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SICHERMAN DICE: EQUIVALENT SUMS WITH A PAIR OF DICE

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Abstract: George Sicherman discovered an interesting pair of dice whose sums have the same probability distribution as a pair of standard dice, and this was reported by Martin Gardner in 1978. This pair of dice is numbered, 1, 3, 4, 5, 6, 8 and 1, 2, 2, 3, 3, 4, and is unique. In order to prove the uniqueness of his combination three methods are shown: trial-and-error with pencil and paper, a Visual Basic program, and factorization of polynomials. The third is the most elegant solution and was presented by Gallian and Rusin, as well as Broline in 1979.

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1. One Particular Problem in Probability

I received a rather interesting mail from Dr. Steve Humble, an English acquaintance from the International Congress on Mathematical Education (ICME10)

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held in Denmark in the summer of 2004. It was regarding a problem in probability which involves finding the sum of the numbers given by two dice. The sums of the numbers on the dice are distributed from 2 to 12, and if we had a different pair of dice with the same probability distribution, they would be marked with the numbers

$$1,\ 3,\ 4,\ 5,\ 6,\ 8 \text{ and } 1,\ 2,\ 2,\ 3,\ 3,\ 4.$$

The problem in particular, was to prove that this is the unique solution (besides the pair of standard dice both marked 1, 2, 3, 4, 5, 6).

Problems involving the probability distributions of dice often come up in exams. For example, 'what is the probability that throwing two dice yields a sum which is even?', '...that the sum is a multiple of 3?', '...that the sum is greater than 5?', *etc.* These questions assume standard dice with the numbers from 1 to 6 on their faces. I read the mail from my acquaintance, thinking to myself that this was another such problem, and tried to confirm the calculation regarding the sums of the numbers on the faces.

Dice are cubes, so they have 6 faces. Taking two standard dice and writing out the frequency distribution of their sum yields the results shown in Figure 1. There are 6 different outcomes from the first die, and another 6 from the second die, giving a total set of 36 outcomes. The sum of 2 occurs 1 time, the sum of 3 occurs 2 times, ..., the sum of 7 occurs 6 times, ..., and the sum of 12 occurs 1 time. The sum of 7 has the most occurrences and the frequency distribution graph has a triangular shape which increases and decreases linearly.

Exchanging the numbers on the first and second dice with the numbers that I received by mail, 1, 3, 4, 5, 6, 8 and 1, 2, 2, 3, 3, 4, and computing the sum of the two dice using some spreadsheet software, I could determine the frequency distribution. What was the result? The sum of the numbers is between 2 and 12, and somehow the frequency distribution is the same as that of normal dice as was shown in Figure 1 (see Figure 2).

I was immediately impressed by this, but then wondered whether it is just the result of chance, and tried to think whether there were some other sets of dice with the same probability distribution. Since the sum of 2 occurs once with normal dice, it is immediately clear that the sum of each die's smallest value must be 2. Writing an x for the numbers which are not yet determined yields the following

$$1,\ x,\ x,\ x,\ x,\ x \quad \text{and} \quad 1,\ x,\ x,\ x,\ x,\ x$$

In addition, the largest sum of 12 also occurs only once, so adding up the

Die 2

Die 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure 1: Normal dice

Die 2

Die 1	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

Figure 2: Dice with the same probability distribution as normal dice

last numbers must result in 12. This results in the following cases.

$$1,\ x,\ x,\ x,\ x,\ 10 \ \text{ and } \ 1,\ x,\ x,\ x,\ x,\ 2$$

$$1,\ x,\ x,\ x,\ x,\ 9 \ \text{ and } \ 1,\ x,\ x,\ x,\ x,\ 3$$

$$1,\ x,\ x,\ x,\ x,\ 8 \ \text{ and } \ 1,\ x,\ x,\ x,\ x,\ 4$$

$$1,\ x,\ x,\ x,\ x,\ 7 \ \text{ and } \ 1,\ x,\ x,\ x,\ x,\ 5$$

It is now sufficient to patiently and mechanically investigate the remaining numbers marked x. Of the 4 x marks in the middle, the smallest value on the left cannot include 1, so it must be at least 2. Also, the large number on the right must not include the largest number so it must be less than this value. It is not therefore necessary to try and find a set of numbers at random, but rather, by thinking about the conditions and only selecting numbers that satisfy the conditions, the problem ceases to be a headache and the solution can be obtained with just pencil and paper.

So with just a little investigation it became clear that the first 2 of the 4 cases shown above were not possible. The numbers in the 4th case are good in one sense, but they are not perfect. For example, in the case of 1, 2, 4, 4, 6, 7 and 1, 2, 3, 3, 4, 5, the frequencies of the sums 5, 6, 8, 9 are out by 1. This means that any solution must exist in case 3, which is $1, x, x, x, x, 8$ and $1, x, x, x, x, 4$. This result was obtained using only pencil and paper.

Taking a look at the sets of numbers in the correct solution 1, 3, 4, 5, 6, 8 and 1, 2, 2, 3, 3, 4, their averages are 4.5 and 2.5 respectively. The sum of these averages is 7. The distribution of these numbers is also symmetric from left to right in the shape of a mountain. Standard dice have an average value of 3.5 and the sum of the averages is 7. Aren't the facts that sums of the average values are equal, and that the numbers are distributed in a well balanced way, essential conditions for the probability distributions to be the same?

While we might not be able to describe this is an elegant solution, we were able to find the dice with the same frequency distributions by ourselves. However, this was a trial-and-error method using paper and pencil, so it's possible that there could have been an oversight. Also, it's hard to say that the method of proof that I thought of is mathematical.

Was the existence of one more set of dice with the same probability distribution chance or necessity? The first round of this investigation ends with various questions left unanswered.

2. For Octahedral Dice, there are 3 Solutions

I didn't think I would want to write a program when I was abroad, so I didn't take my computer software with me. However, to check this problem I did, in fact, need software. This is because there is a high probability of making a mistake with a calculation by hand. I obtained Visual Basic cheaply through an internet auction, installed it, made a simple program and performed the check. If the program is assembled thoughtlessly it takes a lot of computation time and no result is generated. For example, for a single die, choosing the 6 numbers requires 6 loops, and since there are two dice this must be performed twice so there a total of 12 loops. It is then necessary to investigate the 6 possible occurrences of the numbers between 1 and 6. It is sufficient to investigate every single one of these values, but it's not a very good method.

Obtaining the paper and pencil solution before writing the program turned out to be a preliminary investigation for writing an efficient program. There are 6 numbers to decide for the dice, but the first number of 1 and the last number 'max' are fixed so it is sufficient to investigate only those 4 intermediate numbers. Also, the numbers are in ascending order from left to right, so it's not necessary to investigate all possible numbers. If attention is paid to these points the program can obtain a solution efficiently. Executing a program constructed in this way reveals the result that the only set of dice with the same probability distribution for the sum are those with the numbers 1, 3, 4, 5, 6, 8 and 1, 2, 2, 3, 3, 4. There were no mistakes in calculation due to arithmetic by hand, nor oversights.

Satisfied by the result of the program, my interest developed in a different direction. Was it merely a coincidence that 1, 3, 4, 5, 6, 8 and 1, 2, 2, 3, 3, 4 is the only solution? Dice are regular 6-faced bodies. The other regular polyhedra besides this are the tetrahedron with 4 faces, the octahedron with 8 faces, the dodecahedron with 12 faces, and the icosahedron with 20 faces. Wouldn't it be possible to set up similar probability problems by considering these other regular polyhedra as dice? With this in mind, I thought about the octahedron which has 2 extra faces. In the case of 'standard' octahedra, the numbers on the dice are

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \text{ and } 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,$$

so the sum is distributed between 2 and 16, and the sum with the highest frequency is 9, which occurs 8 times. Is there another set of numbers for the octahedron with the same probability distribution, or perhaps not? Maybe not only one solution can be found, but many? I tried to make a prediction. For cubes there is only one case which is a solution. Since the number of sets of resulting numbers is proportional to the product of the number of faces on the dice, with regular octahedra there are more than with the cube, and I wondered if more complicated cases might be found. That is to say, the range to search is not $6 \times 6 = 36$ cases but rather $8 \times 8 = 64$. Doesn't this increase the chances of finding a solution?

I completed a suitable program with just a few modifications to the program used to check the cubes. It was easy make a simple change increasing the number of faces to investigate from 6 to 8.

Running it revealed that my prediction was correct. The following 3 solutions were found for the case of regular octahedra,

$$1,\ 3,\ 3,\ 5,\ 5,\ 7,\ 7,\ 9 \text{ and } 1,\ 2,\ 2,\ 3,\ 5,\ 6,\ 6,\ 7,$$

$$1,\ 2,\ 5,\ 5,\ 6,\ 6,\ 9,\ 10 \text{ and } 1,\ 2,\ 3,\ 3,\ 4,\ 4,\ 5,\ 6,$$

$$1,\ 3,\ 5,\ 5,\ 7,\ 7,\ 9,\ 11 \text{ and } 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ 4,\ 5.$$

I suggest you check for yourself that these sets of numbers have the same probability distributions as normal octagonal dice. There was only 1 solution for the case of cubes, but in the case of octahedra there are 3. It may be thought that the likelihood of finding solutions is proportional to the product of the number of faces after all.

Among the regular polyhedra, there are also dodecahedrons (12 faces) and icosahedra (20 faces). I therefore tried to investigate dodecahedra. However, in order to investigate this using my program, the number of faces must be increased from 8 to 12, and this resulted in a massive increase in computing time. I realized that it was impossible to find a solution using this method and gave up. I found it out later, by means of another method I discovered, that there are 7 solutions for the case of the 12-faced dodecahedron. For the 20-faced icosahedron it is easy to predict that this will be increased even further. Compiling the results above, for the 6-faced cube there is 1 solution, for the 8-faced octahedron there are 3 solutions, and for the 12-faced dodecahedron there are 7 solutions. Plotting this data with the number of faces along the horizontal axis and the number of solutions on the vertical axis reveals that the shape is not linear but rather a 2nd or 3rd order curve.

Giving up on the regular polyhedron with the most faces, I instead investigated the 4-faced tetrahedron. It was unexpected, but I also found 1 solution for tetrahedra. It is the set

$$1,\ 3,\ 3,\ 5\ \text{ and }\ 1,\ 2,\ 2,\ 3.$$

So, I thought to myself, a solution with the same probability distribution exists for tetrahedra. I felt impressed and also satisfied, having achieved some positive results by checking with my Visual Basic program. This check using the program makes no slip-ups as might occur when calculating by hand, and is a convenient method of investigation. However, perhaps because I had a sense of having relied on something other than myself to solve the problem, I didn't feel like I had achieved the solution and I was left feeling somewhat vague.

3. Proof using Polynomials

I organised the results of my program, including the fact that with octagonal dice there are three sets of numbers with the same probability distribution, and mailed them to Dr. Steve Humble. Apparently he didn't already know these results. In reply to my mail he explained the following to me. Regarding a proof, this probability problem appears in an old work by Martin Gardner, which involves something known as a generating function. It has the following form.

$$P(x) = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6). \tag{1}$$

It is possible to make a proof using this formula.

Martin Gardner was a mathematics essayist who was active during the 1970s. He had many readers and was very influential. Some more senior readers may already know the conclusion of this problem. I am poor at probability so I have consciously avoided probability problems, but for some reason I found it interesting to take this one up. The proof is stated below, but it was established in the 1970s so there has been 30 years of progress up to the present day. The mode of expression has changed a little, but it is roughly the same as the original method, which is based on the use of a polynomial. Allow me to explain an outline of the proof.

The following point describes how to read Equation (1), since it is not the type of problem that involves substituting a value for x and obtaining the value of the equation. The proof uses the exponents and coefficients of the polynomial, so I'd like you to get used to paying attention to the exponents and coefficients. There are a total of 6 terms, and each has the form ax^k which is read as meaning that there are a cases when the sum of the face values is k. The exponent is the sum of the face values, and the coefficient corresponds to the frequency. Let's take a look at a concrete example. For a single die there is 1 case when the face value is 1, 1 case when the face value is 2, ..., and 1 case when the face value is 6. This yields $1x^1, 1x^2, \cdots, 1x^6$, *i.e.*, $x, x^2, \cdots, x^6$. Each of the cases occurs with the same probability, and since the overall sum of all the probabilities must be 1, it is divided by 6.

The case of two dice corresponds to squaring the left and right hand sides of Equation (1). Then, when the resulting polynomial is expanded, the exponents and coefficients express the sums of the face values and their frequencies, respectively. The following mathematical expression affirms the above. In order to improve the readability, both sides of Equation (1) are multiplied by 6, thus removing the denominators.

$$6P(x) = x + x^2 + x^3 + x^4 + x^5 + x^6, \tag{1$'$}$$

$$\{6P(x)\}^2 = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}. \tag{2}$$

Paying attention to Equation (2), it can be seen that the exponent and coefficient of every term in the expanded polynomial neatly expresses the relationship between the sums of the face values and their frequencies, and the probability distribution. For example, $5x^6$ is read as meaning that there are 5 cases when the sum of the face values is 6, and $4x^9$ means there are 4 cases when the sum is 9. What is needed to solve this problem is to identify what forms the right hand side of Equation (2) can take if it can be factorized. If the factorization takes the form of the polynomial in Equation (1), then doesn't this correspond to a normal die? This is not true. If it could be factorized as the product of different polynomials, then this constitutes a solution. But can this be done?

Let's return to Equation (1)' and try transforming it as follows. Each term includes the factor x, so first factorize by x and gather the remaining terms,

$(1+x+x^2+x^3+x^4+x^5)$. As is well known, multiplying $\sum_{i=0}^{k-1} x^i$ by $(x-1)$ yields (x^k-1), so let's apply this fact. Multiplying both the numerator and the denominator by $(x-1)$ does not change the value of the formula. In general, the form (x^k-1) is easy to factorize, and as a consequence many factors can be found, *e.g.,* (x^6-1) yields $(x-1)(x^2+x+1)(x+1)(x^2-x+1)$.

Putting the above facts together and writing out the result yields the following.

$$\begin{aligned}6P(x) =&x(x^5+x^4+x^3+x^2+x+1)\\ =&x(x^5+x^4+x^3+x^2+x+1)(x-1)/(x-1)\\ =&x(x^6-1)/(x-1)\\ =&x(x^3-1)(x^3+1)/(x-1)\\ =&x(x-1)(x^2+x+1)(x+1)(x^2-x+1)/(x-1)\\ =&x(x^2+x+1)(x+1)(x^2-x+1). \end{aligned} \tag{3}$$

As shown in Equation (3), it was possible to factorize $P(x)$ into a product of four terms. This is a form that one wouldn't imagine from looking at Equation (1). Simply exponentiating the left and right hand sides of Equation (3) results in the following form

$$\{6P(x)\}^2 = x^2(x^2+x+1)^2(x+1)^2(x^2-x+1)^2. \tag{4}$$

The point is to solve the problem by recomposing Equation (4) into two parts, but it cannot just be partitioned at random. After exponentiating, the total number of terms is 8. There are conditions on the partitioning, so let's investigate them. Returning to Equation (3) and investigating each term reveals firstly that each partition must contain x because this corresponds to the faces of the dice marked 1. Regarding the 3 remaining terms, examining the value in the case when $x=1$ reveals the following.

$$\begin{aligned}(x^2+x+1) =&3,\\ (x+1) =&2,\\ (x^2-x+1) =&1.\end{aligned}$$

The left hand side of $P(x)$ is multiplied by 6, so unless the right hand side also contains the factor 6 then the equation is not balanced. Because of this, the 3 from the term (x^2+x+1) , and the 2 from the term $(x+1)$ must each be included once. With $3 \times 2 = 6$, the equation is balanced with respect to the number 6, and passes this requirement. On the other hand, the value of $(x^2 - x + 1)$ is 1, so it has no effect, regardless of whether it is included or not. As for the way the $(x^2 - x + 1)^2$ terms are partitioned, if one is included in each die, this must be the same as the case of normal dice. If they are included on only one side, it results in dice that differ, which is the solution we were trying to obtain.

The result of this investigation is shown in Equation (5). The left hand side means 2 normal $P(x)$ dice, and the right hand side means a set of differing $Q(x)$ and $R(x)$ dice. Expanding the products of each polynomial, they are the same as the right hand side shown in Equation (2), *i.e.*, the probability distributions of the sums of the face values are the same.

$$\begin{aligned}\{6P(x)\}^2 =&\{6Q(x)\}\{6R(x)\}\\ =&\{x(x^2+x+1)(x+1)(x^2-x+1)^2\}\{x(x^2+x+1)(x+1)\}\\ =&(x^8+x^6+x^5+x^4+x^3+x)(x^4+2x^3+2x^2+x),\end{aligned} \tag{5}$$

The polynomials of the solution are:

$$Q(x) = \frac{1}{6}(x^8 + x^6 + x^5 + x^4 + x^3 + x), \tag{6}$$

$$R(x) = \frac{1}{6}(x^4 + 2x^3 + 2x^2 + x). \tag{7}$$

Since the exponents and coefficients of each of the terms in the polynomials express the sums of the face values and their frequencies, die Q has the numbers 8, 6, 5, 4, 3 and 1 on its faces, and die R has 4, 3, 3, 2, 2 and 1 on its faces.

These numbers are the same as those of the unique solution for cubes presented at the beginning. This method of proving the result by utilizing polynomial exponents and coefficients is nothing short of brilliant. The explanation presented dealt with the proof for cubes, but this method can be applied to the proofs for tetrahedra, octahedra, dodecahedra, and icosahedra. Because of the limits on computational speed when finding solutions using the Visual Basic program, it was not possible to obtain solutions above the dodecahedra, but it is possible using the polynomial method. The discovery of the 7 solutions in the case of the dodecahedron was also due to the polynomial method.

4. Sicherman Dice

Having achieved an understanding of the proof based on polynomials the matter is settled. Who thought of this interesting problem, and who realized there was such an elegant solution method? My interest shifted to searching for the roots of this problem and solution method.

As a result of various research, I was able to track a trail leading back to the 1970s. The problem's first appearance seems to have been in the February 1978 edition of Scientific American, which carried an article by Gardner on page 19 (see Gardner, 1978).[3] Reading this article reveals that the person who initially discovered these peculiar dice was George Sicherman. It is not certain whether or not he knew a proof. He probably just presented the fact that these interesting dice exist. Many letters concerning proofs arrived for Gardner from readers who saw this journal, and Gardner later wrote that the elegant solution method using a polynomial representation was owing to J.A. Gallian and D.M. Broline. Two relevant papers are given in the reference section (see Gallian and Rusin, 1979, and Broline, 1979).[2] [1]

These days the dice are known as 'crazy dice', or alternatively they take the name of their discoverer, *i.e.*, 'Sicherman dice'. Some Company (not Sicherman) sells these dice as merchandise, although it seems that they are not used in actual casinos. This means that while the issue of equivalent sums with a pair of dice is a fascinating topic for mathematicians, it is a different matter in reality!

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