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INCREASING AND DECREASING OF AREAS

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Abstract: This article explains how to help junior-high school students enjoy studying geometry using some card magic. Area may be increased and decreased by arranging pieces of card according to the images on either the front or back of the card. There are some interesting stories associated with this puzzle, including 'Sunflowers Facing the Sun' and 'The UFO-Spotting Brothers.'

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1. The Disappearing Square Piece

It's growing close to 20 years now, but there's a mathematical puzzle I still like to introduce to students every year during seminar time. I am very familiar with the puzzle, so for me it lacks freshness, but students who come upon it for the first time seem impressed all the same. The puzzle that I'll now introduce presents a surprising phenomenon that also makes one want to try to make something similar oneself, as well as try to unlock the secret of the trick mathematically. As a teaching resource for mathematical education it thus kills three birds with just one stone.

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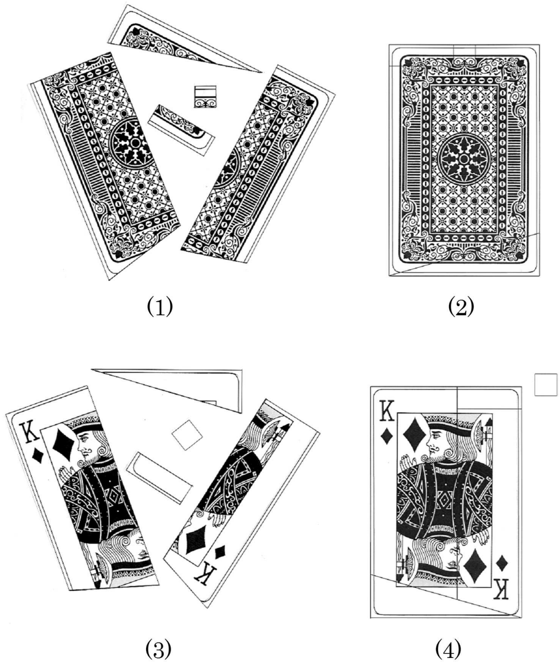
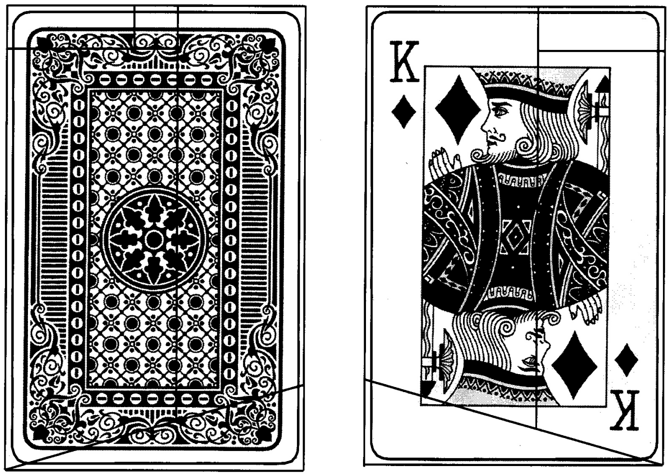


Figure 1: The disappearing square piece

First, allow me to introduce this puzzle I first came across in a magazine. The sliced up playing card on the desk shown in Figure 1(1) is face down. With the help of a student, have these pieces arranged so as to reform the pattern, just as if it were a jigsaw, as shown in Figure 1(2). There are 5 pieces in total: two trapezoids, a right-angled triangle, a rectangle, and one square. Since there are only a few pieces the reverse side is soon completed. After confirming that they are all back together, have the pieces mixed up like at the beginning (1), while keeping them face down.

Then, keeping the 5 pieces where they are, have them turned face up as shown in diagram (3). At this point it is clear that the card was the king of diamonds. Have the pieces arranged together just as before. The design on the front of the card is reformed, but the square piece is left over (4). The puzzle is therefore to work out how this piece can be left over.



(1) Back (2) Front

Figure 2: A playing card for constructing the puzzle

2. Making the Puzzle

When they see this phenomenon, most students are surprised. At that point I explain that it is neither an illusion nor magic, so there is no secret trickery or device involved, everything is open, and that I'd like them to think about an explanation. Next I have them try to make something similar. The students are given a print out resembling the design on a playing card as shown in Figure 2, and some card or thick drawing paper onto which they can glue the design. The image of the playing card used is a copy blown up to an appropriate size. A height of 15 cm with a width of 10 cm should be suitable.

Using the design from a playing card is very convenient during construction. First, place the image from the back side of a playing card on the card (Figure 3). Then mark the grid points on the vertical line with a compass needle, or by pressing down hard with a ball point pen. There are 11 grid points in total. After the marks have been made, remove the design and draw a line joining the marked points. The card is now ready (see the right hand side of Figure 3).

The thin paper playing card shown in Figure 2 (front and back) used for the construction, and the thick card prepared according to Figure 3 are cut out using a pair of scissors. The number of pieces cut is $5 \times 2 + 4 = 14$. These pieces are to be glued onto the card, but since the thin paper is prone to stretch, it seems to be best to apply the glue to the card. The glued paper may also have

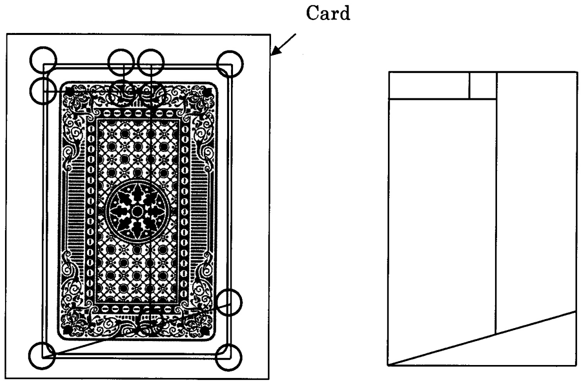


Figure 3: The 11 grid points

a tendency to bend, so it should be kept flat. So now we have the playing card for the puzzle.

3. Thinking about the Explanation

Once the glue has dried and the card is ready to use, group the students into pairs, and have them perform the puzzle using an assistant in the manner that I described earlier. One of the nice things about this puzzle is that anyone can be allowed to assemble the pattern, so any suspicion of hidden trickery or devices is dispelled.

After two or three attempts using trial-and-error, a student typically cries out “I’ve got it!” If the correctly assembled design on the back (Figure 1(2)) is flipped over without adjusting the places of the pieces, the design on the front is not assembled. This may be called an explanation. For me, this is saying no more than restating the phenomenon itself. The trapezoids on the left and right are certainly swapped. So why does swapping the pieces increase or decrease the area? I ask students to think about this, and at this point they become very quiet.

University students in the humanities enter university without doing much mathematics. After explaining that a knowledge of junior high-school geometry is sufficient for an understanding of this mathematical puzzle, I give a hint by a drawing diagram of the dimensions like that shown in Figure 4 on a whiteboard, in order to help students think.

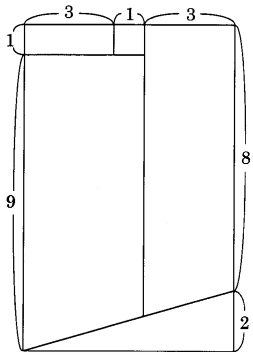


Figure 4: The dimensions on the back of the card

These are the dimensions of the back of a playing card. It is sufficient to think about a playing card with a height of 10 cm and a width of 7 cm. The lengths of the bases of the two trapezoids are 9 cm and 8 cm. The height of the rectangle is 1 cm and its width is 3 cm. The base of the right-angled triangle is 7 cm, and its height is 2 cm. There is an edge in Figure 4 which doesn't have a clearly marked length, and calculating it is the key to solving this problem. Thinking is important for puzzles, so I'll show an example calculation last of all. Please attempt this challenge yourself.

4. 'Sunflowers Facing the Sun'

I first learned of this mathematical puzzle in 1987, which is now 20 years ago. I was watching an NHK television when some footage transmitted by the UK's BBC was broadcast. It was a program called 'Paul Daniel's Magic Show'. Somehow, while I was watching the television I was completely drawn in by this puzzle. I remember hurriedly pressing the record button on the video and saving the footage.

Around that time, there was a page in the Sunday edition of the *Asahi* newspaper called 'The Natural History of Puzzles', by Izuo Sakane, and this puzzle was taken up in this entertaining series of articles on toys and puzzles. In the *Asahi* newspaper they didn't use a design from a playing card, but instead a work by Hiroshi Kondo called 'Sunflowers Facing the Sun' (see Sakane, 1986) [1]. This is shown in Figure 5, and has the same number of pieces as the playing card, *i.e.,* 5. There are sunflowers drawn on the two trapezoid pieces, and a

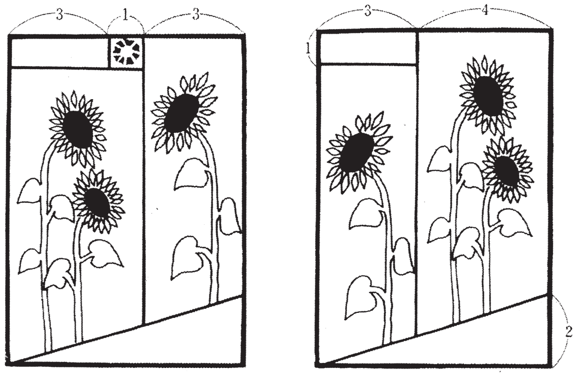


Figure 5: 'Sunflowers facing the sun' by H. Kondo (see Sakane, 1986) [1]

sun on the square. On the left hand side of Figure 5, while the sun is out the sunflowers are facing the sun, but when the sun sets (the square is removed) the sunflowers face where they like, as shown on the right hand side.

The designs on playing cards have a sense of narrative, making the puzzle rather entertaining. Another design was also introduced by the teachers at a high-school mathematics education assembly. This was known as 'The UFO-Spotting Brothers'. The two friendly brothers are gazing at the night sky. The younger brother points and shouts "Look, a UFO!" Next the elder brother looks in the direction of the UFO, but the UFO has suddenly vanished. That is, when the piece with the UFO (the square) is removed, the brothers say to each other, "Where did the UFO go?", and search the sky in different directions.

Isn't this rather nice and romantic? How about devising your own story? Anyway, for both 'Sunflowers Facing the Sun' and 'The UFO-Spotting Brothers', when the design is face up, the trapezoids on the left and right are exchanged. In the case of the playing card, turning over the whole thing reveals that the left and right parts are swapped. When solving this problem from a mathematical perspective all three cases are identical.

5. The Magic Conditions

So, has the hint regarding the dimensions shown in Figure 4 helped you realize the reason why the square piece has disappeared? The reason becomes apparent when the lengths of all the dimensions besides that hinted are calculated.

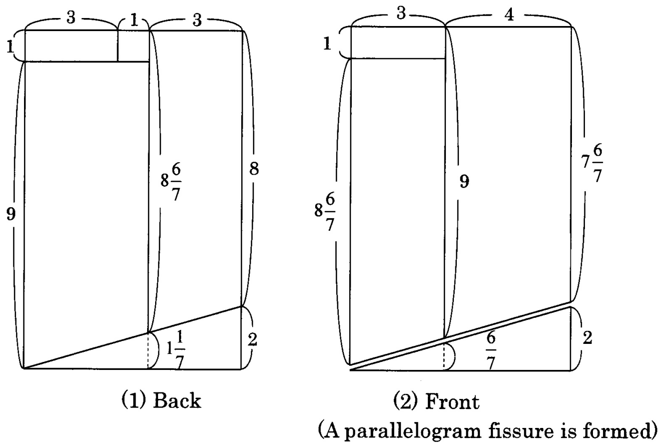


Figure 6: All dimensions of this puzzle

Speaking in terms of the result, the lengths concealed when the playing card is face down are $8\frac{6}{7}$ and $1\frac{1}{7}$ in the vertical direction (Figure 6(1)). Performing the calculation by focusing on the right-angled triangle, and using similar distances, $1\frac{1}{7}$ is first obtained. Junior high-school level geometry is sufficient, so everyone should confirm this for themselves. Summing these up, the result is 10, and the lengths on both the left and right ends are equal. Carefully exchanging the two trapezoids on the left and right produces the front surface of the playing card (Figure 6(2)). 'Carefully' here means taking care not to move the right-angled triangle and the rectangle, which are not repositioned.

By exchanging the trapezoids on the right and left, the width of 7 cm before the repositioning is not changed, but the height is slightly reduced. The 10 cm before the repositioning is slightly reduced to $9\frac{6}{7}$ cm. It's only $\frac{1}{7}$ cm, so no one notices. By the same token, the gradients of the right-angled triangle and the trapezoids are the same, so by trying to place these lines together, the fissure is naturally buried away.

Let's calculate the variation in area. The width and height of the square are 1 cm, so it has an area of $1 \times 1 = 1 \ \ cm^2$. Looking carefully at the fissure, it is a long thin parallelogram. Let's look at the parallelogram outlined in red, in order to emphasize the fissure. This reveals that the parallelogram really is long and thin, with a base of $\frac{1}{7}$ cm, and a height of 7 cm. Since the area of a

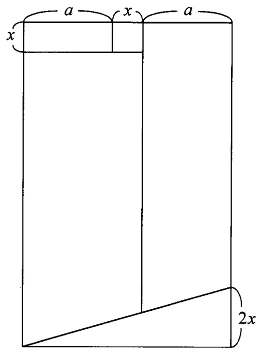


Figure 7: The conditions for magic

parallelogram is equal to its base width × height, the area is $\frac{1}{7} \times 7 = 1\ cm^2$.

We can thus see that the areas of the square piece and the parallelogram are equal. The square piece was reshaped into the parallelogram. In the natural world there is a law dictating the conservation of energy, and there is no such thing as a perpetual motion machine that produces energy eternally. In just the same way, area simply cannot be increased and decreased.

The dimensions of this puzzle were shown in Figure 4, but it's not necessary to construct the puzzle using these dimensions. The conditions of this magic formation are shown as parameters in Figure 7. If the length of the square's edge is x cm, then it is suitable for the rectangle to have a height of x cm, a width of a cm, and for the height of the right-angled triangle to be $2x$ cm. The vertical length of the playing card is not magically related. With these dimensions a parallelogram fissure with a base of $x^2/(2a + x)$ cm and a height of $2a + x$ cm is created. If $x \to 0$ in this diagram, the two trapezoids become congruent rectangles, and the whole thing fits together correctly.

And finally, there is one more piece of ingenuity to this piece of magic. It is in the shape of the card. Real playing cards have their corners cut into round curves. However, if the corners are cut then the trick will be given away when the card is flipped over with the left and right pieces exchanged. The card is shaped with square corners, but the design itself on the playing card expresses the curvature of the corners. I think the readers who have made it this far are potential magicians...

References

[1] I. Sakane, *Shin Asobino Hakubutushi* [New Natural History of Puzzles], Tokyo: Asahi Newspaper (Includes a description of [Sunflowers Facing the Sun] by H. Kondo) (1986).