International Journal of Pure and Applied Mathematics
Volume 80 No. 3 2012, 375-384
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

NUMERICAL PALINDROMES AND THE 196 PROBLEM

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Abstract: The numbers 727, 1991, 38483 and so on, are the same when read forwards or backwards. These numbers are known as numerical palindromes. Let's pick an arbitrary number, reorder the digits in reverse and add it to the original number. It is said that repeating this operation eventually leads to a palindrome. Numbers in the sequence derived from 196 however, do not yield a palindrome. This is an unresolved mathematical problem.

AMS Subject Classification: 11A02, 00A09, 97A20
Key Words: numerical palindromes, 196 problem, 196 and 879, seed

1. Numerical Palindromes

There is a certain concept referred to as a 'palindrome'. Palindromes are phrases that are identical when read both forwards and backwards. In Japanese, the phrases "TA KE YA BU YA KE TA" and "U TSU I KE N SHI HA SHI N KE I TSU U" are palindromes (meaning "The bamboo bush burnt" and "Mr. Ken Utsui has neuralgia" respectively). There are also some outstanding literary creations such as "NA KA KI YO NO TO O NO NE FU RI NO MI NA ME SA ME NA MI NO RI FU NE NO O TO NO YO KI KA NA" which takes the form of a traditional 31 syllable Japanese poem known as a tanka. It is written "長き夜の遠の眠りの皆目覚め波乗り船の音の良きかな", and means "For those who awaken after a long night of deep sleep, how comforting is the sound of a boat on the waves."

Received: June 5, 2012

Numbers can also be palindromes. For example, 727, 1991, 38483 and so on, are the same when read forwards or backwards. Numbers which are symmetrical in this way are known as numerical palindromes.

Now, let's pick an arbitrary number, reorder the digits in reverse and add it to the original number. It is said that repeating this operation eventually leads, at some point, to a palindrome. For example, suppose we pick 59.

$59 + 95 = 154, \quad 154 + 451 = 605, \quad 605 + 506 = 1111$

Let's try again with another number, 183.

$183 + 381 = 564, \quad 564 + 465 = 1029, \quad 1029 + 9201 = 10230,$

$10230 + 3201 = 13431$

After 3 repetitions of the operation the number 59 arrived at the palindrome 1111. After 4 repetitions the number 183 arrived at the palindrome 13431. Please try this for yourself with another number.

Almost all numbers arrive at a palindrome when this operation is repeated, but numbers in the sequence derived from 196 do not arrive at a palindrome. Also, for a given number, it is not known whether it will arrive at a palindrome. This problem is both old and new. It has been discussed in *Scientific American*, and although I wasn't particularly interested at that time, I developed an interest which prompted me to investigate it in this chapter.

2. All 2-Digit Numbers Lead to Palindromes

As a starter, let's try some 2-digit numbers. It can be confirmed that starting from any 2-digit number from 10 to 99 leads to a palindrome. The number 89 doesn't seem to yield a palindrome, but eventually after 24 iterations, the following 13-digit number is the first palindrome obtained.

8813200023188

Meticulously investigating every one of the 2-digit numbers to confirm that it yields a palindrome is not very mathematical. Paying attention to certain digits, and observing their combined sum reveals that the result is a palindrome if all the sums are less than or equal to 9. For example, for 35, $3+5=8$, which is less than 9 so it is not necessary to investigate further. There are 90 2-digit numbers, from 10 up to 99. These numbers can be written $ab (1 \leq a \leq 9, 0 \leq b \leq 9)$.

(1) When the units' digit is 0, the result is a palindrome. There are 9 such cases. ($a0$)

(2) When the units' digit and the tens' digits are the same, the number is already a palindrome. There are 9 such cases. (aa)

(3) When the units' digit and the tens' digit are symmetrical there is no need to investigate any further. There are 36 such cases. (ab, ba)

(4) When the sum of the units' digit and the tens' digit is less than or equal to 9 there is no carry to another digit, so the result is a palindrome. There are 16 such cases. $(a + b \leq 9)$

(5) When the sum of the units' digit and the tens' digit is less than or equal to 13, the result is a 3-digit number and may be denoted abc. Then each digit is less than or equal to 4, so in the next step it is certain to yield a palindrome. There are 14 such cases. $(10 \leq a + b \leq 13)$

(6) When the units' digit and the tens' digit sum to 14, the numbers may be 59 or 68. Both $59+95 = 154$, and $68+86 = 154$ so it is sufficient to investigate only one case. Likewise, when the sum is 15, the number may be 69 or 78 but $69 + 96 = 165$, and $78 + 87 = 165$ so again, it is sufficient to investigate only one case.

According to the results above, since $90 - 9 - 9 - 36 - 16 - 14 - 2 = 4$, it is sufficient to investigate only the following 4 cases.

$59, 69, 79, 89$

Drawing up a table of these observations reveals the following (Figure 1). The 4 cases mentioned above are located in the lower right-hand side of the table. The numbers that are difficult to turn into palindromes are those for which the units' digit and the tens' digit are close to 9. It is therefore predictable that it is difficult with the number 89.

10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Figure 1: Validating the 2-digit numbers

3. The 3-Digit Numbers 196 and 879

Next, let's consider 3-digit numbers. A similar method to that used for 2-digit numbers can be used for 3-digit numbers, but along with the increase in the number of digits, the degree of refinement gets worse. There are 900 numbers with 3-digits, from 100 to 999. At present the following 13 from among them are known to not yield palindromes.

$196, 295, 394, 493, 592, 689, 691, 788, 790, 879, 887, 978, 986$

The first of these numbers which do not yield palindromes is 196, so this issue is also known as the '196 problem'. I found this an interesting problem and began to investigate myself, but I needed the help of a computer program. Table 1 shows the number of iterations needed to convert the numbers into a palindrome.

Among the 900 cases, 90 of them are already palindromes so an investigation is not needed. Investigating the process by which the remaining 810 numbers yield palindromes, there are 213 which yield palindromes in 1 step, 281 in 2 steps, and 145 which require 3 steps. This reveals that there are many numbers which yield palindromes in an unexpectedly small number of steps. The slowest require 23 steps, and there are 7 such cases. There are also the 13 cases mentioned above which do not yield palindromes.

The 13 3-digit numbers that do not yield palindromes can be divided into two groups. The first group contains 196, 295, 394, 493, 592, 689, 691, 788, 790, 887 and 986, and the second group contains 879 and 978. These are shown in the schematic diagram in Figure 2. The numbers 196 and 879 are known as 'seeds', and it can be seen that all the other numbers can be represented by these seeds. The reason is because the number 691 is the reverse ordering of the number 196 so they are members of the same group ($196+691 = 691+196 = 887$). 295 and 592 yield the same number, 887, after one iteration so they are also in the same group ($295 + 592 = 196 + 691 = 887$).

I do not know whether these two groups are forever separate, or whether they eventually unify.

4. The Root of the 196 Problem

I wonder when the number 196 was first recognized as problematic with regard to numerical palindromes. Regarding literature in Japan, 'Asimov's Collection of General Knowledge' (see Asimov, trans. Hoshi, 1986) [2] contained an explanation about 196. The original document corresponding to the Japanese

Iterations	Frequency	Proportion
0	90	10.0%
1	213	23.7%
2	281	31.2%
3	145	16.1%
4	63	7.0%
5	31	3.4%
6	9	1.0%
7	17	1.9%
8	7	0.8%
10	2	0.2%
11	7	0.8%
14	2	0.2%
15	7	0.8%
17	4	0.4%
22	2	0.2%
23	7	0.8%
Over 100	13	1.4%
Total	900	100%

Table 1: The number of iterations required to reach a palindrome (3-digit numbers)

translation was published in 1979, so it is clear that the topic has been known about for quite some time (see Asimov, 1979).[1] Gardner and other authors also took up the topic a number of times in *Scientific American* around this time.

I was wondering whether it was Asimov who first mentioned it, and therefore investigated a little further. I wrote this manuscript while I was visiting Cambridge, so I was able to lay my hands on some precious documents in the library there. They revealed that Trigg had already taken up the problem of 196 in *Mathematics Magazine* in 1967 (see Trigg, 1967),[6] and that in addition, Lehmer had raised the issue in the *Sphinx* magazine published in Brussels in 1938 (see Lehmer, 1938).[5] He had performed 73 iterations without reaching a palindrome. It may be possible to trace back even further, but I could find no other existing documents.

The Japanese word for palindrome is *kaibun*. The word 'palindrome' originated in the 17th century, and it was adopted from Greek. The period when

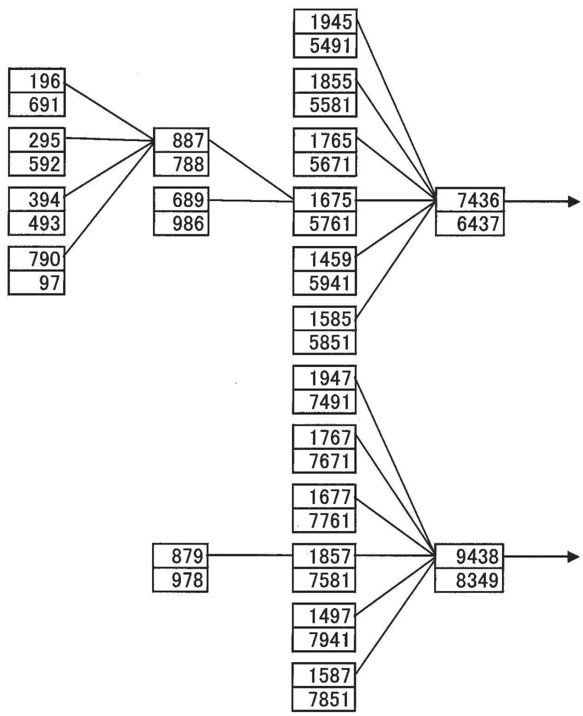

Figure 2: Schematic diagram showing the 13 unresolved cases (for 3-digit numbers)

Descartes and Newton were active was between the 17th and 18th centuries, and although it cannot be confirmed in the literature, it is possible that the problem of 196 was a topic of discussion among the mathematicians of this time. At any rate, in the 70 years or so since 1938 many mathematicians have pitted themselves against the problem of 196, and it is as yet a long-lived unsolved problem.

5. A World Record and Still Counting

Let's take a look at the records achieved by the mathematicians who have wrestled with this problem. In 1938 Lehmer calculated 73 iterations starting with 196 without reaching a palindrome, and obtained the following 35 digit number.

45747 6603920132 8565933091 8416673654

This was the highest record at that time. When I repeated the calculation using a Visual Basic program, I found the following slightly different number.

45747 6591819132 8565933092 7106673654

Speaking of 1938, it was a time when computers had not yet appeared, and printed on the back cover of the magazine there was an advertisement for a desktop calculator, *i.e.*, a cash register. It could handle up to 12 digits. The mathematicians of that time pitted themselves against the problem of 196 using tools that could only handle calculations up to 12 digits! In 1967 Trigg computed 3556 iterations yielding a 1700 digit number and confirmed that no palindrome was reached. He used the latest computer of the time, which was an IBM1401.

More recent data was provided by Walker in 1990, who computed 2,415,836 iterations, obtained a 1,000,000 digit number and confirmed the absence of a palindrome. At present 1,000,000 has been exceeded, and since the numbers are so long I will express them in terms of millions. In February 2006 Landingham completed 699 million iterations achieving a 289 million digit number without finding a palindrome, and the computation continues (see Landingham, 2006).[4] In order to show just how big a number this is, expressing it using an exponent reveals that although it is as large as $10^{289,430,478}$, yet there is still no palindrome. Since $289 \div 699 = 0.413 \cdots$, each calculation increases the number of digits by approximately 0.4, or roughly 1 digit for every 2 iterations.

The original version of Asimov's book was published in 1979, which was before a million iterations had been achieved. It is now almost 30 years since then. Technical innovation in computing has advanced, and the capabilities of home computers have improved, but despite reaching 289 million digits the problem remains unsolved.

The discussion above concerns the world record for which the number 196 has been confirmed as not yielding a palindrome, but there is another record. It is shown in Table 2, and is the largest number of iterations required by a number before it eventually *does* yield a palindrome.

This table is read as indicating that the 2-digit number 89 requires 24 iterations before it yields a palindrome, and that the 3-digit number 187 requires 23 iterations. Among the largest numbers of iterations so far obtained, the current world record is for the 17-digit number 10,442,000,392,399,900 which requires 236 iterations. The record is held by Doucette and was calculated in 2005 (see Doucette, 2005).[3]

Digits	Number	Number of iterations
2	89	24
3	187	23
4	1,297	21
5	10,911	55
6	150,296	64
7	9,008,299	96
8	10,309,988	95
9	140,669,390	98
10	1,005,499,526	109
11	10,087,799,570	149
12	100,001,987,765	143
13	1,600,005,969,190	188
14	14,104,229,999,995	182
15	100,120,849,299,260	201
16	1,030,020,097,997,900	197
17	10,442,000,392,399,900	236

Table 2: The largest number of iterations required to yield a palindrome (Doucette, 2005)

6. Will the Problem of 196 ever be Solved?

The world record is still open, but will the problem of 196 eventually be solved? At this point let's shift our perspective and consider what happens to the ratio of numbers that are palindromes when the number of digits increases. There are 9 1-digit numbers from 1 to 9, and these can all be regarded as palindromes. There are 90 2-digit numbers from 10 to 99, and there are 9 palindromes, 11, 22, 33, 44, 55, 66, 77, 88 and 99. The ratio of palindromes is $9/90 = 0.1$.

There are 900 3-digit numbers between 100 and 999, and the number of palindromes can be calculated as follows. The 3-digit palindromes all take one of the following forms

$$1x1, 2x2, 3x3, 4x4, 5x5, 6x6, 7x7, 8x8, 9x9$$

There are 10 digits from 0 to 9 that can be substituted for x, so the number of palindromes is $9 \times 10 = 90$. The ratio of palindromes is $90/900 = 0.1$.

For 4-digit numbers, there are 9000 between 1000 and 9999. Palindromes with 4-digits have one of the following forms.

$$1xx1, 2xx2, 3xx3, 4xx4, 5xx5, 6xx6, 7xx7, 8xx8, 9xx9$$

There are 10 ways of replacing with the digits 00 to 99, so the number of palindromes is $9 \times 10 = 90$. The ratio of palindromes is thus . The ratios can be computed in the same way for 5 and 6 digits, and the results are shown in Table 3.

Digits	Total	Palindromes	Ratio
2	90	9	0.1
3	900	90	0.1
4	9000	90	0.01
5	90000	900	0.01
6	900000	900	0.001

Table 3. The ratio of palindromes

In general, for $2n$ digits the ratio of palindromes is $\frac{1}{10^n}$. When the number of digits is $2n+1$, the ratio is the same as for $2n$. The ratio of palindromes can therefore be summarized as $\frac{1}{10^n}$ for both $2n$-digit (an even number of digits) and $2n+1$-digit (an odd number of digits) numbers.

Thus, as the number of digits increases, for every two new digits the proportion of palindromes decreases by a factor of 10. The number 196 has at present been taken as far as 289 million digits, and I suppose you can imagine just how small this makes the ratio of palindromes. However, no matter how small the ratio, palindromes do exist, and that's why it's such a frustrating problem for mathematicians.

The ratio of palindromes certainly grows extremely small, but before this happens there are many addition operations. There are 81 2-digit numbers that are not palindromes, but among them 49 (60%) yield a palindrome after one operation. These are good odds. For the 3-digit numbers, 810 of them are not palindromes, and 213 (26%) of these yield a palindrome after one operation. The proportion that yield a palindrome after one operation surely decreases as the number of digits increases, but compared to the ratio of palindromes discussed above it is large.

At present the calculations for 196 following on from 289 million digits are continuing on and on, but as to whether those calculations are drawing closer to a palindrome or whether they will continue forever as non-palindromes, it is unclear. Neither is it known through what states the numbers transition. The four-color problem was ultimately solved by resorting to the power of computers, but wouldn't it be nice if there were a more mathematical approach that did not require their use? I certainly hope those readers whose interest has been tempted by numerical palindromes and the problem of 196 would attempt the challenge of finding a proof.

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