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AN IDENTITY OF THE TWISTED Q-EULER POLYNOMIALS ASSOCIATED WITH THE p-ADIC q-INTEGRALS ON $\mathbb{Z}_p$

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Abstract: In [6], we studied the twisted q-Euler numbers and polynomials. By using these numbers and polynomials, we investigate the alternating sums of powers of consecutive integers. By applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the twisted q-Euler polynomials.

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1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an

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indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-8])} .$$

Hence, $\lim_{q \to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [1-8] .} \quad (1.1)$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \quad (1.2)$$

Let $T_p = \cup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{\zeta | \zeta^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$.

In [6], we defined the twisted q-Euler numbers and polynomials and investigate their properties. For $q \in \mathbb{C}_p$ with $|1-q|_p \le 1$, and $\zeta \in T_p$, the twisted q-Euler polynomials $\widetilde{E}_{n,q,\zeta}(x)$ are defined by

$$\widetilde{F}_{q,\zeta}(x,t) = \sum_{n=0}^{\infty} \widetilde{E}_{n,q,\zeta}(x) \frac{t^n}{n!} = \frac{[2]_q}{\zeta q e^t + 1} e^{xt}. \quad (1.3)$$

The twisted q-Euler numbers $\widetilde{E}_{n,q,\zeta}$ are defined by the generating function:

$$\widetilde{F}_{q,\zeta}(t) = \sum_{n=0}^{\infty} \widetilde{E}_{n,q,\zeta} \frac{t^n}{n!} = \frac{[2]_q}{\zeta q e^t + 1}. \quad (1.4)$$

The following elementary properties of the q-Euler numbers $\widetilde{E}_{n,q,\zeta}$ and polynomials $\widetilde{E}_{n,q,\zeta}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4) (see, for details, [6]). We, therefore, choose to omit details involved.

Theorem 1. *(Witt formula). For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $\zeta \in T_p$, we have*

$$\widetilde{E}_{n,q,\zeta} = \int_{\mathbb{Z}_p} \zeta^x x^n d\mu_{-q}(x),$$

$$\widetilde{E}_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \zeta^x (x+y)^n d\mu_{-q}(y).$$

Theorem 2. *For any positive integer n, we have*

$$\widetilde{E}_{n,q,\zeta}(x) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{E}_{k,q,\zeta} x^{n-k}.$$

In this paper, by using the symmetry of p-adic q-integral on $\mathbb{Z}_p$, we obtain the recurrence identities the twisted q-Euler polynomials .

2. The Alternating Sums of Powers of Consecutive q-Integers

Let q be a complex number with $|q| < 1$ and ζ be the p^N-th root of unity. By using (1.3), we give the alternating sums of powers of consecutive q-integers as follows:

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,q,\zeta} \frac{t^n}{n!} = \frac{[2]_q}{\zeta q e^t + 1} = [2]_q \sum_{n=0}^{\infty} (-1)^n \zeta^n q^n e^{nt}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty} (-1)^n \zeta^n q^n e^{(n+k)t} + \sum_{n=0}^{\infty} (-1)^{n-k} \zeta^{n-k} q^{n-k} e^{nt} = \sum_{n=0}^{k-1} (-1)^{n-k} \zeta^{n-k} q^{n-k} e^{nt}.$$

Thus, we have

$$\begin{aligned} &- [2]_q \sum_{n=0}^{\infty} (-1)^n \zeta^n q^n e^{(n+k)t} + [2]_q (-1)^{-k} \zeta^{-k} q^{-k} \sum_{n=0}^{\infty} (-1)^n \zeta^n q^n e^{nt} \\ &= [2]_q (-1)^{-k} \zeta^{-k} q^{-k} \sum_{n=0}^{k-1} (-1)^n \zeta^n q^n e^{nt}. \end{aligned} \tag{2.1}$$

By using (1.3)and (1.4), and (2.1), we obtain

$$-\sum_{j=0}^{\infty} \widetilde{E}_{j,q,\zeta}(k) \frac{t^j}{j!} + (-1)^{-k} \zeta^{-k} q^{-k} \sum_{j=0}^{\infty} \widetilde{E}_{j,q,\zeta} \frac{t^j}{j!}$$

$$= [2]_q \sum_{j=0}^{\infty} \left((-1)^{-k} \zeta^{-k} q^{-k} \sum_{n=0}^{k-1} (-1)^n \zeta^n q^n n^j \right) \frac{t^j}{j!}.$$

By comparing coefficients of $\frac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1} (-1)^n \zeta^n q^n n^j = \frac{(-1)^{k+1} \zeta^k q^k \widetilde{E}_{j,q,\zeta}(k) + \widetilde{E}_{j,q,\zeta}}{[2]_q}.$$

By using the above equation we arrive at the following theorem:

Theorem 3. *Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we obtain*

$$\widetilde{T}_{j,q,\zeta}(k-1) = \sum_{n=0}^{k-1} (-1)^n \zeta^n q^n n^j = \frac{(-1)^{k+1} \zeta^k q^k \widetilde{E}_{j,q,\zeta}(k) + \widetilde{E}_{j,q,\zeta}}{[2]_q}.$$

Corollary 4. *For $\zeta = 1$, we have*

$$\lim_{q \to 1} \widetilde{T}_{j,q,\zeta}(k-1) = \sum_{n=0}^{k-1} (-1)^n n^j = \frac{(-1)^{k+1} E_j(k) + E_j}{2},$$

where $E_j(x)$ and E_j denote the Euler polynomials and Euler numbers, respectively.

Next, we assume that $q \in \mathbb{C}_p$ and $\zeta \in T_p$. We obtain recurrence identities the second q-Euler polynomials and the q-analogue of alternating sums of powers of consecutive integers. By using (1.1), we have

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l),$$

where $g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$q^n I_{-q}(g_n) + I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l) \text{ (cf. [1-5])}. \tag{2.2}$$

It will be more convenient to write (2.2) as the equivalent integral form

$$q^n \int_{\mathbb{Z}_p} g(x+n) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = [2]_q \sum_{k=0}^{n-1} (-1)^k q^k g(k). \tag{2.3}$$

Substituting $g(x) = \zeta^x e^{xt}$ into the above, we obtain

$$\zeta^n q^n \int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} \zeta^x e^{xt} d\mu_{-q}(x) = [2]_q \sum_{j=0}^{n-1} (-1)^j \zeta^j q^j e^{jt}. \quad (2.4)$$

After some elementary calculations, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \zeta^x e^{xt} d\mu_{-q}(x) &= \frac{[2]_q}{\zeta q e^t + 1}, \\ \int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t} d\mu_{-q}(x) &= e^{nt} \frac{[2]_q}{\zeta q e^t + 1}. \end{aligned} \quad (2.5)$$

By using (2.4) and (2.5), we have

$$\zeta^n q^n \int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} \zeta^x e^{xt} d\mu_{-q}(x) = \frac{[2]_q (1 + \zeta^n q^n e^{nt})}{\zeta q e^t + 1}.$$

From the above, we get

$$\frac{[2]_q (1 + \zeta^n q^n e^{nt})}{\zeta q e^t + 1} = \frac{[2]_q \int_{\mathbb{Z}_p} \zeta^x e^{xt} d\mu_{-q}(x)}{\int_{\mathbb{Z}_p} \zeta^{nx} q^{(n-1)x} e^{ntx} d\mu_{-q}(x)}. \quad (2.6)$$

By substituting Taylor series of e^{xt} into (2.4), we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty} \left(\zeta^n q^n \int_{\mathbb{Z}_p} \zeta^x (x+n)^m d\mu_{-q}(x) + \int_{\mathbb{Z}_p} \zeta^x x^m d\mu_{-q}(x) \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left([2]_q \sum_{j=0}^{n-1} (-1)^j \zeta^j q^j j^m \right) \frac{t^m}{m!}. \end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$\zeta^n q^n \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \int_{\mathbb{Z}_p} \zeta^x x^k d\mu_{-q}(x) + \int_{\mathbb{Z}_p} \zeta^x x^m d\mu_{-q}(x) = [2]_q \sum_{j=0}^{n-1} (-1)^j \zeta^j q^j j^m.$$

By using Theorem 3, we have

$$\zeta^n q^n \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \int_{\mathbb{Z}_p} \zeta^x x^k d\mu_{-q}(x) + \int_{\mathbb{Z}_p} \zeta^x x^m d\mu_{-q}(x) = [2]_q \widetilde{T}_{m,q,\zeta}(n-1). \quad (2.7)$$

By using (2.6) and (2.7), we arrive at the following theorem:

Theorem 5. *Let n be odd positive integer. Then we have*

$$\frac{\int_{\mathbb{Z}_p} \zeta^x e^{xt} d\mu_{-q}(x)}{\int_{\mathbb{Z}_p} \zeta^{nx} q^{(n-1)x} e^{ntx} d\mu_{-q}(x)} = \sum_{m=0}^{\infty} \left(\widetilde{T}_{m,q,\zeta}(n-1) \right) \frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. By (2.5), Theorem 5, and after some elementary calculations, we obtain the following theorem.

Theorem 6. *Let w_1 and w_2 be odd positive integers. Then we have*

$$\frac{\int_{\mathbb{Z}_p} \zeta^{w_2 x} e^{w_2 xt} d\mu_{-q^{w_2}}(x)}{\int_{\mathbb{Z}_p} \zeta^{w_1 w_2 x} q^{(w_1 w_2 -1)x} e^{w_1 w_2 tx} d\mu_{-q}(x)} = \frac{[2]_{q^{w_2}}}{[2]_q} \sum_{m=0}^{\infty} \left(\widetilde{T}_{m,q^{w_2},\zeta^{w_2}}(w-1) w_2^m \right) \frac{t^m}{m!}. \tag{2.8}$$

By (1.1), we obtain

$$\begin{aligned} &\frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \zeta^{w_1 x_1 + w_2 x_2} e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x)t} d\mu_{-q^{w_1}}(x_1) d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} \zeta^{w_2 x_2} q^{(w_1 w_2 -1)x} e^{w_1 w_2 xt} d\mu_{-q}(x)} \\ &= \frac{e^{w_1 w_2 xt} \int_{\mathbb{Z}_p} \zeta^{w_1 x_1} e^{w_1 x_1 t} d\mu_{-q^{w_1}}(x_1) \int_{\mathbb{Z}_p} \zeta^{w_2 x_2} e^{w_2 x_2 t} d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} \zeta^{w_1 w_2 x} q^{(w_1 w_2 -1)x} e^{w_1 w_2 xt} d\mu_{-q}(x)}. \end{aligned} \tag{2.9}$$

By using (2.8) and (2.9), after elementary calculations, we obtain

$$\begin{aligned} a &= \left(\int_{\mathbb{Z}_p} \zeta^{w_1 x_1} e^{(w_1 x_1 + w_1 w_2 x)t} d\mu_{-q^{w_1}}(x_1) \right) \\ &\qquad \times \left(\frac{\int_{\mathbb{Z}_p} \zeta^{w_2 x_2} e^{x_2 w_2 t} d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} \zeta^{w_1 w_2 x} q^{(w_1 w_2 -1)x} e^{w_1 w_2 xt} d\mu_{-q}(x)} \right) \\ &= \left(\sum_{m=0}^{\infty} \widetilde{E}_{m,q^{w_1},\zeta^{w_1}}(w_2 x) w_1^m \frac{t^m}{m!} \right) \left(\frac{[2]_{q^{w_2}}}{[2]_q} \sum_{m=0}^{\infty} \widetilde{T}_{m,q^{w_2},\zeta^{w_2}}(w_1 - 1) w_2^m \frac{t^m}{m!} \right). \end{aligned} \tag{2.10}$$

By using Cauchy product in the above, we have

$$a = \sum_{m=0}^{\infty} \left(\frac{[2]_{q^{w_2}}}{[2]_q} \sum_{j=0}^{m} \binom{m}{j} \widetilde{E}_{j,q^{w_1},\zeta^{w_1}}(w_2 x) w_1^j \widetilde{T}_{m-j,q^{w_2},\zeta^{w_2}}(w_1 - 1) w_2^{m-j} \right) \frac{t^m}{m!}. \tag{2.11}$$

By using the symmetry in (2.10), we obtain

$$a = \left(\int_{\mathbb{Z}_p} \zeta^{w_2 x_2} e^{(w_2 x_2 + w_1 w_2 x)t} d\mu_{-q^{w_2}}(x_2)\right) \times \left(\frac{\int_{\mathbb{Z}_p} \zeta^{w_1 x_1} e^{x_1 w_1 t} d\mu_{-q^{w_1}}(x_1)}{\int_{\mathbb{Z}_p} \zeta^{w_1 w_2 x} q^{(w_1 w_2 - 1)x} e^{w_1 w_2 x t} d\mu_{-q}(x)}\right)$$
$$= \left(\sum_{m=0}^{\infty} \widetilde{E}_{m,q^{w_1},\zeta^{w_2}}(w_1 x) w_2^m \frac{t^m}{m!}\right)\left(\frac{[2]_{q^{w_1}}}{[2]_q} \sum_{m=0}^{\infty} \widetilde{T}_{m,q^{w_1},\zeta^{w_1}}(w_2 - 1) w_1^m \frac{t^m}{m!}\right).$$

Thus we obtain

$$a = \sum_{m=0}^{\infty} \left(\frac{[2]_{q^{w_1}}}{[2]_q} \sum_{j=0}^{m} \binom{m}{j} \widetilde{E}_{j,q^{w_2},z^{w_2}}(w_1 x) w_2^j \widetilde{T}_{m-j,q^{w_1},\zeta^{w_1}}(w_2 - 1) w_1^{m-j}\right) \frac{t^m}{m!}. \tag{2.12}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.11) and (2.12), we arrive at the following theorem.

Theorem 7. *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$[2]_{q^{w_2}} \sum_{j=0}^{m} \binom{m}{j} \widetilde{E}_{j,q^{w_1},\zeta^{w_1}}(w_2 x) w_1^j \widetilde{T}_{m-j,q^{w_2},\zeta^{w_2}}(w_1 - 1) w_2^{m-j}$$
$$= [2]_{q^{w_1}} \sum_{j=0}^{m} \binom{m}{j} \widetilde{E}_{j,q^{w_2},z^{w_2}}(w_1 x) w_2^j \widetilde{T}_{m-j,q^{w_1},\zeta^{w_1}}(w_2 - 1) w_1^{m-j},$$

where $\widetilde{E}_{k,q,\zeta}(x)$ and $\widetilde{T}_{m,q,\zeta}(k)$ denote the twisted q-Euler polynomials and the q-analogue of alternating sums of powers of consecutive integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$[2]_{q^{w_1}} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j x^{j-k} \widetilde{E}_{k,q,\zeta^{w_2}} \widetilde{T}_{m-j,q^{w_1},\zeta^{w_1}}(w_2 - 1)$$
$$= [2]_{q^{w_2}} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^j w_2^{m-k} x^{j-k} \widetilde{E}_{k,q,\zeta^{w_1}} \widetilde{T}_{m-j,q^{w_2},\zeta^{w_2}}(w_1 - 1).$$

By using (2.9), we have

$$a = \left(e^{w_1w_2xt}\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{x_1w_1t}d\mu_{-q^{w_1}}(x_1)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{x_2w_2t}d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q}(x)}\right)$$

$$=\frac{[2]_{q^{w_2}}}{[2]_q}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2j}\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{\left(x_1+w_2x+j\frac{w_2}{w_1}\right)(w_1t)}d\mu_{-q^{w_1}}(x_1)$$

$$=\sum_{n=0}^{\infty}\left(\frac{[2]_{q^{w_2}}}{[2]_q}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2j}\widetilde{E}_{n,q^{w_1},\zeta^{w_1}}\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}. \quad (2.13)$$

By using the symmetry property in (2.13), we also have

$$a = \left(e^{w_1w_2xt}\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{x_2w_2t}d\mu_{-q^{w_2}}(x_2)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{x_1w_1t}d\mu_{-q^{w_1}}(x_1)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q}(x)}\right)$$

$$=\frac{[2]_{q^{w_1}}}{[2]_q}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}q^{w_1j}\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{\left(x_2+w_1x+j\frac{w_1}{w_2}\right)(w_2t)}d\mu_{-q^{w_2}}(x_2)$$

$$=\sum_{n=0}^{\infty}\left(\frac{[2]_{q^{w_1}}}{[2]_q}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}q^{w_1j}\widetilde{E}_{n,q^{w_2},\zeta^{w_2}}\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.2.14$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.13) and (2.14), we have the following theorem.

Theorem 9. *Let w_1 and w_2 be odd positive integers. Then we have*

$$\begin{aligned}&[2]_{q^{w_2}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2j}\widetilde{E}_{n,q^{w_1},\zeta^{w_1}}\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n\\ =&[2]_{q^{w_1}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}q^{w_1j}\widetilde{E}_{n,q^{w_2},\zeta^{w_2}}\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n.\end{aligned} \quad (2.15)$$

Corollary 10. *Let w_1 and w_2 be odd positive integers. If $q \to 1$ and $\zeta = 1$, we have*

$$\sum_{j=0}^{w_1-1}(-1)^jE_n\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n=\sum_{j=0}^{w_2-1}(-1)^jE_n\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n.$$

Substituting $w_1 = 1$ into (2.15), we arrive at the following corollary.

Corollary 11. *Let w_2 be odd positive integer. Then we obtain*

$$\widetilde{E}_{n,q,\zeta}(x) = \frac{[2]_q}{[2]_{q^{w_2}}} \sum_{j=0}^{w_2-1} (-1)^j \zeta^j q^j \widetilde{E}_{n,q^{w_2},\zeta^{w_2}} \left(\frac{x+j}{w_2} \right) w_2^n.$$

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