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ON CONNECTIONS BETWEEN DOMINATING SETS AND TRANSVERSALS IN SIMPLE HYPERGRAPHS

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Abstract: This article focuses on characterizing the hypergraphs in which dominating sets are transversals.

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1. Introduction

The cardinality (or, size) of a finite nonempty set V is denoted by $\mid V \mid$. The set of all subsets (including the empty set ϕ) of V is denoted by 2^V which is called the *power set* [7] of V. The set of all nonempty subsets of V is denoted by 2^{V*}; that is, $2^{V*} = 2^V - \{\phi\}$.

Let E be a family of nonempty subsets of V. If $\bigcup_{X \in E} X = V$, we say E *fills out* V. A *hypergraph* [2] on V is a pair (or, couple) $H = (V, E)$ where V is a nonempty finite set and E is a family of nonempty subsets of V that fills out V. The set V is called the *vertex set* of H and each member of E is called a *hyperedge* of H. If the members of E are all distinct (that is, no two members are equal as subsets of V; or, $E \subseteq 2^{V*}$) then H is called *simple.* If no member of E is a subset (proper or otherwise) of another, then H is called a Sperner

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hypergraph. Some authors (instances: [2] and [3]) take Sperner hypergraphs to be simple and vice versa but there is distinction [6] between the two: Sperner hypergraphs are necessarily simple but not conversely. See 1.1 that follows.

Example 1.1. Let $H = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$, $E = \{X_1, X_2, X_3\}$ with $X_1 = \{2, 3\}$, $X_2 = \{2\}$ and $X_3 = \{1, 3, 4, 5\}$. H is simple because the three hyperedges are all distinct as subsets of V. But H is not Sperner because $X_2 \subset X_1$.

All the hypergraphs in the coming discussion are assumed simple unless there is some unambiguous indication to the contrary. The motivation for this research work comes principally from [4] that discusses transversal number and dominating number in simple hypergraphs in substantial detail.

2. Transversals and Dominating Sets

Let $H = (V, E)$ be a simple hypergraph and $T \in 2^{V*}$. Then T is called a *transversal* [2] in H iff $T \cap A \neq \phi$ for each $A \in E$ - which criterion is also rephrased as: T intersects every hyperedge of H. If T is a transversal and $T \neq V$ then T is called a *proper transversal* in H.

Two vertices x and y in V are said to be *adjacent* if there is a hyperedge that contains x and y; that is: $x, y \in A$ for some $A \in E$. Evidently every vertex is adjacent to itself. Let $D \in 2^{V*}$. Then D is called a *dominating set* [1] in H iff: (i) $D \neq V$, and (ii) each $x \in V - D$ is adjacent to some $y \in D$.

Proposition 2.1. *If T is a proper transversal in $H = (V, E)$, then T is a dominating set in H.*

Proof. Let $x \in V - T$. Let $A \in E$ be such that $x \in A$. Then $T \cap A \neq \phi$ since T is a transversal. Let $y \in T \cap A$. At once we have $x \neq y$ and x is adjacent to y, whence T is a dominating set.

Example 2.2. Not every dominating set in H is a transversal, though. Consider $H = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$, $E = \{X_1, X_2, X_3, X_4\}$ with $X_1 = \{1, 2\}$, $X_2 = \{2, 3, 4\}$, $X_3 = \{4, 5\}$ and $X_4 = \{3, 5\}$. Let $D = \{1, 5\}$. Each element of $V - D$ $(= \{2, 3, 4\})$ is adjacent to either 1 or 5, and so D is a dominating set in H. But D is not a transversal because $D \cap X_2 = \phi$.

Every hypergraph has a transversal - the vertex set is always one. But there are hypergraphs without dominating sets. When can a hypergraph have a dominating set?

Proposition 2.3. *$H = (V, E)$ has a dominating set if and only if $\mid X \mid\geq 2$ for some $X \in E$.*

Proof. Assume H has a dominating set, say D. Let $x \in V - D$ be given. Were $\mid X \mid = 1$ for every $X \in E$ then $\{x\}$ is the only hyperedge containing x, and so x is not adjacent to any member of D, contradicting the dominating nature of D.

Conversely, suppose $\mid X \mid\geq 2$ for some $X \in E$. Let $a, b \in X$ and $a \neq b$. Let $D = V - \{a\}$. Then (i) $D \neq V$, (ii) $b \in D$, (iii) $V - D = \{a\}$ and (iv) a is adjacent to b, whence D is a dominating set in H.

We are looking to characterize the hypergraphs with the property that every dominating set is a transversal. Section 3 deals with this, culminating in Proposition 3.4.

3. Dominating Sets in Trim Hypergraphs

This section is devoted to trim hypergraphs. Let $H = (V, E)$. A hyperedge X in H is called *redundant* in H (or, redundant in E) if there exists $S \subseteq E - \{X\}$ such that S covers X; that is, $X \subseteq \bigcup_{Y \in S} Y$. If $H = (V, E)$ has no redundant hyperedges then we call E a minimal hyperedge cover for H and we call H a *trim* hypergraph [5]. For a vertex $x \in V$, the number of hyperedges that contain x is defined to be the *degree* of x in H, and this number is denoted by $dx(H)$ or dx.

Proposition 3.1. *A hyperedge X is redundant in H if and only if no vertex of X is of degree 1. In other words, $H = (V, E)$ is trim if and only if each hypergdge has a vertex of degree 1.*

The proof of 3.1 is discussed in [5].

Proposition 3.2. *If H is trim then every dominating set in H is a transversal.*

Proof. Let D be a given dominating set in the trim hypergraph H, and let X be a given hyperedge in H. Then $dz = 1$ for some $z \in X$. If $z \in D$ then the conclusion follows at once. If $z \notin D$, then z is adjacent to some $y \in D$. Then $y \in X$, in view of $dz = 1$, whence $y \in D \cap X$.

Proposition 3.3. *If H is a Sperner hypergraph and if every dominating set in H is a transversal, then H is trim.*

Proof. Suppose H is not trim, and so let Y be a redundant hyperedge in H. Clearly $V-Y$ is nonempty. Let $D = V-Y$. Given $y \in Y$, there is a hyperedge X ($\neq Y$) such that $y \in X$. Since H is Sperner, there is $z \in X$ with $z \neq y$ and $z \in V-Y$. Then y is adjacent to z, and so D is a dominating set in H. But then D fails to be a transversal because $D \cap Y = \phi$.

Proposition 3.4. *Let H be Sperner. Then every dominating set in H is a transversal if and only if H is trim.* (3.4 is a consequence of 3.2 and 3.3.)

4. Summing Up

(i) In a simple hypergraph (not necessarily Sperner), each proper transversal is a dominating set (2.1), though not conversely (2.2), and

(ii) in a Sperner hypergraph, each dominating set is a transversal if and only if the hypergraph is trim (3.4).

Thus, it is precisely in the class of trim hypergraphs that every dominating set is a transversal. This is of theoretical interest at this point, and possibilities of applications are being studied.

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