2} \cup \{(\alpha_2, \alpha_3), (\alpha_3, \alpha_3)\}$
We have, $A_1 \subset A_{1/2} = A_{1/3} = ... = A_{1/k} = ...$
$(iv)A'_{1/2} = \{(\alpha_i, \alpha_j) \in X/ \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/2\} = A_1$
$(v)A'_{1/3} = \{(\alpha_i, \alpha_j) \in X/ \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/3\} = A_{1/2}$
$(vi)A'_{1/4} = \{(\alpha_i, \alpha_j) \in X/ \ \ 1/max(|a_{ij}|, |a_{ji}|) > 1/4\} = A_{1/3}$
We see that, $A'_1 \subset A'_{1/2} \subset A'_{1/3} = A'_{1/4} = ... = A'_{1/k} = ...$
$(vii)$ By the above relations, $A_{1/3}... = A_{1/l} = ... = A'_{1/4} = ... = A'_{1/k}$ $|A_1| = 2, |A_{1/2}| = 5, |A_{1/3}| = 7, |A'_{1/4}| = 7.$
For $k = 4, 5, ...,$ $|A_{1/k}| = |A'_{1/k}| = 7.$ □

**Lemma 33.** *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the affine type of Kac-Moody algebra $D_4^{(3)}$ by equation (1) then $\tilde{A}$ has the following properties:*
*(i)The cardinality* $|\tilde{A}| = 4.167.$
*(ii)Relative cardinality* $\|\tilde{A}\| = 0.463.$
*(iii)The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $D_4^{(3)}$, $\forall$ $(\alpha_i, \alpha_j) \in X$ are listed below:*
$\mu_{\complement\tilde{A}}(\alpha_1, \alpha_2) = \mu_{\complement\tilde{A}}(\alpha_2, \alpha_1) = 0.$
*For* $i = 1, 2, 3,$ $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1/2.$
*For* $i \neq j,$ $i \neq j + 1$ *and* $i \neq j - 1,$ $\mu_{\complement\tilde{A}}(\alpha_1, \alpha_3) = \mu_{\complement\tilde{A}}(\alpha_3, \alpha_1) = 1$ *and* $\mu_{\complement\tilde{A}}(\alpha_2, \alpha_3) = \mu_{\complement\tilde{A}}(\alpha_3, \alpha_2) = 2/3.$

*Proof.* (i) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $D_4^{(3)}$ contains 2 elements in X having membership grade 1 ,3 elements in X having membership grade 1/2, 2 elements in X having membership grade 1/3 and all the other elements in X having membership grade 0. By the definition of cardinality, $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = 4.167.$

(ii)$\|\tilde{A}\| = |\tilde{A}|/|X| = 0.463$.

(iii)The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $D_4^3(l \geq 2)$, is normal. For $(\alpha_i, \alpha_j) \in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ are listed below:

$\mu_{\complement\tilde{A}}(\alpha_1, \alpha_2) = \mu_{\complement\tilde{A}}(\alpha_2, \alpha_1) = 1 - 1 = 0$.

For $i = 1, 2, 3$, $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1 - 1/2 = 1/2$.

For $i \neq j$, $i \neq j + 1$ and $i \neq j - 1$, $\mu_{\complement\tilde{A}}(\alpha_1, \alpha_3) = \mu_{\complement\tilde{A}}(\alpha_3, \alpha_1) = 1 - 0 = 1$ and $\mu_{\complement\tilde{A}}(\alpha_2, \alpha_3) = \mu_{\complement\tilde{A}}(\alpha_3, \alpha_2) = 1 - 1/3 = 2/3$. □

**Theorem 34.** *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$, where $\Pi$ denotes the root basis for the affine Kac-Moody algebras given by the equation (1). Then the $\alpha$- cut decomposition for the fuzzy set $\tilde{A}$ on the affine families $A_l^{(2)}, A_{2l-2}^{(2)}(l \geq 3), A_{2l-3}(2)(l \geq 4)$, $D_l^{(2)}(l \geq 3), E_6^{(2)}, D_4^{(3)}$ are given as follows:*

*(i)*$A_l^{(2)}$ : $1/2\ A_{1/2} \cup 1/3\ A_{1/3} \cup 1/4\ A_{1/4}$.

*(ii)*$A_{2l-2}^{(2)}(l \geq 3) : 1\ A_1 \cup 1/2\ A_{1/2}$.

*(iii)*$A_{2l-3}^{(2)}(l \geq 4) : 1\ A_1 \cup 1/2\ A_{1/2}$.

*(iv)*$D_l^{(2)}(l \geq 3) : 1\ A_1 \cup 1/2\ A_{1/2}$.

*(v)*$E_6^{(2)} : 1\ A_1 \cup 1/2\ A_{1/2}$.

*(vi)*$D_4^{(3)} : 1\ A_1 \cup 1/2\ A_{1/2} \cup 1/3\ A_{1/3}$.

*Proof.* (*i*) From theorem 22, for the affine family $A_l^{(2)}$, $A_1 \subset A_{1/2} = A_{1/3} \subset A_{1/4}$ and $A_{1/4} = X$.

By definition, the $\alpha$-cut decomposition for the fuzzy set $\tilde{A}$ is $\cup\ \alpha\ A_\alpha$.

Hence $\tilde{A} : 1/2\ A_{1/2} \cup 1/3\ A_{1/3} \cup 1/4\ A_{1/4}$.

(*ii*) From theorem 24, for the affine family $A_{2l-2}^{(2)}(l \geq 3)$, $A_1 \subset A_{1/2}$ and $A_{1/2} = X$. Hence $\tilde{A} : 1\ A_1 \cup 1/2\ A_{1/2}$.

(*iii*) From theorem 26, for the affine family $A_{2l-3}^{(2)}(l \geq 4)$, $A_1 \subset A_{1/2}$ and $A_{1/2} = X$. Hence $\tilde{A} : 1\ A_1 \cup 1/2\ A_{1/2}$.

(*iv*) From theorem 28, for the affine family $D_l^{(2)}(l \geq 3)$, $A_1 \subset A_{1/2}$ and $A_{1/2} = X$. Hence $\tilde{A} : 1\ A_1 \cup 1/2\ A_{1/2}$.

(*v*) From theorem 30, for the affine family $E_6^{(2)}$, $A_1 \subset A_{1/2}$ and $A_{1/2} = X$. Hence $\tilde{A} : 1\ A_1 \cup 1/2\ A_{1/2}$.

(*vi*) From theorem 32, for the affine family $D_4^{(3)}$, $A_1 \subset A_{1/2} \subset A_{1/3}$ and $A_{1/3} = X$.

Hence $\tilde{A} : 1\ A_1 \cup 1/2\ A_{1/2} \cup 1/3\ A_{1/3}$. □

## 3. Conclusion

We can further compute the level sets and determine the $\alpha$-cut decomposition for various families of affine, indefinite, hyperbolic, extended hyperbolic and non hyperbolic type of Kac-Moody algebras; Other interesting structural properties on the fuzzy nature of these algebras can also be studied.

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