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ON A STONE-WEIERSTRASS TYPE THEOREM FOR $C(X)$

Liaqat Ali Khan[1 §], Asma Thabit Al-Hamidi[2]

[1]Department of Mathematics
King Abdulaziz University
Jeddah, SAUDI ARABIA
[2]Department of Mathematics
King Abdulaziz University, Girls Section
Jeddah, SAUDI ARABIA

Abstract: We give a simple and short proof of a weaker form of the Stone-Weierstrass theorem for $C(X)$. We also discuss some of its consequences regarding the characterization of closed and maximal ideals in $C(X)$.

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1. Introduction

Throughout this note, X denotes a compact Hausdorff space and $C(X) = C(X, \mathbb{C})$, the algebra of all complex-valued continuous functions on X. The *uniform topology* u on $C(X)$ is given by the sup norm

$$\|f\| = \sup_{x \in X} |f(x)|, f \in C(X).$$

A subset A of $C(X)$ is said to *separates the points of* X if, for any $x \neq y$ in X, there exists an $f \in A$ such that $f(x) \neq f(y)$; we say that A *does not vanish on* X if, for each $x \in X$, there exists some $g \in A$ such that $g(x) \neq 0$; A is said to be *self-adjoint* if, for each $g \in A$, its complex conjugate $\overline{g}$ also belongs to A; A is called an *ideal* in $C(X)$ if $fg \in A$ for all $f \in A$ and $g \in C(X)$. The u-closure of A in $C(X)$ is denoted by $\bar{A}^u$.

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§Correspondence author

The classical Weierstrass approximation theorem, proved in 1885 by K. Weierstrass, states that the set of all polynomials in $t \in [0,1]$ with real coefficients is uniformly dense in $C([0,1],\mathbb{R})$. In 1948, M. H. Stone [7] obtained the following far reaching generalization of this theorem (see Rudin [6], p. 152, Theorem 7.31).

Theorem 1.1. (Stone-Weierstrass) *Suppose a subset A of $C(X)$ satisfies the following conditions:*

(i) A is a self-adjoint subalgebra of $C(X)$;

(ii) A separates the point of X;

(iii) A does not vanish on X.

Then A is u-dense in $C(X)$.

Several mathematicians have later generalized it or given its new proofs. Most of the proofs given in the text books (see [2, 4, 5, 6]) depend on the following arguments: (1) the classical Weierstrass theorem (or its special case of uniformly approximating $f(t) = |t|$ on $[-1,1]$ by polynomials); (2) the closure of a subalgebra in $C(X)$ is a subalgebra; (3) the closure of a subalgebra in $C(X)$ is a sublattice. An elegent proof, due to Brosowski and Deutsch [1], is elementary in the sense that it does not use any of the above arguments but uses only some basic results of continuity and compactness and the Bernoulli's inequality. However, this proof does involve quite a few manipulations. In a note [3], the first author had considered a weaker form of the Stone-Weierstrass theorem and proved it by using a lemma on *partition of unity* [5] (whose proof depends on Urysohn's lemma). In the present note, we give a more elementary and direct proof of this result.

2. Main Results

Theorem 2.1. *Let X be a compact Hausdorff space. Suppose a subset A of $C(X)$ satisfies the following conditions:*

(a) A is a vector subspace of $C(X)$;

(b) A is closed under multiplication by all functions in $C(X)$ with values in $[0,1]$;

(c) A does not vanish on X.

Then A is u-dense in $C(X)$.

Remark. Note that the conditions (a) and (b) together are easily seen to be equivalent to the condition that A is an ideal in $C(X)$. Further, using the Urysohn's Lemma, the conditions (b) and (c) imply that A separates the points of X. [Let $x \neq y$ be in X. Choose a function $\varphi \in C(X)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 0$, and $\varphi(y) = 1$. By (c), there exists a $g \in A$ such that $g(y) \neq 0$. Let $f = \varphi g$. Then $f \in A$ and $f(x) \neq f(y)$.] The above theorem is thus a weaker form of the usual Stone-Weierstrass theorem.

Proof of Theorem 2.1. Let $f \in C(X)$, and let $\varepsilon > 0$. Put $U = \{x \in X : |f(x)| < \varepsilon\}$, an open set in X . Since A does not vanish on X, for each $x \in X$, choose $g_x \in A$ such that $g_x(x) \neq 0$. By continuity of g_x, there exists an open neighbourhood $N(x)$ of x such that $g_x(y) \neq 0$ for all $y \in N(x)$. Since $X \backslash U$, being a closed subbset of a compact space, is compact, its open cover $\{N(x) : x \in X \backslash U\}$ has a finite subcover $\{N(x_1), ..., N(x_k)\}$ (say). Put

$$g = g_{x_1}\overline{g_{x_1}} + ... + g_{x_k}\overline{g_{x_k}}.$$

Then $g \in A$ (since $g_{x_1} \in A$ and A is an ideal), $g \geq 0$ and $g \neq 0$ on $X \backslash U$. Further, $\mu = \inf\{g(y) : y \in X \backslash U\} > 0$. Then, for any integer $n \geq 1$, $(1 + ng)(x) \neq 0$ for all $x \in X$ and so $\frac{1}{1+ng} \in C(X)$. Choose an integer $m \geq 1$ such that

$$\frac{\|f\|}{1 + m\mu} < \varepsilon.$$

Define $h = \frac{mfg}{1+mg}$. Then $h \in A$ (since $g \in A$ and A is an ideal). Let $y \in X$. If $y \in U$, then

$$|h(y) - f(y)| = |f(y)||\frac{mg(y)}{1 + mg(y)} - 1| < \varepsilon;$$

if $y \in X \backslash U$, then

$$|h(y) - f(y)| = |f(y)|.\frac{1}{1 + mg(y)} \leq \|f\|.\frac{1}{1 + m\mu} < \varepsilon.$$

Hence $\|h - f\| \leq \varepsilon$, and consequently $f \in \bar{A}^u$. Thus A is u-dense in $C(X)$. □

We mention that the above proof is a modified form of the arguments used in ([4], Theorem 2, p. 52). It depends only on some basic properties of continuity and compactness, and the fact that a continuous real-valued function attains its infimum on a compact set. It also does not require A to be self-adjoint. Indeed, the proof is simple at the cost of loosing the generality of Theorem 1.1.

An equivalent and more useful version of Theorem 2.1 is as follows:

Theorem 2.2. *Let X be a compact Hausdorff space and A an ideal in $C(X)$, and let $f \in C(X)$. Then the following coditions are equivalent.*

(i) $f \in \bar{A}^u$.

(ii) For any $x \in X$, $g(x) = 0$ for all $g \in A$ implies that $f(x) = 0$.

Proof. (i) $\Rightarrow$ (ii). Suppose $f \in \bar{A}^u$. Then there exists a sequence $\{f_n\} \subseteq A$ such that $f_n \longrightarrow f$. Let $x \in X$ with $g(x) = 0$ for all $g \in A$. Then, in particular, $f_n(x) = 0$ for all $n \geq 1$; hence $f(x) = \lim_{n\to\infty} f_n(x) = 0$.

(ii) $\rightarrow$ (i). Suppose that (ii) holds, and let $\varepsilon > 0$. Put $U = \{x \in X : |f(x)| < \varepsilon\}$. For each $x \in X \backslash U$, $f(x) \neq 0$ and so , by (ii), there exists some $g_x \in A$ such that $g_x(x) \neq 0$. Now, proceeding exactly as in the proof of Theorem 2.1, there exists an $h \in A$ such that $\|h - f\| \leq \varepsilon$. Hence $f \in \bar{A}^u$, as required. □

The above two theorems have the major disadvantage that the classical Weierstrass theorem cannot be deduced from them. However, it has some other useful consequences. For instance, we obtain the following characterizations of closed and maximal ideals in $C(X)$ (see [2, 4]). We sketch its proof for the sake of completeness.

Corollary 2.3. *(i) There is a one-one correspondence between the u-closed ideals in $C(X)$ and the closed subsets of X. In particular, each maximal ideal in $C(X)$ is u-closed and there is one-one correspondence between the proper maximal ideals in $C(X)$ and the points of X.*

(ii) Each u-closed ideal in $C(X)$ is the intersection of all maximal ideals which contain it.

Proof. (i) For any $F \subseteq X$, we let $\mathcal{I}_F = \{f \in C(X) : f = 0 \text{ on } F\}$. If F is closed, then $\mathcal{I}_F$ is easily seen to be a u-closed ideal in $C(X)$. Now, let A be a u-closed ideal in $C(X)$. Put $F = \cap\{f^{-1}(0) : f \in A\}$, a closed set in X. Clearly, $A \subseteq \mathcal{I}_F$. If $f \in \mathcal{I}_F$, then, for any $x \in X$, $f(x) = 0$ whenever $g(x) = 0$ for all $g \in A$. So, by Theorem 2.2, $f \in \bar{A} = A$. Thus $A = \mathcal{I}_F$. Similarly, if A is a maximal ideal in $C(X)$, then $A = \mathcal{I}_{\{x\}}$ for some $x \in X$.

(ii) Let A be a u-closed ideal in $C(X)$, and let $g \in C(X)$ with $g \notin A$. We need to show that g does not belong to some maximal ideal containing A. By Theorem 2.2, there exists an $x_0 \in X$ such that $h(x_0) = 0$ for all $h \in A$. Clearly, $A \subseteq \mathcal{I}_{\{x_0\}}$, but $g \notin \mathcal{I}_{\{x_0\}}$ since $g(x_0) \neq 0$. □

The following classical example also serves as a counter-example in the setting of Theorem 2.1. This shows that, even if A is a subalgebra containing 1, the condition (b) of Theorem 2.1 cannot be dropped.

Example 2.4. Let X be the closed unit disc $\{z : |z| \leq 1\}$ in the complex plane, and let A be the set of all functions in $C(X)$ which are analytic on the interior, $\{z : |z| < 1\}$, of X. Then A satisfies the conditions (a) and (c), but not the condition (b) of Theorem 2.1 (since the product of a continuous function and an analytic function need not be analytic). By a standard result of *Complex Analysis*, A is u-closed in $C(X)$; so $A \neq C(X)$ since there are certainly non-analytic continuous functions on X.

Finally, we mention that, if X is a locally compact Hausdorff space, the above results remain true for $C(X)$ replaced by its subalgebra $C_0(X)$ consisting of those functions which *vanish at infinity* (see [2]).

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