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HERMITE-HADAMARD-LIKE AND SIMPSON-LIKE TYPE INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE STRONGLY φ-CONVEX FUNCTIONS

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Abstract: In this article, several inequalities of the Hermite-Hadamard-like and Simpson-like type integral inequalities are obtained for the class of functions whose second derivatives in absolute value at certain powers are strongly φ-convex with modulus $c > 0$.

AMS Subject Classification: 26D10, 26A51
Key Words: Hermite-Hadamard inequality, Simpson's inequality, φ-convexity

1. Introduction

The following double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions: For a convex function $f : I \subset R \to R$ defined on the interval I of real numbers and $a, b \in I$ with $a < b$, the following double inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This inequality plays an important role in convex analysis and it has a huge lit-

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erature dealing with its applications, various generalizations and refinements [1, 2, 3, 4, 6, 9, 10, 11, 14].

In the sequel, let us consider a function $\varphi : [a,b] \subset R \to [a,b]$.

In [15], Youness have defined the φ-convex functions as follows:

Definition 1. A function $f : [a,b] \to R$ is said to be φ-convex on $[a,b]$ if, for every $x, y \in [a,b]$ and $t \in [0,1]$, the following inequality holds:

$$f(t\varphi(x) + (1-t)\varphi(y)) \le tf(\varphi(x)) + (1-t)f(\varphi(y)).$$

Obviously, if $\varphi(x) = x$, then the classical convexity is obtained from the previous definition.

In [12], Polyak have defined the strongly convex functions with modulus $c > 0$ as follows:

Definition 2. A function $f : [a,b] \to R$ is said to be strongly convex with modulus $c > 0$, if for every $x, y \in [a,b]$ and $t \in (0,1)$ the following inequality holds:

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2.$$

The notion of strongly convex functions with modulus $c > 0$ plays an important role in optimization theory and mathematical economics. Various elementary properties and applications of them can be found in the literature see [7, 8, 12].

Let $(X, \| \,.\, \|)$ be a real normed space, D stands for a convex subset of X, $\varphi : D \to D$ is a given function and c is a positive constant. In [13], Sarikaya have introduced the notion of the strongly φ-convex functions with modulus c and some properties of them. Moreover in his article, Sarikaya have presented Hermite-Hadamard-type inequalities for strongly φ-convex functions as follows:

Definition 3. A function $f : D \to R$ is said to be strongly φ-convex with modulus $c > 0$, if, for every $x, y \in D$ and $t \in [0,1]$, the following inequality holds:

$$f(t\varphi(x) + (1-t)\varphi(y)) \le tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t) \parallel \varphi(x) - \varphi(y) \parallel^2 .$$

The notion of φ-convex function corresponds to the case $c = 0$. If φ is an identity function, then the strongly convex function with modulus $c > 0$ is obtained from the previous definition.

Theorem 1.1. *If $f : [a,b] \to R$ is a strongly φ-convex function with modulus $c > 0$ for a continuous function $\varphi : [a,b] \to [a,b]$, then the following* *double inequality holds:*

$$f(\frac{\varphi(a)+\varphi(b)}{2})+\frac{c}{12}(\varphi(b)-\varphi(a))^2$$
$$\leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx$$
$$\leq \frac{f(\varphi(a))+f(\varphi(b))}{2}+\frac{c}{6}(\varphi(b)-\varphi(a))^2.$$

In this article, the main aim is to establish several new inequalities of the Hermite-Hadamard-like and Simpson-like type integral inequalities are obtained for the class of functions whose second derivatives in absolute value at certain powers are strongly φ-convex with modulus $c>0$.

2. Main Results

To prove our main results, we need the following lemma [13]:

Lemma 1. Let $f: I\subset R\to R$ be a twice differentiable mapping on I^0, the interior of an interval I, such that $f'' \in L_1[a,b]$, where $a,b\in I$ with $a<b$, and $\varphi:[a,b]\to[a,b]$ a continuous function with $\varphi(a)<\varphi(b)$. Then for $r\geq 2$ the following identities hold:

$$(a) H_{\varphi(a)}^{\varphi(b)}(f)(r)$$
$$=^{\text{put}} (\frac{1}{2}-\frac{1}{r})(\frac{f(\varphi(a))+f(\varphi(b))}{2})$$
$$+(\frac{1}{2}+\frac{1}{r})f(\frac{\varphi(a)+\varphi(b)}{2})-\frac{1}{b-a}\int_{\varphi(a)}^{\varphi(b)} f(x)dx$$
$$=\frac{(b-a)^2}{16}\int_0^1 (1-t)(t-\frac{2}{r})\Big\{f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)$$
$$+f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\Big\}dt, \tag{1}$$

$$(b) R_{\varphi(a)}^{\varphi(b)}(f)(r)$$
$$=^{\text{put}} (\frac{1}{2}+\frac{1}{r})(\frac{f(\varphi(a))+f(\varphi(b))}{2})$$

$$+(\frac{1}{2}-\frac{1}{r})f(\frac{\varphi(a)+\varphi(b)}{2})-\frac{1}{b-a}\int_{\varphi(a)}^{\varphi(b)}f(x)dx$$
$$=\frac{(b-a)^2}{16}\int_0^1(1-t)(t+\frac{2}{r})\Big\{f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)$$
$$+f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\Big\}dt. \quad (2)$$

Proof. By using partial integrations the proofs are obvious.

The next theorem gives Hermite-Hadamard inequality and Simpson inequality for strongly φ-convex functions with modulus c as follows:

Theorem 2.1. *Let $f : I \subset R \to R$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $a,b \in I$ with $a < b$. If $|f''|^q$ is a strongly φ-convex function with modulus c on $[a,b]$ for a continuous function $\varphi : [a,b] \to [a,b]$ with $\varphi(a) < \varphi(b)$. Then for $r \geq 2$ the following inequalities hold:*

$$(a)|H_{\varphi(a)}^{\varphi(b)}(f)(r)|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{16}$$
$$\times\Big\{\mu_{01}(|f''(\varphi(a))|+|f''(\varphi(b))|)-c\mu_{02}(\varphi(b)-\varphi(a))^2\Big\}, \quad (3)$$

where

$$\mu_{01}=\frac{r^3-6r^2+24r-16}{6r^3},$$
$$\mu_{02}=\frac{7r^5-50r^4+240r^3-160r^2-160r+192}{120r^5}.$$

$$(b)|R_{\varphi(a)}^{\varphi(b)}(f)(r)|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{16}$$
$$\times\Big\{\nu_{01}(|f''(\varphi(a))|+|f''(\varphi(b))|)-c\nu_{02}(\varphi(b)-\varphi(a))^2\Big\}, \quad (4)$$

where

$$\nu_{01}=\frac{r+6}{6r}, \qquad \nu_{02}=\frac{7r+50}{120r}.$$

Proof. By using the strongly φ-convex functions with modulus c of $| f'' |$, we have

$$
\begin{aligned}
(i)\big|f''\big(\big(\frac{1+t}{2}\big)\varphi(a)+\big(\frac{1-t}{2}\big)\varphi(b)\big)\big|
&\le \big(\frac{1+t}{2}\big)\big|f''(\varphi(a))\big|+\big(\frac{1-t}{2}\big)\big|f''(\varphi(b))\big| \\
&\quad - c\big(\frac{1-t^2}{4}\big)\big(\varphi(b)-\varphi(a)\big)^2, \qquad (5)\\
(ii)\big|f''\big(\big(\frac{1-t}{2}\big)\varphi(a)+\big(\frac{1+t}{2}\big)\varphi(b)\big)\big|
&\le \big(\frac{1-t}{2}\big)\big|f''(\varphi(a))\big|+\big(\frac{1+t}{2}\big)\big|f''(\varphi(b))\big| \\
&\quad - c\big(\frac{1-t^2}{4}\big)\big(\varphi(b)-\varphi(a)\big)^2. \qquad (6)
\end{aligned}
$$

(a) From Lemma 1(1) and using (5) and (6), we

$$
\begin{aligned}
&|H_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\le \frac{(\varphi(b)-\varphi(a))^2}{16} \\
&\quad\times \Big[\Big\{\int_0^1 |\,(1-t)(t-\frac{2}{r})\,|\,dt\Big\}\big(\,|\,f''(\varphi(a))\,|+|\,f''(\varphi(b))\,|\,\big) \\
&\quad - \frac{c}{2}(\varphi(b)-\varphi(a))^2\Big\{\int_0^1 |\,(1-t)(t-\frac{2}{r})\,|\,(1-t^2)dt\Big\}\Big] \\
&= \frac{(\varphi(b)-\varphi(a))^2}{16} \\
&\quad\times \Big\{\mu_{01}(|\,f''(\varphi(a))\,|+|\,f''(\varphi(b))\,|)-c\mu_{02}(\varphi(b)-\varphi(a))^2\Big\}. \qquad (7)
\end{aligned}
$$

(b) From Lemma 1(2) and using (5) and (6), we

$$
\begin{aligned}
&|R_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\le \frac{(\varphi(b)-\varphi(a))^2}{16}\Big[\int_0^1 \big|(1-t)(t+\frac{2}{r})\big|\Big\{\big|f''\big(\big(\frac{1+t}{2}\big)\varphi(a) \\
&\quad + \big(\frac{1-t}{2}\big)\varphi(b)\big)\big| + \big|f''\big(\big(\frac{1-t}{2}\big)\varphi(a)+\big(\frac{1+t}{2}\big)\varphi(b)\big)\big|\Big\}dt\Big] \\
&\le \frac{(\varphi(b)-\varphi(a))^2}{16} \\
&\quad\times \Big\{\nu_{01}(|\,f''(\varphi(a))\,|+|\,f''(\varphi(b))\,|)-c\nu_{02}(\varphi(b)-\varphi(a))^2\Big\}. \qquad (8)
\end{aligned}
$$

Theorem 2.2. *Let $f : I \subset R \to R$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $| f'' |^q$ is a strongly φ-convex function with modulus c on $[a,b]$ for a continuous function $\varphi : [a,b] \to [a,b]$ with $\varphi(a) < \varphi(b)$ and $q > 1$. Then for $r \geq 2$ the following inequalities hold:*

$$\begin{aligned}
&(a) |H_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\quad \leq \frac{(\varphi(b)-\varphi(a))^2}{16} \mu_{10}^{\frac{1}{p}} \\
&\quad \times \Big[\Big\{ \mu_{11} \mid f''(\varphi(a)) \mid^q + \mu_{12} \mid f''(\varphi(b)) \mid^q - \mu_{13} c(\varphi(b)-\varphi(a))^2 \Big\}^{\frac{1}{q}} \\
&\quad + \Big\{ \mu_{12} \mid f''(\varphi(a)) \mid^q + \mu_{11} \mid f''(\varphi(b)) \mid^q - \mu_{13} c(\varphi(b)-\varphi(a))^2 \Big\}^{\frac{1}{q}} \Big],
\end{aligned}$$

where

$$\begin{aligned}
\mu_{10} &= \frac{r^3 - 6r^2 + 24r - 16}{6r^3}, \\
\mu_{11} &= \frac{3r^4 - 16r^3 + 48r^2 - 32}{24r^4}, \\
\mu_{12} &= \frac{r^4 - 8r^3 + 48r^2 - 64r + 32}{24r^4}, \\
\mu_{13} &= \frac{7r^5 - 50r^4 + 240r^3 - 160r^2 - 160r + 192}{240r^5}.
\end{aligned}$$

$$\begin{aligned}
&(b) |R_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\quad \leq \frac{(\varphi(b)-\varphi(a))^2}{16} \nu_0^{\frac{1}{p}} \\
&\quad \times \Big[\Big\{ \nu_1 \mid f''(\varphi(a)) \mid^q + \nu_2 \mid f''(\varphi(b)) \mid^q - \nu_3 c(\varphi(b)-\varphi(a))^2 \Big\}^{\frac{1}{q}} \\
&\quad + \Big\{ \nu_2 \mid f''(\varphi(a)) \mid^q + \nu_1 \mid f''(\varphi(b)) \mid^q - \nu_3 c(\varphi(b)-\varphi(a))^2 \Big\}^{\frac{1}{q}} \Big],
\end{aligned}$$

where

$$\begin{aligned}
\nu_{10} &= \frac{r^3 - 6r^2 + 24r - 16}{6r^3}, \quad \nu_{11} = \frac{3r + 16}{24r}, \\
\nu_{12} &= \frac{r + 8}{24r}, \qquad \nu_{13} = \frac{7r + 50}{120r}.
\end{aligned}$$

Proof. By using the strongly φ-convex functions with modulus c of $|\ f^{\prime\prime}\ |^q$, we know that for every $t \in [0,1]$

$$\begin{aligned}(i)\big|f^{\prime\prime}\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q\\ \leq \big(\frac{1+t}{2}\big)\big|f^{\prime\prime}(\varphi(a))\big|^q+\big(\frac{1-t}{2}\big)\big|f^{\prime\prime}(\varphi(b))\big|^q\\ -c\big(\frac{1-t^2}{4}\big)\big(\varphi(b)-\varphi(a)\big)^2,\end{aligned}\tag{9}$$

$$\begin{aligned}(ii)\big|f^{\prime\prime}\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q\\ \leq \big(\frac{1-t}{2}\big)\big|f^{\prime\prime}(\varphi(a))\big|^q+\big(\frac{1+t}{2}\big)\big|f^{\prime\prime}(\varphi(b))\big|^q\\ -c\big(\frac{1-t^2}{4}\big)\big(\varphi(b)-\varphi(a)\big)^2.\end{aligned}\tag{10}$$

(a) From Lemma 1(1) and using (9) and (10), we

$$\begin{aligned}&|H_{\varphi(a)}^{\varphi(b)}(f)(r)|\\ &\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big(\int_0^1\big|(1-t)(t-\frac{2}{r})\big|dt\Big)^{\frac{1}{p}}\\ &\times\Big[\Big\{\int_0^1\big|(1-t)(t-\frac{2}{r})\big|\big|f^{\prime\prime}\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\Big\}^{\frac{1}{q}}\\ &+\Big\{\int_0^1\big|(1-t)(t-\frac{2}{r})\big|\big|f^{\prime\prime}\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q dt\Big\}^{\frac{1}{q}}\Big].\end{aligned}\tag{11}$$

By (9) and (10), we have

$$(i)\int_0^1\big|(1-t)(t-\frac{2}{r})\big|dt=\frac{1}{6r^3}(r^3-6r^2+24r-16),$$

$$\begin{aligned}(ii)\int_0^1\big|(1-t)(t-\frac{2}{r})\big|\big|f^{\prime\prime}\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\\ \leq \mu_{11}\big|f^{\prime\prime}(\varphi(a))\big|^q+\mu_{12}\big|f^{\prime\prime}(\varphi(b))\big|^q-c\mu_{13}\big(\varphi(b)-\varphi(a)\big)^2,\end{aligned}\tag{12}$$

$$\begin{aligned}(iii)\int_0^1\big|(1-t)(t-\frac{2}{r})\big|\big|f^{\prime\prime}\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q dt\\ \leq \mu_{12}\big|f^{\prime\prime}(\varphi(a))\big|^q+\mu_{11}\big|f^{\prime\prime}(\varphi(b))\big|^q-c\mu_{13}\big(\varphi(b)-\varphi(a)\big)^2.\end{aligned}\tag{13}$$

By (12) and (13) in (11), the part (a) in this theorem is proved.
(b) From Lemma 1(2) and using (9) and (10), we

$$\begin{aligned}
&|R^{\varphi(b)}_{\varphi(a)}(f)(r)| \\
&\le \frac{(\varphi(b)-\varphi(a))^2}{16}\Big(\int_0^1 \big|(1-t)(t+\frac{2}{r})\big|dt\Big)^{\frac{1}{p}} \\
&\times \Big[\Big\{\int_0^1 \big|(1-t)(t+\frac{2}{r})\big|\big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\Big\}^{\frac{1}{q}} \\
&+\Big\{\int_0^1 \big|(1-t)(t+\frac{2}{r})\big|\big|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q dt\Big\}^{\frac{1}{q}}\Big]. \qquad (14)
\end{aligned}$$

By (9) and (10), we have

$$\begin{aligned}
&(i) \int_0^1 \big|(1-t)(t+\frac{2}{r})\big|\big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt \\
&\le \nu_{11}\big|f''(\varphi(a))\big|^q+\nu_{12}\big|f''(\varphi(b))\big|^q - c\nu_{13}\big(\varphi(b)-\varphi(a)\big)^2, \qquad (15)
\end{aligned}$$

$$\begin{aligned}
&(ii) \int_0^1 \big|(1-t)(t-\frac{2}{r})\big|\big|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q dt \\
&\le \nu_{12}\big|f''(\varphi(a))\big|^q+\nu_{11}\big|f''(\varphi(b))\big|^q - c\mu_{13}\big(\varphi(b)-\varphi(a)\big)^2. \qquad (16)
\end{aligned}$$

By (15) and (16) in (14), the part (b) in this theorem is proved.

Recall that the Gamma function Γ, the Beta function β, the incomplete Beta function β_x and the hypergeometric function ${}_2F_1(a,b;c:x)$ are respectively defined by

$$\begin{aligned}
&(a)\ \ \Gamma(a)=\int_0^\infty e^{-t}t^{a-1}dt, \quad a>0, \\
&(b)\ \ \beta(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt, \quad a,b>0, \\
&(c)\ \ {}_2F_1(a,b,c,x)=\Sigma_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, 0<x<1,
\end{aligned}$$

where the Pochhammer symbol $(z)_n$ of z is defined by

$$(z)_n=\begin{cases}1 & \text{if } \ n=0\\ z(z+1)\cdots(z+n-1) & \text{if } \ n>0,\end{cases}$$

where c is not $0,-1,-2,\cdots$.

Theorem 2.3. *Let $f : I \subset R \to R$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $| f'' |^q$ is a strongly φ-convex function with modulus c on $[a,b]$ for a continuous function $\varphi : [a,b] \to [a,b]$ with $\varphi(a) < \frac{\varphi(a)+\varphi(b)}{2} < \varphi(b)$ and $q > 1$. Then for $r \geq 2$ the following inequalities hold:*

$$
\begin{aligned}
(a)&|H_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{16} \\
&\times \Big[\mu_{21}^{\frac{1}{p}}\Big\{\frac{1}{r}\Big(| f''(\frac{\varphi(a)+\varphi(b)}{2}) |^q + | f''\big((\frac{r+2}{2r})\varphi(a) + (\frac{r-2}{2r})\varphi(b)\big) |^q \Big) \\
&\qquad - \frac{c}{3r^3}\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}} \\
&+ \mu_{22}^{\frac{1}{p}}\Big\{(\frac{r-2}{2r})\Big(| f''\big((\frac{r+2}{2r})\varphi(a) + (\frac{r-2}{2r})\varphi(b)\big) |^q + | f''(\varphi(a)) |^q \Big) \\
&\qquad - \frac{c}{3}(\frac{r-2}{2r})^3\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}} \\
&+ \mu_{21}^{\frac{1}{p}}\Big\{(\frac{1}{r})\Big(| f''\big((\frac{r-2}{2r})\varphi(a) + (\frac{r+2}{2r})\varphi(b)\big) |^q + | f''(\frac{\varphi(a)+\varphi(b)}{2}) |^q \Big) \\
&\qquad - \frac{c}{3r^3}\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}} \\
&+ \mu_{22}^{\frac{1}{p}}\Big\{(\frac{r-2}{2r})\Big(| f''\big((\frac{r-2}{2r})\varphi(a) + (\frac{r+2}{2r})\varphi(b)\big) |^q + | f''(\varphi(b)) |^q \Big) \\
&\qquad - \frac{c}{3}(\frac{r-2}{2r})^3\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}}\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{21} &= (\frac{2}{r})^{p+1}(\frac{1}{p+1}) \ \ {}_2F_1[1, -p, 2+p, \frac{2}{r}], \\
\mu_{22} &= (\frac{r-1}{r})^{2p+1}\beta(1+p, 1+p).
\end{aligned}
$$

$$
\begin{aligned}
(b)&|R_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\nu_{21}^{\frac{1}{p}} \\
&\times \Big[\Big\{\frac{1}{2}\Big(| f''(\varphi(a)) |^q + | f''(\frac{\varphi(a)+\varphi(b)}{2}) |^q - \frac{c}{24}(\varphi(b)-\varphi(a))^2\Big\}^{\frac{1}{q}}
\end{aligned}
$$

$$+\Big\{\frac{1}{2}\Big(\mid f''(\frac{\varphi(a)+\varphi(b)}{2})\mid^q+\mid f''(\varphi(b))\mid^q\Big)-\frac{c}{24}(\varphi(b)-\varphi(a))^2\Big\}^{\frac{1}{q}}\Big],$$

where

$$\nu_{21}=(\frac{2}{r})^p(\frac{1}{p+1})\ \ {}_2F_1[1,-p,2+p,-\frac{2}{r}].$$

Proof. (a) From Lemma 1(1) and using (9) and (10), we

$$\begin{aligned}
&|H^{\varphi(b)}_{\varphi(a)}(f)(r)|\\
&\le \frac{(\varphi(b)-\varphi(a))^2}{16}\Big[\int_0^1\big|(1-t)(t-\frac{2}{r})\big|\Big\{\big|f''\big((\frac{1+t}{2})\varphi(a)\\
&\quad+(\frac{1-t}{2})\varphi(b)\big)\big|+\big|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|\Big\}dt\Big]\\
&\le \frac{(\varphi(b)-\varphi(a))^2}{16}\Big[\mu_{21}^{\frac{1}{p}}\Big(\int_0^{\frac{2}{r}}\big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\mu_{22}^{\frac{1}{p}}\Big(\int_{\frac{2}{r}}^1\big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\mu_{21}^{\frac{1}{p}}\Big(\int_0^{\frac{2}{r}}\big|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\mu_{22}^{\frac{1}{p}}\Big(\int_{\frac{2}{r}}^1\big|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q dt\Big)^{\frac{1}{q}}\Big].
\end{aligned}\tag{17}$$

By Theorem 1.1, we have

$$\begin{aligned}
(i)\ &\int_0^{\frac{2}{r}}\big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\\
&\le\frac{1}{r}\Big\{\big|f''(\frac{\varphi(a)+\varphi(b)}{2})\big|^q+\big|f''\big((\frac{r+2}{2r})\varphi(a)+(\frac{r-2}{2r})\varphi(b)\big)\big|^q\Big\}\\
&\qquad-\frac{c}{3r^3}\big(\varphi(b)-\varphi(a)\big)^2,
\end{aligned}\tag{18}$$

$$\begin{aligned}
(ii)\ &\int_{\frac{2}{r}}^1\big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\\
&\le\frac{r-2}{2r}\Big\{\big|f''\big((\frac{r+2}{2r})\varphi(a)+(\frac{r-2}{2r})\varphi(b)\big)\big|^q+\big|f''(\varphi(a))\big|^q\Big\}\\
&\qquad-\frac{c}{3}(\frac{r-2}{2r})^3\big(\varphi(b)-\varphi(a)\big)^2,
\end{aligned}\tag{19}$$

$$(iii) \int_0^{\frac{2}{r}} \left|f''\left(\left(\frac{1-t}{2}\right)\varphi(a) + \left(\frac{1+t}{2}\right)\varphi(b)\right)\right|^q dt$$
$$\leq \frac{1}{r}\Big\{\left|f''\left(\left(\frac{r-2}{2r}\right)\varphi(a) + \left(\frac{r+2}{2r}\right)\varphi(b)\right)\right|^q + \left|f''\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|^q\Big\}$$
$$- \frac{c}{3r^3}\big(\varphi(b)-\varphi(a)\big)^2, \tag{20}$$

$$(iv) \int_{\frac{2}{r}}^1 \left|f''\left(\left(\frac{1-t}{2}\right)\varphi(a) + \left(\frac{1+t}{2}\right)\varphi(b)\right)\right|^q dt$$
$$\leq \frac{r-2}{2r}\Big\{\left|f''\left(\left(\frac{r-2}{2r}\right)\varphi(a) + \left(\frac{r+2}{2r}\right)\varphi(b)\right)\right|^q + \left|f''(\varphi(a))\right|^q\Big\}$$
$$- \frac{c}{3}\left(\frac{r-2}{2r}\right)^3\big(\varphi(b)-\varphi(a)\big)^2. \tag{21}$$

By (18)-(21) in (17), the part (a) in this theorem is proved.

(b) From Lemma 1(2) and using (9) and (10), we

$$|R_{\varphi(a)}^{\varphi(b)}(f)(r)|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big(\int_0^1 (1-t)^p\left(t+\frac{2}{r}\right)^p dt\Big)^{\frac{1}{p}}$$
$$\times \Big[\Big\{\int_0^1 \left|f''\left(\left(\frac{1+t}{2}\right)\varphi(a) + \left(\frac{1-t}{2}\right)\varphi(b)\right)\right|^q dt\Big\}^{\frac{1}{q}}$$
$$+ \Big\{\int_0^1 \left|f''\left(\left(\frac{1-t}{2}\right)\varphi(a) + \left(\frac{1+t}{2}\right)\varphi(b)\right)\right|^q dt\Big\}^{\frac{1}{q}}\Big]. \tag{22}$$

By Theorem 1.1, we have

$$(i) \int_0^1 (1-t)^p\left(t+\frac{2}{r}\right)^p dt = \left(\frac{2}{r}\right)^p \frac{1}{p+1} \; {}_2F_1\left[1,-p,2+p,-\frac{r}{2}\right], \tag{23}$$

$$(ii) \int_0^1 \left|f''\left(\left(\frac{1+t}{2}\right)\varphi(a) + \left(\frac{1-t}{2}\right)\varphi(b)\right)\right|^q dt$$
$$\leq \frac{1}{2}\Big\{\left|f''(\varphi(a))\right|^q + \left|f''\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|^q\Big\} - \frac{c}{24}\big(\varphi(b)-\varphi(a)\big)^2, \tag{24}$$

$$(iii) \int_0^1 \left|f''\left(\left(\frac{1-t}{2}\right)\varphi(a) + \left(\frac{1+t}{2}\right)\varphi(b)\right)\right|^q dt$$
$$\leq \frac{1}{2}\Big\{\left|f''\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|^q + \left|f''(\varphi(b))\right|^q\Big\} - \frac{c}{24}\big(\varphi(b)-\varphi(a)\big)^2 \tag{25}$$

By (23)-(25) in (22), the part (b) in this theorem is proved.

Theorem 2.4. *Let $f : I \subset R \to R$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $\mid f'' \mid^q$ is a strongly φ-convex function with modulus c on $[a,b]$ for a continuous function $\varphi : [a,b] \to [a,b]$ with $\varphi(a) < \varphi(b)$ and $q > 1$. Then for $r \geq 2$ the following inequalities hold:*

$$
\begin{aligned}
&(a)|H_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\quad \leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big\{\mu_{31}^{\frac{1}{p}} + \mu_{32}^{\frac{1}{p}}\Big\} \\
&\quad \times \Big[\Big\{\frac{3}{4} \mid f''(\varphi(a)) \mid^q + \frac{1}{2} \mid f''(\varphi(b)) \mid^q - \frac{c}{6}(\varphi(b)-\varphi(a))^2\Big\}^{\frac{1}{q}} \\
&\qquad + \Big\{\frac{1}{2} \mid f''(\varphi(a)) \mid^q + \frac{3}{4} \mid f''(\varphi(b)) \mid^q - \frac{c}{6}(\varphi(b)-\varphi(a))^2\Big\}^{\frac{1}{q}}\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{31} &= (\frac{2}{r})^{p+1}(\frac{1}{p+1}) \ \ {}_2F_1[1, -p, 2+p, \frac{2}{r}], \\
\mu_{32} &= (\frac{r-2}{r})^{2p+1}\beta(1+p, 1+p).
\end{aligned}
$$

$$
\begin{aligned}
&(b)|R_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\quad \leq \frac{(\varphi(b)-\varphi(a))^2}{16}\nu_{31}^{\frac{1}{p}} \\
&\quad \times \Big[\Big\{\frac{3}{4} \mid f''(\varphi(a)) \mid^q + \frac{1}{2} \mid f''(\varphi(b)) \mid^q - \frac{c}{6}(\varphi(b)-\varphi(a))^2\Big\}^{\frac{1}{q}} \\
&\qquad + \Big\{\frac{1}{2} \mid f''(\varphi(a)) \mid^q + \frac{3}{4} \mid f''(\varphi(b)) \mid^q - \frac{c}{6}(\varphi(b)-\varphi(a))^2\Big\}^{\frac{1}{q}}\Big],
\end{aligned}
$$

where

$$
\nu_{31} = (\frac{2}{r})^{p}(\frac{1}{p+1}) \ \ {}_2F_1[1, -p, 2+p, -\frac{2}{r}].
$$

Proof. (a) From Lemma 1(1) and using (9) and (10), we

$$
\begin{aligned}
&|H_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big(\int_0^1 \big|(1-t)(t-\frac{2}{r})\big|^p dt\Big)^{\frac{1}{p}}
\end{aligned}
$$

$$\times \Big[\Big\{\int_0^1 \big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\Big\}^{\frac{1}{q}}$$

$$+\Big\{\int_0^1 \big|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q dt\Big\}^{\frac{1}{q}}\Big]$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big\{\mu_{31}^{\frac{1}{p}}+\mu_{32}^{\frac{1}{p}}\Big\}$$
$$\times \Big[\Big\{\frac{3}{4}\mid f''(\varphi(a))\mid^q+\frac{1}{2}\mid f''(\varphi(b))\mid^q-\frac{c}{6}\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{1}{2}\mid f''(\varphi(a))\mid^q+\frac{3}{4}\mid f''(\varphi(b))\mid^q-\frac{c}{6}\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}}\Big]. \quad (26)$$

(b) From Lemma 1(2) and using (9) and (10), we

$$|R_{\varphi(a)}^{\varphi(b)}(f)(r)|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big[\int_0^1 \big|(1-t)(t+\frac{2}{r})\big|\Big\{\big|f''\big((\frac{1+t}{2})\varphi(a)$$
$$+(\frac{1-t}{2})\varphi(b)\big)\big|+\big|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|\Big\}dt\Big]$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big(\int_0^1 (1-t)^p(t+\frac{2}{r})^p dt\Big)^{\frac{1}{p}}$$
$$\times \Big[\Big\{\int_0^1 \big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt\Big\}^{\frac{1}{q}}$$
$$+\Big\{\int_0^1 \big|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\big|^q dt\Big\}^{\frac{1}{q}}\Big]. \quad (27)$$

Note that

$$(i) \int_0^1 (1-t)^p(t+\frac{2}{r})^p dt=(\frac{2}{r})^p\frac{1}{p+1}\;\; {}_2F_1[1,-p,2+p,-\frac{r}{2}], \quad (28)$$

$$(ii) \int_0^1 \big|f''\big((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b)\big)\big|^q dt \quad (29)$$

$$\leq \frac{3}{4}\big|f''(\varphi(a))\big|^q+\frac{1}{2}\big|f''(\varphi(b))\big|^q-\frac{c}{6}\big(\varphi(b)-\varphi(a)\big)^2, \quad (30)$$

$$(iii)\int_0^1 \left|f''\big((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b)\big)\right|^q dt$$
$$\leq \frac{1}{2}\left|f''(\varphi(a))\right|^q+\frac{3}{4}\left|f''(\varphi(b))\right|^q-\frac{c}{6}\big(\varphi(b)-\varphi(a)\big)^2. \tag{31}$$

By (28)-(30) in (27), the part (b) in this theorem is proved.

Theorem 2.5. *Let $f : I \subset R \to R$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $| f'' |^q$ is a strongly φ-convex function with modulus c on $[a,b]$ for a continuous function $\varphi : [a,b] \to [a,b]$ with $\varphi(a) < \varphi(b)$ and $q > 1$. Then for $r \geq 2$ the following inequalities hold:*

$$(a)|H_{\varphi(a)}^{\varphi(b)}(f)(r)|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big\{\frac{2^{p+1}+(r-2)^{p+1}}{r^{p+1}(p+1)}\Big\}^{\frac{1}{p}}$$
$$\times\Big[\Big\{(\frac{q+3}{2(q+1)(q+2)})\mid f''(\varphi(a))\mid^q+(\frac{1}{2(q+2)})\mid f''(\varphi(b))\mid^q$$
$$-\frac{c}{4}(\frac{q+4}{2(q+2)(q+3)})\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}}$$
$$+\Big\{(\frac{1}{2(q+2)})\mid f''(\varphi(a))\mid^q+(\frac{q+3}{2(q+1)(q+2)})\mid f''(\varphi(b))\mid^q$$
$$-\frac{c}{4}(\frac{q+4}{2(q+2)(q+3)})\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}}\Big].$$

$$(b)|R_{\varphi(a)}^{\varphi(b)}(f)(r)|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big\{\frac{(r+2)^{p+1}-2^{p+1}}{r^{p+1}(p+1)}\Big\}^{\frac{1}{p}}$$
$$\times\Big[\Big\{(\frac{q+3}{2(q+1)(q+2)})\mid f''(\varphi(a))\mid^q+(\frac{1}{2(q+2)})\mid f''(\varphi(b))\mid^q$$
$$-\frac{c}{4}(\frac{q+4}{2(q+2)(q+3)})\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}}$$
$$+\Big\{(\frac{1}{2(q+2)})\mid f''(\varphi(a))\mid^q+(\frac{q+3}{2(q+1)(q+2)})\mid f''(\varphi(b))\mid^q$$
$$-\frac{c}{4}(\frac{q+4}{2(q+2)(q+3)})\big(\varphi(b)-\varphi(a)\big)^2\Big\}^{\frac{1}{q}}\Big].$$

Proof. (a) From Lemma 1(1) and using (9) and (10), we

$$
\begin{aligned}
&|H_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big(\int_0^1 |t-\frac{2}{r}|^p dt\Big)^{\frac{1}{p}} \\
&\quad \times \Big[\Big\{\int_0^1 |1-t|^q |f''((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b))|^q dt\Big\}^{\frac{1}{q}} \\
&\quad\quad + \Big\{\int_0^1 |1-t|^q |f''((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b))|^q dt\Big\}^{\frac{1}{q}}\Big]. \qquad (32)
\end{aligned}
$$

Since $| f'' |^q$ is a strongly φ-convex function with modulus c on $[a, b]$ for a continuous function $\varphi : [a, b] \to [a, b]$ with $\varphi(a) < \varphi(b)$, by using the above inequality (32) we get the desired result.

(b) From Lemma 1(2) and using (9) and (10), we

$$
\begin{aligned}
&|H_{\varphi(a)}^{\varphi(b)}(f)(r)| \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{16}\Big(\int_0^1 |t+\frac{2}{r}|^p dt\Big)^{\frac{1}{p}} \\
&\quad \times \Big[\Big\{\int_0^1 |1-t|^q |f''((\frac{1+t}{2})\varphi(a)+(\frac{1-t}{2})\varphi(b))|^q dt\Big\}^{\frac{1}{q}} \\
&\quad\quad + \Big\{\int_0^1 |1-t|^q |f''((\frac{1-t}{2})\varphi(a)+(\frac{1+t}{2})\varphi(b))|^q dt\Big\}^{\frac{1}{q}}\Big]. \qquad (33)
\end{aligned}
$$

Since $| f'' |^q$ is a strongly φ-convex function with modulus c on $[a, b]$ for a continuous function $\varphi : [a, b] \to [a, b]$ with $\varphi(a) < \varphi(b)$, by using the above inequality (33) we get the desired result.

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