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# COMPUTING CONSTANTS IN SOME WEIGHT SUBSPACES OF FREE ASSOCIATIVE COMPLEX ALGEBRA

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**Abstract:** Let $\mathcal{N} = \{i_1, i_2, \ldots, i_N\}$ be a fixed subset of nonnegative integers and let $q_{ij}$, $i, j \in \mathcal{N}$ be given complex numbers. We consider a free unital associative complex algebra $\mathcal{B}$ generated by $N$ generators $\{e_i\}_{i\in\mathcal{N}}$ (each of degree one) together with $N$ linear operators $\partial_i \colon \mathcal{B} \to \mathcal{B}$, $i \in \mathcal{N}$ that act as twisted derivations on $\mathcal{B}$. The algebra $\mathcal{B}$ is graded by total degree. More generally $\mathcal{B}$ could be considered as multigraded. Then it has a direct sum decomposition into multigraded (weight) subspaces $\mathcal{B}_Q$, where $Q$ runs over multisets (over $\mathcal{N}$). An element $C$ in $\mathcal{B}$ is called a *constant* if it is annihilated by all operators $\partial_i$. Then the fundamental problem is to describe the space $\mathcal{C}$ of all constants in algebra $\mathcal{B}$. The space $\mathcal{C}$ also inherits the direct sum decomposition into multigraded subspaces $\mathcal{C}_Q = \mathcal{B}_Q \cap \mathcal{C}$. Thus it is enough to determine the finite dimensional spaces $\mathcal{C}_Q$.

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## 1. Introduction

Following the paper [3] by C. Frønsdal, where he studied the classification of the ideals of 'free differential algebras' and $\boldsymbol{q}$-algebras, our study here is

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modeled on a unital associative complex algebra $\mathcal{B} = \mathbb{C}\langle e_{i_1}, e_{i_2}, \dots, e_{i_N}\rangle$ with a multiparametric $\boldsymbol{q}$-differential structure. In the study of the universal R-matrix of quantum groups, the generators $\{e_i\}_{i\in\mathcal{N}}$ could be regarded as positive Serre generators and the negative Serre generators are represented by $\boldsymbol{q}$-differential operators $\{\partial_i\}_{i\in\mathcal{N}}$, which act on $\mathcal{B}$ according to the twisted Leibniz rule $\partial_i(e_j x) = \delta_{ij}x + q_{ij}e_j\partial_i(x)$ for each $x \in \mathcal{B}$, where the parameters $q_{ij}$ are (complex) values of a function $\boldsymbol{q}\colon \mathcal{N}\times\mathcal{N} \to \mathbb{C}\backslash\{0\}$, $(i,j) \mapsto q_{ij}$. In this twisted Leibniz rule we 'mark' each passing of $\partial_i$ through $e_i$ (from the left) by additional factor $q_{ij}$, so $\partial_i$ is a kind of generalized $i$-th partial derivative. This rule is in direct relation to $q_{ij}$-canonical commutation relations (see [6, 1.1]), where the authors examine the Hilbert space realizability of the $\{q_{ij}\}$-*multiparametric quon algebras*. By comparing these two approaches it can be easily seen that the generator $e_i$ should be regarded as the i-th creation operator and $\partial_i$ as the i-th annihilation operator in the Fock representation. Note that the algebra $\mathcal{B}$ can also be considered as multigraded, and then the operators $\partial_i$, of degree $-1$, respects the direct sum decomposition of $\mathcal{B}$ into multigraded subspaces $\mathcal{B}_Q$ ($Q$ a multiset over $\mathcal{N}$). The action of $\partial_i$ on any monomial $e_{j_1\dots j_n} \in \mathbf{B}_Q$ (where $\mathbf{B}_Q$ denotes the monomial basis of $\mathcal{B}_Q$) is given explicitly by

$$\partial_i(e_{j_1\dots j_n}) = \sum_{1\le p\le n,\, j_p=i} q_{ij_1}\cdots q_{ij_{p-1}}e_{j_1\dots j_{p-1}j_{p+1}\dots j_n}.$$

The number of terms in this sum is equal to the number of appearances (multiplicity) of the generator $e_i$ in monomial $e_{j_1\dots j_n} = e_{j_1}\cdots e_{j_n}$. An important special case is the following

$$\partial_i(e_i^n) = \left(1 + q_{ii} + q_{ii}^2 + \dots + q_{ii}^{n-1}\right)e_i^{n-1} = [n]_{q_{ii}}\, e_i^{n-1},$$

where $[n]_q = 1 + q + \dots + q^{n-1}$ is a $q$-analogue of a natural number $n$.

We define a *constant* $C \in \mathcal{B}$ to be any element of $\mathcal{B}$ with the property $\partial_{i_p}C = 0$ for each $1 \le p \le N$ (i.e $\partial_i C = 0$ for every $i \in \mathcal{N}$). Denote by $\mathcal{C}$ the space of all constants in $\mathcal{B}$. In our approch to determine constants we define a multidegree operator $\partial$ on $\mathcal{B}$ by $\partial = \sum_{i\in\mathcal{N}} e_i\partial_i$, which preserves the multigrading. Then $C$ is a constant iff $\partial C = 0$ i.e $\partial_i C = 0$ for each $i \in \mathcal{N}$.

Now we can study the restrictions $\partial^Q$ of $\partial$ to $\mathcal{B}_Q$. If we denote by $\mathcal{C}_Q$ the space of all constants in $\mathcal{B}_Q$, then $\mathcal{C}_Q = \mathcal{B}_Q \cap \mathcal{C}$. In the case Card $Q = 1$, zero is the only constant in $\mathcal{B}_Q$. Hence nontrivial constants might exist only in the spaces $\mathcal{B}_Q$, Card $Q \ge 2$. Our procedure of computing nontrivial constants in $\mathcal{B}_Q$ is as follows. Let $\mathrm{B}_Q$ denote the matrix of $\partial^Q$. Its entries are given by (19) i.e by the polynomials in $q_{ij}$*'s*, so $\det \mathrm{B}_Q$ is also a polynomial in $q_{ij}$*'s.* Of particular interest is the study of $\det \mathrm{B}_Q$. Namely, if $\det \mathrm{B}_Q \neq 0$ (or equivalently in terminology of Frønsdal's if the parameters $q_{ij}$*'s* are in general position) then $\mathcal{C}_Q = \{0\}$. The space $\mathcal{C}_Q$ is nonzero only for *singular parameters* $q_{ij}$*'s* for which $\det \mathrm{B}_Q = 0$. In view of the fact that $\det \mathrm{B}_Q$ has a nice factorization (c.f. Remark 10) with factors $\beta_T$ for each $T \subseteq Q$, $|T| \geq 2$, we are going to distinguish two types of singular parameters (c.f. (20) resp. (21)), which we shall call *$Q$-cocycle condition* or *top cocycle condition* resp. $(Q;T)$*-cocycle condition.* In the description of certain basic nontrivial constants belonging to $\mathcal{C}_Q$ we shall use certain iterated $\boldsymbol{q}$-commutators $Y_{\underline{j}}$ and certain simple $\boldsymbol{q}$-commutators $X_{\underline{j}}$ and also some binomials $X^{\underline{j}}$ defined in the Section 3. Next we study some singular orbits (long and short) and explain the dimension of $\mathcal{C}_Q$ (differently than in [3]). Our motivation is to show that the basic constants in degenerated $\mathcal{B}_Q$*'s* can be constructed from those in the generic case by a certain specialization procedure. This leads us to the conclusion that the fundamental problem of description the constants in $\mathcal{C}$ can be reduced to the problem of determining the constants $C_Q$ in generic subspaces $\mathcal{C}_Q$, under the top cocycle condition $c_Q$. Further studies show that each 'generic basic constant' $C_Q \in \mathcal{C}_Q$, $Q = l_1 \dots l_n$ under the top cocycle condition can be expressed in terms of $(n-1)!$ iterated $\boldsymbol{q}$-commutators $Y_{l_1\xi}$, where $l_1 \in Q$ is fixed and the remaining $n-1$ indices $\xi = j_2 \dots j_n$ vary. The cases $n = 3, 4$ are treated in Remark 11. The cases $n \geq 5$ are more complicated and will not be considered here.

## 2. Free Associative Complex Algebra $\mathcal{B}$

Let $\mathbb{N}_0 = \{0, 1, \dots\}$ be the set of nonnegative integers and let $\mathcal{N} = \{i_1, \dots, i_N\}$ be a fixed subset of $\mathbb{N}_0$. Then we denote by $\mathcal{B} = \mathcal{B}_\mathcal{N} = \mathbb{C}\langle e_{i_1}, \dots, e_{i_N}\rangle$ the free (unital) associative $\mathbb{C}$-algebra with $N$ generators $\{e_i\}_{i \in \mathcal{N}}$, where degree of each generator $e_i$ is equal to one. We can think of $\mathcal{B}$ as an algebra of noncommutative polynomials in $N$ noncommuting variables $e_{i_1}, \dots, e_{i_N}$. Every sequence $l_1, \dots, l_n \in \mathcal{N}$ such that $l_1 \leq \dots \leq l_n$ we can think of as a multiset $Q = \{l_1 \leq \dots \leq l_n\}$ over $\mathcal{N}$ of size $n = |Q|$, where $|Q| = \mathrm{Card}\, Q$ denotes the cardinality of the multiset $Q$. Sometimes, we will simply write $Q = l_1 \dots l_n$.

The algebra $\mathcal{B}$ is naturally graded by the total degree

$$\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}^n, \tag{1}$$

where $\mathcal{B}^0 = \mathbb{C}$ and $\mathcal{B}^n$ consists of all homogeneous noncommuting polynomials of total degree $n$ in variables $e_{i_1}, \dots, e_{i_N}$. We also have a finer decomposition of $\mathcal{B}$ into multigraded components (= weight subspaces)

$$\mathcal{B} = \bigoplus_{n\geq 0,\, l_1\leq\cdots\leq l_n,\, l_j\in\mathcal{N}} \mathcal{B}_{l_1\ldots l_n}, \tag{2}$$

where each weight subspace $\mathcal{B}_Q = \mathcal{B}_{l_1\ldots l_n}$, corresponding to a multiset $Q$, is given by

$$\mathcal{B}_Q = span_{\mathbb{C}} \left\{ e_{j_1\ldots j_n} := e_{j_1}\cdots e_{j_n} \mid j_1\ldots j_n \in \widehat{Q} \right\}. \tag{3}$$

Here $\widehat{Q} = S_n Q = \{\sigma(l_1\ldots l_n) \mid \sigma \in S_n\}$ denotes the set of all rearrangements of the sequence $l_1,\ldots,l_n$ (i.e $\widehat{Q}$ is the set of all distinct permutations of the multiset $Q$). Thus $\dim \mathcal{B}_Q = \left|\widehat{Q}\right|$.

If $l_1,\ldots,l_n \in \mathcal{N}$ satisfy $l_1 < \cdots < l_n$, then $Q$ is a set, $Q = \{l_1,\ldots,l_n\} \subseteq \mathcal{N}$ and the corresponding weight subspace $\mathcal{B}_Q$ we call *generic*. Any other weight subspaces $\mathcal{B}_Q$ (i.e nongeneric) we call *degenerate*.

Denote by $\mathcal{B}^{\mathrm{gen}}$ the (generic) subspace of $\mathcal{B}$ spanned by all multilinear monomials and by $\mathcal{B}^{\mathrm{deg}}$ the (degenerate) subspace of $\mathcal{B}$ spanned by all monomials which are nonlinear in at least one variable. Then the direct sum decomposition (1) can be written in the form: $\mathcal{B} = \mathcal{B}^{\mathrm{gen}} \oplus \mathcal{B}^{\mathrm{deg}}$, where

$$\mathcal{B}^{\mathrm{gen}} = \bigoplus_{Q \text{ a set}} \mathcal{B}_Q^{\mathrm{gen}}, \qquad \mathcal{B}^{\mathrm{deg}} = \bigoplus_{Q \text{ a multiset (not set)}} \mathcal{B}_Q^{\mathrm{deg}}.$$

Fix a map $\boldsymbol{q}\colon \mathcal{N}\times\mathcal{N} \to \mathbb{C}$, $(i,j) \mapsto q_{ij}$, $i,j \in \mathcal{N}$. Complex numbers $q_{ij}$'$s$ are treated as parameters and $\boldsymbol{q}$ can be interpreted as a point in the parameter space $\mathbb{C}^{\mathcal{N}^2}$.

On the algebra $\mathcal{B}$ we introduce $N$ linear operators $\partial_i = \partial_i^{\boldsymbol{q}}\colon \mathcal{B} \to \mathcal{B}$, $i \in \mathcal{N}$, defined recursively, as follows:

$$\partial_i(1) = 0, \quad \partial_i(e_j) = \delta_{ij}, \tag{4}$$

$$\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x) \quad \text{for each } x \in \mathcal{B},\ i,j \in \mathcal{N}. \tag{5}$$

(Here $\delta_{ij} = 1$ if $i = j$, and 0 otherwise is a standard Kronecker delta.)

From (5) we see that the operators $\partial_i$, $i \in \mathcal{N}$ act as a generalized $i$-th partial derivative on the algebra $\mathcal{B}$. As a result, they depend on additional parameters (complex numbers) $q_{ij}$, so we say that $\partial_i$ is a multiparametrically deformed $i$-th partial derivative or shortly $\boldsymbol{q}$-deformed $i$-th partial derivative. In particular, if all $q_{ij}$'$s$ are equal to one, then $\partial_i$ coincides with a usual $i$-th partial derivative.

In what follows we shall consider $\mathcal{B}$ equiped with this '$\boldsymbol{q}$-differential structure'.

By abbreviating $j_1 \dots j_n$ by $\underline{j}$ let us denote by $\mathbf{B}_Q = \left\{ e_{\underline{j}} \mid \underline{j} \in \widehat{Q} \right\}$ the monomial basis of $\mathcal{B}_Q$. Then by applying the formula (5) several times we get an explicit formula for the action of $\partial_i$ on a typical monomial $e_{\underline{j}} \in \mathbf{B}_Q$ as follows:

$$\partial_i(e_{\underline{j}}) = \sum_{1 \le k \le n,\, j_k = i} q_{ij_1} \cdots q_{ij_{k-1}} e_{j_1 \dots \widehat{j_k} \dots j_n} \tag{6}$$

(c.f. eq. (21) in [5]). Here $\widehat{j_k}$ denotes the omission of the corresponding index $j_k$.

Eg. $\quad \partial_2(e_{1321212}) = q_{21}q_{23}e_{131212} + q_{21}^2 q_{22}q_{23}e_{132112} + q_{21}^2 q_{22}^2 q_{23}e_{132121}.$

In special case (where there is only one $k, 1 \le k \le n$ such that $j_k = i$) the formula (6) is reduced to:

$$\partial_i(e_{\underline{j}}) = q_{ij_1} \cdots q_{ij_{k-1}} e_{j_1 \dots \widehat{j_k} \dots j_n}. \tag{7}$$

Similarly, if $j_k = i$ for all $1 \le k \le n$, then the formula (6) reads as:

$$\partial_i(e_i^n) = [n]_{q_{ii}}\, e_i^{n-1}, \tag{8}$$

where

$$[n]_q := \sum_{0 \le k \le n-1} q^k = 1 + q + \cdots + q^{n-1}, \;\; n \ge 1. \tag{9}$$

Note that formula (9) is a $q$-analogue of the natural number $n$, therefore, for $q_{ii} = 1$ from the formula (8) we get the classical formula $\partial_i(e_i^n) = n \cdot e_i^{n-1}$.

Suppose that $x \in \mathcal{B}_{l_1 \dots l_n}$. Then for any $y \in \mathcal{B}$ we have a formula more general than (5):

$$\partial_i(xy) = \partial_i(x)y + q_{il_1} \cdots q_{il_n} x \partial_i(y) \quad \text{for each } i \in \mathcal{N}. \tag{10}$$

## 3. Commutators and Constants in Algebra $\mathcal{B}$

In order to write efficiently some constants in the algebra $\mathcal{B}$ we first introduce the following abbreviations:

($i$) for any subset $T \subseteq Q$, $|T| \ge 2$:

$$q_T := \prod_{a \ne b \in T} q_{ab} \tag{11}$$

(c.f. eq. (4.1) in [4]); in particular $\;q_{\{i,j\}} = q_{ij}q_{ji}$;

*(ii)* for any sequence $j_1 \dots j_p$ we define $X^{j_1\cdots j_p}$ to be the following binomials:

$$X^{j_1\cdots j_p} := e_{j_1\dots j_p} + (-1)^{p-1} \prod_{1\le a<b\le p} q_{j_b j_a} e_{j_p\dots j_1}. \tag{12}$$

(with $X^{j_1} := e_{j_1}$ for $p=1$);

*(iii)* for any sequence $j_1 \dots j_p$ we define $X_{j_1\dots j_p}$ to be the following simple $\boldsymbol{q}$-commutators:

$$X_{j_1} := e_{j_1}, \qquad X_{j_1\dots j_p} := \left[e_{j_1\dots j_{p-1}}, e_{j_p}\right]_{q_{j_p j_1}\cdots q_{j_p j_{p-1}}} \tag{13}$$

and let the iterated $\boldsymbol{q}$-commutators $Y_{j_1\dots j_p}$ be defined recursively by

$$Y_{j_1} := e_{j_1}, \qquad Y_{j_1\dots j_p} := \left[Y_{j_1\dots j_{p-1}}, e_{j_p}\right]_{q_{j_p j_1}\cdots q_{j_p j_{p-1}}}. \tag{14}$$

**Remark 1.** For $p=2$ we have:

$$X^{j_1 j_2} = X_{j_1 j_2} = Y_{j_1 j_2} = e_{j_1 j_2} - q_{j_2 j_1} e_{j_2 j_1}.$$

In the following three propositions we show how to compute the action of $\partial_i$ on the simple $\boldsymbol{q}$-commutators, the iterated $\boldsymbol{q}$-commutators and binomials $X^{j_1\cdots j_p}$. (Note that for $p=1$ we get: $\partial_i(e_{j_1}) = \delta_{ij_1}$ for each $i \in \mathcal{N}$.)

**Proposition 2.** *Let $p \ge 2$, $j_1, \dots, j_p \in \mathcal{N}$. Then for each $i \in \mathcal{N}$ we have*

$$\partial_i\left(X_{j_1\dots j_p}\right) = \left[\partial_i\left(e_{j_1\dots j_{p-1}}\right), e_{j_p}\right]_{q_{ij_p} q_{j_p j_1}\cdots q_{j_p j_{p-1}}}. \tag{15}$$

*Proof.* By using (10) we get

$$\begin{aligned}
\partial_i\left(X_{j_1\dots j_p}\right) &= \partial_i\left(e_{j_1\dots j_{p-1}} e_{j_p} - q_{j_p j_1}\cdots q_{j_p j_{p-1}} e_{j_p} e_{j_1\dots j_{p-1}}\right)\\
&= \left[\partial_i\left(e_{j_1\dots j_{p-1}}\right) e_{j_p} + q_{ij_1}\cdots q_{ij_{p-1}} e_{j_1\dots j_{p-1}} \partial_i\left(e_{j_p}\right)\right]\\
&\quad - q_{j_p j_1}\cdots q_{j_p j_{p-1}}\left[\partial_i\left(e_{j_p}\right) e_{j_1\dots j_{p-1}} + q_{ij_p} e_{j_p} \partial_i\left(e_{j_1\dots j_{p-1}}\right)\right]\\
&= \partial_i\left(e_{j_1\dots j_{p-1}}\right) e_{j_p} - q_{ij_p} q_{j_p j_1}\cdots q_{j_p j_{p-1}} e_{j_p} \partial_i\left(e_{j_1\dots j_{p-1}}\right)\\
&= \left[\partial_i\left(e_{j_1\dots j_{p-1}}\right), e_{j_p}\right]_{q_{ij_p} q_{j_p j_1}\cdots q_{j_p j_{p-1}}}.
\end{aligned}$$

It is clear that $\partial_i\left(X_{j_1\dots j_p}\right) = 0$ for each $i \notin \{j_1, \dots, j_{p-1}\}$. □

**Proposition 3.** *Let $p \ge 2$, $j_1, \dots, j_p \in \mathcal{N}$. Then for each $i \in \mathcal{N}$ we have*

$$\partial_i\left(Y_{j_1\dots j_p}\right) = \begin{cases} \left(1 - q_{\{j_1,j_2\}}\right) Y^{j_1}_{j_2\dots j_p} & \textit{if} \quad i = j_1 \\ 0 & \textit{if} \quad i \ne j_1 \end{cases} \tag{16}$$

*where*

$$Y^{j_1}_{j_2} := e_{j_2}, \qquad Y^{j_1}_{j_2\dots j_p} := \left[Y^{j_1}_{j_2\dots j_{p-1}}, e_{j_p}\right]_{q_{\{j_1,j_p\}} q_{j_p j_2}\cdots q_{j_p j_{p-1}}}. \tag{17}$$

*Proof.* For $p = 2$, $Y_{j_1j_2} = [e_{j_1}, e_{j_2}]_{q_{j_2j_1}} = e_{j_1j_2} - q_{j_2j_1}e_{j_2j_1}$ and by using (15) it follows that $\partial_i (Y_{j_1j_2}) = \delta_{ij_1} (1 - q_{ij_2}q_{j_2j_1}) e_{j_2}$. If we apply (15) several times, then for any $2 \le k \le p$ we get $\partial_i (Y_{j_1\dots j_k}) = \delta_{ij_1} (1 - q_{ij_2}q_{j_2j_1}) Y^i_{j_2\dots j_k}$, where $Y^i_{j_2\cdots j_k}$ is given by (17) for $j_1 = i$. Finally, it follows (16).

Clearly, if $q_{\{j_1,j_2\}} = 1$, then $\partial_i \left(Y_{j_1\dots j_p}\right) = 0$ for each $i \in \mathcal{N}$. □

**Remark 4.** The expressions $q_{j_pj_1} \cdots q_{j_pj_{p-1}}$ resp. $q_{\{j_1,j_p\}}q_{j_pj_2} \cdots q_{j_pj_{p-1}}$ appearing in (13) and (14) resp. (17) are in Frønsdal *[3, Subsections 2.2. and 3.1.]* denoted by $a\,(j_1 \dots j_p)$ resp. $b_{j_1} (j_2 \dots j_p) = q_{j_1j_p}a\,(j_1j_2 \dots j_p)$ and are called the commutation factors.

**Proposition 5.** *Let $p \ge 2$, $j_1, \dots, j_p \in \mathcal{N}$. Then for each $i \in \mathcal{N}$ such that $i = j_k$, we have*

$$\partial_i \left(X^{j_1\cdots j_p}\right)$$
$$= q_{ij_1} \cdots q_{ij_{k-1}} \left( e_{j_1\dots\widehat{j_k}\dots j_p} + (-1)^{p-1} \prod_{1\le a<b\le p-1} q_{j_bj_a}\sigma_{i|j_{k+1},\dots,j_p} e_{j_p\dots\widehat{j_k}\dots j_1} \right)$$

*and $\partial_i \left(X^{j_1\cdots j_p}\right) = 0$ otherwise. Here we have used the notation*

$$\sigma_{i|j_{k+1},\dots,j_p} := \prod_{k+1\le m\le p} q_{\{i,j_m\}}.$$

**Proposition 6.** *If for some $i \ne j \in \mathcal{N}$ $q_{\{i,j\}} = 1$, then $Y_{ji} = -q_{ij}Y_{ij}$.*

*Proof.* From $q_{\{i,j\}} = 1$ we obtain $q_{ji} = 1/q_{ij}$ and then $Y_{ji} = e_je_i - q_{ij}e_ie_j = -q_{ij} (e_ie_j - q_{ji}e_je_i) = -q_{ij}Y_{ij}$. □

**Corrolary 7.** *Let $j_1, \dots, j_p \in \mathcal{N}$, $2 \le p \le N$ and $j_1 \ne j_2$. If $q_{\{i,j\}} = 1$ then $Y_{j_2j_1j_3\dots j_p} = -q_{j_1j_2}Y_{j_1j_2j_3\dots j_p}$.*

**Definition 8.** A **constant** in $\mathcal{B}$ is any element $C$ in $\mathcal{B}$ annihilated by all $\partial_i$'s $(i \in \mathcal{N})$ i.e $\partial_i(C) = 0$ for every $i \in \mathcal{N}$.

Denote by $\mathcal{C} = \{C \in \mathcal{B} \mid \partial_i(C) = 0, \ \text{ for all } i \in \mathcal{N}\}$ the space of all constants in $\mathcal{B}$.

Observe that $\mathcal{B}^0 = \mathbb{C}$ consists of trivial constants and in $\mathcal{B}^1$ the only constant is zero. Thus, nontrivial constants could exist only in the space $\bigoplus_{n\ge2} \mathcal{B}^n$.

**Definition 9.** We define a **multidegree operator** $\partial\colon \mathcal{B} \to \mathcal{B}$ by the formula:

$$\partial := \sum_{i\in\mathcal{N}} e_i\,\partial_i,$$

where $e_i\colon \mathcal{B}\to\mathcal{B}$ are considered as (multiplication by $e_i$) operators on $\mathcal{B}$.

Note that $\partial$ is the operator of degree zero. Clearly,

$$\partial C=\sum_{i\in\mathcal{N}} e_i\,\partial_i\,C=0 \qquad \text{iff} \qquad \partial_i\,C=0 \quad \text{for all} \quad i\in\mathcal{N}.$$

Therefore $\mathcal{C}=\ker\partial$, where $\ker\partial$ denotes the kernel of the multidegree operator $\partial$. The operator $\partial$ preserves the direct sum decomposition of the algebra $\mathcal{B}$, i.e $\partial\mathcal{B}_Q\subset\mathcal{B}_Q$. In other words, each subspace $\mathcal{B}_Q$ is an invariant subspace of $\partial$. Denote by $\partial^Q\colon \mathcal{B}_Q\to\mathcal{B}_Q$ the restriction of $\partial\colon \mathcal{B}\to\mathcal{B}$ to the subspace $\mathcal{B}_Q$ i.e

$$\partial^Q x=\partial x \qquad \text{for every} \quad x\in\mathcal{B}_Q. \tag{18}$$

Let $\mathcal{C}_Q$ be the space of all constants belonging to $\mathcal{B}_Q$. Thus $\mathcal{C}_Q=\ker\partial^Q$ and $\mathcal{C}_Q=\mathcal{B}_Q\cap\mathcal{C}$. The space $\mathcal{C}$ also inherits the direct sum decomposition into multigraded subspaces $\mathcal{C}_Q$. Hence the fundamental problem to determine the space $\mathcal{C}$ can be reduced to determine the finite dimensional spaces $\mathcal{C}_Q$ $(=\ker\partial^Q)$ for all multisets $Q$ over $\mathcal{N}$.

Let $|Q|=n\geq 2$ and let $e_{j_1\ldots j_n}$ be any basis element from a monomial basis $\mathbf{B}_Q$ of $\mathcal{B}_Q$. Then by definition of $\partial^Q$ and using the formula (6) it follows that

$$\begin{aligned}\partial^Q\left(e_{j_1\ldots j_n}\right)=\sum_{i\in\mathcal{N}} e_i\partial_i\left(e_{j_1\ldots j_n}\right)&=\sum_{i\in\mathcal{N}} e_i \sum_{1\leq k\leq n,\, j_k=i} q_{ij_1}\cdots q_{ij_{k-1}}e_{j_1\ldots\widehat{j_k}\ldots j_n}\\ &=\sum_{1\leq k\leq n}\ \sum_{i\in\mathcal{N},\, i=j_k} q_{ij_1}\cdots q_{ij_{k-1}}e_{ij_1\ldots\widehat{j_k}\ldots j_n}\end{aligned}$$

i.e

$$\partial^Q\left(e_{j_1\ldots j_n}\right)=\sum_{1\leq k\leq n} q_{j_kj_1}\cdots q_{j_kj_{k-1}}e_{j_kj_1\ldots\widehat{j_k}\ldots j_n}, \tag{19}$$

for each $j_1\ldots j_n\in\widehat{Q}$.

Let $\mathrm{B}_Q$ denotes the matrix of $\partial^Q$ w.r.t $\mathbf{B}_Q$ (considered with the Johnson-Trotter ordering on permutations c.f. [7]).

For any muliset $Q=\left\{k_1^{n_1},\ldots,k_p^{n_p}\right\}$ ($k_i$ distinct) of cardinality $|Q|=n_1+\cdots+n_p=:n$ the size of the matrix $\mathrm{B}_Q$ is equal to the following multinomial coefficient

$$\frac{n!}{n_1!\cdots n_p!}=\binom{n}{n_1,\ldots,n_p}\ (=\dim\mathcal{B}_Q).$$

The entries of $\mathrm{B}_Q$ are polynomials in $q_{ij}$*'s*, hence its determinant is also a polynomial in $q_{ij}$*'s.* It turns out that the polynomial $\det\mathrm{B}_Q$ has a nice factorization (which, in case $Q$ is a set, has only binomial factors, see (26)) with factors $\beta_T$

for each $T \subseteq Q$, $|T| \geq 2$. Thus, $\det \mathrm{B}_Q = 0$ implies that $\beta_T$ vanishes for at least one $T \subseteq Q$.

Of particular interest are the actual values of parameters $q_{ij}$'$s$ (called *singular values* or *singular parameters*) for which at least one $\beta_T = 0$. In other words, we say that parameters $q_{ij}$'$s$ are singular parameters if $\det \mathrm{B}_Q = 0$, otherwise they are regular (i.e parameters in general position). We have that there are no nontrivial constants in $\mathcal{B}_Q$ (i.e $\mathcal{C}_Q = \{0\}$) when the parameters $q_{ij}$'$s$ are in general position. The space $\mathcal{C}_Q$ is nonzero only for singular parameters. Thus singular parameters play the crucial role in computing (nontrivial) constants in $\mathcal{B}_Q$.

In this paper we shall distinguish two types of singular parameters satisfying

**Type 1** ($Q$-*cocycle condition*):

$$c_Q := \{\beta_Q = 0, \;\; \beta_T \neq 0, \;\;\; \forall\, T \subsetneqq Q\} \tag{20}$$

or

**Type 2** (($Q;T$)-*cocycle condition*): for fixed $T \subsetneqq Q$

$$c_{Q;T} := \{\beta_Q = 0, \;\; \beta_T = 0, \;\; \beta_S \neq 0, \;\;\; \forall\, S \subsetneqq Q, \, S \neq T\} \tag{21}$$

respectively. Notice that

1. the $Q$-cocycle condition implies
$$\mathcal{C}_Q \neq \{0\}, \;\; \mathcal{C}_T = \{0\}, \; \forall\, T \subsetneqq Q \quad \text{and}$$
2. the ($Q;T$)-*cocycle condition* implies
$$\mathcal{C}_Q \neq \{0\}, \;\; \mathcal{C}_T \neq \{0\}, \;\; \mathcal{C}_S = \{0\}, \; \forall\, S \subsetneqq Q, \, S \neq T.$$

Sometimes, the $Q$-cocycle condition we shall simply call 'top cocycle condition'.

From our experience in finding constants in the cases $|Q| \leq 4$ we guess that Type 2 constants could be obtained from Type 1 constants by certain specialization procedure. Thus in $\mathcal{B}_Q$ it is enough to determine the Type 1 constants.

**Remark 10.** We can rewrite the operator $\partial^Q$ (c.f. (19)) in terms of simpler operators acting on $\mathcal{B}_Q$. Let $T_{1,1} = id$ and let $T_{k,1} = T^Q_{k,1}$ be given as follows

$$T_{k,1}\, e_{j_1 \ldots j_n} := q_{j_k j_1} \cdots q_{j_k j_{k-1}} e_{j_k j_1 \ldots \widehat{j_k} \ldots j_n}$$

for each $j_1 \ldots j_n \in \widehat{Q}$, $2 \leq k \leq n$. Then $\partial^Q$ can be rewritten as

$$\partial^Q = \sum_{1 \leq k \leq n} T_{k,1}.$$

Moreower, we have the following (specialized) factorization (a special case of the braid factorization from *[2, Proposition 4.7]* c.f. matrix factorization from *[6]*)

$$\partial^Q \cdot C_Q = D_Q. \tag{22}$$

where

$$C_Q := (id - T_{n,1})\,(id - T_{n-1,1}) \cdots (id - T_{3,1})\,(id - T_{2,1})\,,$$
$$D_Q := \left(id - T_{2,1}^2 T_{n,2}\right)\left(id - T_{2,1}^2 T_{n-1,2}\right) \cdots \left(id - T_{2,1}^2 T_{3,2}\right)\left(id - T_{2,1}^2 T_{2,2}\right).$$

Observe that the operators $T_{2,1}^2 T_{k,2}$ appearing in $D_Q$ act as

$$T_{2,1}^2 T_{k,2}\, e_{j_1 \dots j_n} = q_{\{j_1, j_k\}} q_{j_k j_2} \cdots q_{j_k j_{k-1}} e_{j_1 j_k j_2 \dots \widehat{j_k} \dots j_n}.$$

Then (22) can be rewritten as

$$\partial^Q\,(id - T_{n,1})\,M = \left(id - T_{2,1}^2 T_{n,2}\right) N \tag{23}$$

where

$$M = \overleftarrow{\prod_{2 \le k \le n-1}} (id - T_{k,1})\,, \qquad N = \overleftarrow{\prod_{2 \le k \le n-1}} \left(id - T_{2,1}^2 T_{k,2}\right).$$

Under the top cocycle condition $N$ is invertible and we can rewrite (23) further as

$$\partial^Q\,(id - T_{n,1})\,MN^{-1} = \left(id - T_{2,1}^2 T_{n,2}\right)$$

i.e.

$$\partial^Q\,(id - T_{n,1})\,MN^{-1}(Y) = \left(id - T_{2,1}^2 T_{n,2}\right)(Y) \qquad \text{for each} \quad Y \in \mathcal{B}.$$

From this last formula we can relate: $\ker\left(id - T_{2,1}^2 T_{n,2}\right) \subset \mathcal{B}_Q$ to $\ker \partial^Q =$ the space of constants in $\mathcal{B}_Q$. To each $Y \in \ker\left(id - T_{2,1}^2 T_{n,2}\right)$ the right hand side is zero, so the corresponding vector

$$X := \left((id - T_{n,1}) \cdot M \cdot N^{-1} \cdot Y\right) \in \ker \partial^Q \tag{24}$$

belongs to $\ker \partial^Q$, hence is a constant in $\mathcal{B}_Q$. It turns out that

$$\dim\left(\ker \partial^Q\right) = \dim\left(\ker\left(id - T_{2,1}^2 T_{n,2}\right)\right) - \dim\left(\ker\,(id - T_{n,1})\right). \tag{25}$$

This gives an alternative proof of a result of Frønsdal and Galindo [4, Theorem 4.1.2] that the space of constants has dimension $(n-2)!$ in the generic case.

When $Q$ is a set, then the matrix $\mathrm{B}_Q$ of the operator $\partial^Q$ (w.r.t monomial basis $\mathbf{B}_Q$) is a $n!$ by $n!$ (monomial) matrix. Its determinant is given explicitly as product of binomial factors $\beta_T$:

$$\det \mathrm{B}_Q = \prod_{\substack{T\subseteq Q\\ 2\le |T|\le n}} (\beta_T)^{(|T|-2)!(n-|T|)!} \tag{26}$$

(c.f. [6, Theorem 1.9.2]), where $\beta_T = 1 - q_T$, with $q_T = \prod_{a\neq b\in T} q_{ab}$ given by (11). Here $Q$-cocycle resp. $(Q;T)$-cocycle condition take the form

$$c_Q = \{q_Q = 1,\ \ q_T \neq 1 \ \text{ for all } \ T \subsetneqq Q\}\,, \tag{27}$$

resp.

$$c_{Q;T} = \{q_Q = 1,\ \ q_T = 1,\ \ q_S \neq 1 \ \text{ for all } \ S \subsetneqq Q,\, S\neq T\}\,. \tag{28}$$

## 4. Computation of Nontrivial Constants

In this section we are going to determine the explicit formulas for nontrivial constants depending on the top cocycle condition, but also for the appropriate $(Q;T)$-cocycle conditions. Here we shall not examine the constants depending on all singular parameters (see [4, Subsection 4.2.] for a detailed overview in the case $|Q| = 3$). In what follows we shall give the dimension of the space $\mathcal{C}_Q$ in the generic and degenerate cases (for Type 1 and Type 2 constants).

### 4.1. Generic Case

Let us examine the basic constants in generic subspaces $\mathcal{B}_Q$, $2 \le |Q| \le 4$.

#### 4.1.1. Basic Constants in the Space $\mathcal{B}_{\mathbf{Q}}$, $|Q| = 2$.

Let $Q = \{l_1, l_2\}$, $(l_1 < l_2)$. Then the matrix of $\partial^Q$ w.r.t the monomial basis $\mathbf{B}_{l_1l_2} = \{e_{l_1l_2}, e_{l_2l_1}\}$ is

$$\mathrm{B}_{l_1l_2} = \begin{pmatrix} 1 & q_{l_1l_2} \\ q_{l_2l_1} & 1 \end{pmatrix}$$

and hence $\det \mathrm{B}_{l_1l_2} = 1 - q_{\{l_1,l_2\}}$.

So $Q$-cocycle condition is given by $c_{l_1 l_2} = \left\{ q_{\{l_1,l_2\}} = 1 \right\}$. If $c_{l_1 l_2}$ holds, then

$$C_{l_1 l_2} = e_{l_1 l_2} - q_{l_2 l_1} e_{l_2 l_1} = Y_{l_1 l_2}$$

is a nontrivial constant in $\mathcal{B}_{l_1 l_2}$, where $Y_{l_1 l_2}$ is the iterated $\boldsymbol{q}$-commutator. Thus the space of constants in $\mathcal{B}_{l_1 l_2}$ is the following 1-dimensional space $\mathcal{C}_{l_1 l_2} = \mathbb{C}\left\{Y_{l_1 l_2}\right\}$.

It is easy to see that $\mathcal{C}_{l_1 l_2} = \mathbb{C}\left\{Y_{l_1 l_2}\right\} = \mathbb{C}\left\{Y_{l_2 l_1}\right\}$ (c.f. Proposition 6).

Here $\det \mathrm{B}_{l_1 l_2}$ has only one factor of the binomial form $1 - q_{\{l_1,l_2\}}$, so we have only the $Q$-cocycle condition. In general, when $Q$-cocycle condition does not hold, the space $\mathcal{C}_{l_1 l_2}$ is zero.

#### 4.1.2. Basic Constants in the Space $\mathcal{B}_{\mathbf{Q}}$, $|Q| = 3$.

Let $Q = \{l_1, l_2, l_3\}$, $(l_1 < l_2 < l_3)$. Then the matrix $\mathrm{B}_{l_1 l_2 l_3}$ of $\partial^Q$ w.r.t basis $\mathbf{B}_{l_1 l_2 l_3} = \{e_{l_1 l_2 l_3}, e_{l_1 l_3 l_2}, e_{l_3 l_1 l_2}, e_{l_3 l_2 l_1}, e_{l_2 l_3 l_1}, e_{l_2 l_1 l_3}\}$ is given by

$$\mathrm{B}_{l_1 l_2 l_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & q_{l_1 l_2} q_{l_1 l_3} & q_{l_1 l_2} \\ 0 & 1 & q_{l_1 l_3} & q_{l_1 l_3} q_{l_1 l_2} & 0 & 0 \\ q_{l_3 l_1} q_{l_3 l_2} & q_{l_3 l_1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & q_{l_3 l_2} & q_{l_3 l_2} q_{l_3 l_1} \\ 0 & 0 & q_{l_2 l_3} q_{l_2 l_1} & q_{l_2 l_3} & 1 & 0 \\ q_{l_2 l_1} & q_{l_2 l_1} q_{l_2 l_3} & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its determinant is equal to

$$\det \mathrm{B}_{l_1 l_2 l_3} = \left(1 - q_{\{l_1,l_2,l_3\}}\right) \prod_{1 \le i < j \le 3} \left(1 - q_{\{l_i,l_j\}}\right).$$

Under the $Q$-cocycle condition

$$c_{l_1 l_2 l_3} = \left\{ q_{\{l_1,l_2,l_3\}} = 1,\ \ q_{\{l_i,l_j\}} \neq 1 \ \text{ for all } 1 \le i < j \le 3 \right\}$$

we get that the space of constants is 1-dimensional $\mathcal{C}_{l_1 l_2 l_3} = \mathbb{C}\left\{C_{l_1 l_2 l_3}\right\}$, where a basic constant $C_{l_1 l_2 l_3}$ can be written as

$$\begin{aligned} C_{l_1 l_2 l_3} = {} & q_{l_1 l_2} q_{l_2 l_3} \left(1 - q_{\{l_1,l_3\}}\right) X^{l_1 l_2 l_3} + q_{l_2 l_3} q_{l_3 l_1} \left(1 - q_{\{l_1,l_2\}}\right) X^{l_2 l_3 l_1} \\ & + q_{l_3 l_1} q_{l_1 l_2} \left(1 - q_{\{l_2,l_3\}}\right) X^{l_3 l_1 l_2} \end{aligned}$$

or shortly

$$C_{l_1 l_2 l_3} = \sum_{cyc} q_{l_1 l_2} q_{l_2 l_3} \left(1 - q_{\{l_1,l_3\}}\right) X^{l_1 l_2 l_3} \tag{29}$$

where $X^{ijk}\,(= e_{ijk} + q_{ji}q_{ki}q_{kj}e_{kji})$ are defined in (12) and $\sum_{cyc}$ denotes the cyclic sum.

On the other hand, under $(Q;T)$-cocycle condition ($T = \{l_i, l_j\} \subset Q$ is fixed) we obtain that the constant $C_{l_1l_2l_3}$ further reduces to the iterated $\boldsymbol{q}$-commutator $Y_{l_il_jl_k}$, where $\{i,j,k\} = \{1,2,3\}$.

For example, assume that $T = \{l_1, l_2\}$. Then the condition (28) now reads as follows

$$c_{l_1l_2l_3;l_1l_2} = \left\{q_{\{l_1,l_2,l_3\}} = 1,\;\; q_{\{l_1,l_2\}} = 1,\;\; q_{\{l_1,l_3\}} \neq 1,\;\; q_{\{l_2,l_3\}} \neq 1\right\},$$

what implies $\;q_{\{l_1,l_3\}}q_{\{l_2,l_3\}} = 1\;$ i.e $\;q_{l_1l_3}q_{l_3l_1}q_{l_2l_3}q_{l_3l_2} = 1$. In this case, the constant (29) reduces to

$$\begin{aligned} C_{l_1l_2l_3} &= q_{l_1l_2}q_{l_2l_3}\left(1 - q_{\{l_1,l_3\}}\right)\left(e_{l_1l_2l_3} + q_{l_2l_1}q_{l_3l_1}q_{l_3l_2}e_{l_3l_2l_1}\right) \\ &\quad + q_{l_3l_1}q_{l_1l_2}\left(1 - q_{\{l_2,l_3\}}\right)\left(e_{l_3l_1l_2} + q_{l_1l_3}q_{l_2l_3}q_{l_2l_1}e_{l_2l_1l_3}\right). \end{aligned}$$

By using that

$$\begin{aligned} q_{l_3l_1}q_{l_1l_2}\left(1 - q_{\{l_2,l_3\}}\right) &= q_{l_3l_1}q_{l_1l_2}\left(1 - \frac{1}{q_{l_1l_3}q_{l_3l_1}}\right) = -\frac{q_{l_1l_2}\left(1 - q_{\{l_1,l_3\}}\right)}{q_{l_1l_3}} \\ &= -q_{l_1l_2}q_{l_2l_3}q_{l_3l_2}q_{l_3l_1}\left(1 - q_{\{l_1,l_3\}}\right) = -q_{l_3l_2}q_{l_3l_1}\left(q_{l_1l_2}q_{l_2l_3}\left(1 - q_{\{l_1,l_3\}}\right)\right) \end{aligned}$$

and $\;q_{l_3l_2}q_{l_3l_1}q_{l_1l_3}q_{l_2l_3}q_{l_2l_1} = q_{l_2l_1}\;$ we obtain

$$C_{l_1l_2l_3} = q_{l_1l_2}q_{l_2l_3}\left(1 - q_{\{l_1,l_3\}}\right)Y_{l_1l_2l_3},$$

where $\;Y_{l_1l_2l_3} = e_{l_1l_2l_3} + q_{l_2l_1}q_{l_3l_1}q_{l_3l_2}e_{l_3l_2l_1} - q_{l_3l_2}q_{l_3l_1}e_{l_3l_1l_2} - q_{l_2l_1}e_{l_2l_1l_3}$.

### 4.1.3. Basic Constants in the Space $\mathcal{B}_{\mathbf{Q}}$, $|Q| = 4$.

Let $Q = \{l_1, l_2, l_3, l_4\}$, $(l_1 < l_2 < l_3 < l_4)$. The matrix $\mathrm{B}_{l_1l_2l_3l_4}$ of $\partial^Q$ in the monomial basis $\mathbf{B}_{l_1l_2l_3l_4}$ has determinant given by

$$\det \mathrm{B}_{l_1l_2l_3l_4} = \left(1 - q_{\{l_1,l_2,l_3,l_4\}}\right)^2 \prod_{1\le i<j\le 4}\left(1 - q_{\{l_i,l_j\}}\right)^2 \prod_{1\le i<j<k\le 4}\left(1 - q_{\{l_i,l_j,l_k\}}\right).$$

1) The space of $Q$-constants, under the $Q$-cocycle condition

$$c_{l_1l_2l_3l_4} = \left\{q_{\{l_1,l_2,l_3,l_4\}} = 1,\;\; q_{\{l_i,l_j,l_k\}} \neq 1,\;\; q_{\{l_i,l_j\}} \neq 1 \;\text{ for all }\; 1 \le i < j < k \le 4\right\}$$

is 2-dimensional $\mathcal{C}_{l_1l_2l_3l_4} = \mathbb{C}\{C_{l_1l_2l_3l_4}, C_{l_1l_2l_4l_3}\}$ with the following basis elements

$$\begin{aligned}C_{l_1l_2l_3l_4} =& Z_{l_1l_2l_3l_4} + q_{l_4l_2}q_{l_4l_3}q_{\{l_1,l_4\}}\, Z_{l_1l_4l_2l_3} + q_{l_3l_2}q_{l_4l_2}q_{\{l_1,l_3,l_4\}}\, Z_{l_1l_3l_4l_2},\\ C_{l_1l_2l_4l_3} =& q_{l_3l_4}q_{\{l_1,l_2,l_3\}}\, Z'_{l_1l_2l_3l_4} + q_{l_4l_2}q_{\{l_1,l_4\}}\, Z'_{l_1l_4l_2l_3}\\ &+ q_{l_3l_2}q_{l_3l_4}q_{l_4l_2}q_{\{l_1,l_3\}}q_{\{l_1,l_3,l_4\}}\, Z'_{l_1l_3l_4l_2}.\end{aligned}$$

where

$$\begin{aligned}Z_{i_1i_2i_3i_4} :=& \zeta_{i_1i_2i_3i_4}\left(\frac{1-q_{\{i_1,i_3\}}}{q_{i_3i_1}}V^{i_4}_{i_1i_2i_3} + \frac{1-q_{\{i_1,i_2\}}}{q_{i_1i_2}}V^{i_4}_{i_2i_3i_1}\right.\\ &\left.+q_{i_3i_2}\left(q_{\{i_1,i_2\}}q_{\{i_1,i_3\}}-1\right)W^{i_4}_{i_3i_1i_2}\right),\\ Z'_{i_1i_2i_3i_4} :=& \zeta_{i_1i_2i_3i_4}\left(\frac{1-q_{\{i_1,i_3\}}}{q_{i_3i_1}}W^{i_4}_{i_1i_2i_3} + \frac{1-q_{\{i_1,i_2\}}}{q_{i_1i_2}}W^{i_4}_{i_2i_3i_1}\right.\\ &\left.+\frac{q_{\{i_1,i_2\}}q_{\{i_1,i_3\}}-1}{q_{i_2i_3}q_{\{i_1,i_2\}}q_{\{i_1,i_3\}}}V^{i_4}_{i_3i_1i_2}\right),\end{aligned}$$

$$\begin{aligned}\zeta_{i_1i_2i_3i_4} :=& q_{i_3i_1}(1-q_{\{i_1,i_4\}})(1-q_{\{i_1,i_2,i_4\}})(1-q_{\{i_1,i_3,i_4\}}),\\ V^m_{ijk} :=& X^{ijkm} - q_{mi}q_{mj}q_{mk}q_{\{i,j,k\}}\, X^{mijk},\\ W^m_{ijk} :=& X^{ijkm} - q_{mi}q_{mj}q_{mk}\, X^{mijk}\end{aligned}$$

with $X^{ijkm} = e_{ijkm} - q_{ji}q_{ki}q_{kj}q_{mi}q_{mj}q_{mk}e_{mkji}$ (c.f. (12)).

2) $(Q;T)$-constants:

a) Let $|T| = 2$, $T = \{l_i, l_j\} \subset Q$ and let $\{l_k, l_m\} = Q\setminus T$. Then, by using the additional condition $q_{\{l_i,l_j\}} = 1$, the two expressions $C_{l_1l_2l_3l_4}$ and $C_{l_1l_2l_4l_3}$ turn out to be proportional. But we have two independent constants given by simpler expressions as the following iterated $\boldsymbol{q}$-commutators $Y_{l_il_jl_kl_m}$ and $Y_{l_il_jl_ml_k}$ (c.f. (14)).

b) Let $|T| = 3$, $T = \{l_i, l_j, l_k\} \subset Q$ and let $\{l_m\} = Q\setminus T$. Then, by using the additional condition $q_{\{l_i,l_j,l_k\}} = 1$, the two expressions $C_{l_1l_2l_3l_4}$ and $C_{l_1l_2l_4l_3}$ turn out to be proportional. In this case we obtain one constant given by simpler expression $\left[C_{l_il_jl_k}, e_{l_m}\right]_{q_{l_ml_i}q_{l_ml_j}q_{l_ml_k}}$, where $C_{l_il_jl_k}$ is given by (29). Thus the space $\mathcal{C}_{(Q;T)}$ is one-dimensional.

**Remark 11.** According to Remark 10 for $Q = \{l_1, l_2, l_3\}$, $(l_1 < l_2 < l_3)$ under the $Q$-cocycle condition, if we take the following three linearly independent vectors $y_1, y_2, y_3 \in \ker\left(id - T_{2,1}^2 T_{3,2}\right)$ given by:
$y_1 = e_{l_1 l_2 l_3} + q_{l_3 l_2} q_{\{l_1,l_3\}}\, e_{l_1 l_3 l_2}$, $y_2 = e_{l_3 l_1 l_2} + q_{l_2 l_1} q_{\{l_2,l_3\}}\, e_{l_3 l_2 l_1}$, $y_3 = e_{l_2 l_3 l_1} + q_{l_1 l_3} q_{\{l_2,l_1\}}\, e_{l_1 l_2 l_3}$, then their images $x_i$ under the correspondence (24) in Remark 10 give the following three constants

$D_{l_1 l_2 l_3} = (1 - q_{\{l_1,l_3\}}) Y_{l_1 l_2 l_3} + q_{l_3 l_2} q_{\{l_1,l_3\}} (1 - q_{\{l_1,l_2\}}) Y_{l_1 l_3 l_2}$,
$D_{l_2 l_3 l_1} = (1 - q_{\{l_1,l_2\}}) Y_{l_2 l_3 l_1} + q_{l_1 l_3} q_{\{l_1,l_2\}} (1 - q_{\{l_2,l_3\}}) Y_{l_2 l_1 l_3}$,
$D_{l_3 l_1 l_2} = (1 - q_{\{l_2,l_3\}}) Y_{l_3 l_1 l_2} + q_{l_2 l_1} q_{\{l_2,l_3\}} (1 - q_{\{l_1,l_3\}}) Y_{l_3 l_2 l_1}$

written in terms of **q**-iterated commutators (c.f. (14)). It is easy to check that all three constants above are proportional i.e $D_{l_2 l_3 l_1} = q_{l_1 l_2}/q_{l_3 l_1} D_{l_1 l_2 l_3}$, $D_{l_3 l_1 l_2} = q_{l_2 l_3}/q_{l_3, l_1} D_{l_1 l_2 l_3}$, hence the space of constants is one-dimensional. Therefore, we can take that a basic constant in $\mathcal{C}_{l_1 l_2 l_3}$ is given by $D_{l_1 l_2 l_3} = (1 - q_{\{l_1,l_3\}}) Y_{l_1 l_2 l_3} + q_{l_3 l_2} q_{\{l_1,l_3\}} (1 - q_{\{l_1,l_2\}}) Y_{l_1 l_3 l_2}$ (compare with (29)).

Similarly, for $Q = \{l_1, l_2, l_3, l_4\}$, $(l_1 < l_2 < l_3 < l_4)$ under the $Q$-cocycle condition there are eight linearly independent vectors $y_{\mathbf{j}} \in \ker\left(id - T_{2,1}^2 T_{4,2}\right)$ given by $y_{\mathbf{j}} = \left(id - T_{2,1}^2 T_{4,2} + \left(T_{2,1}^2 T_{4,2}\right)^2\right) e_{\mathbf{j}}$ for $\mathbf{j} = l_1 l_2 l_3 l_4$; $l_1 l_2 l_4 l_3$; $l_2 l_1 l_3 l_4$; $l_2 l_1 l_4 l_3$; $l_3 l_1 l_2 l_4$; $l_3 l_1 l_4 l_2$; $l_4 l_1 l_2 l_3$; $l_4 l_1 l_3 l_2$.

Their images $x_i$ under the correspondence (24) in Remark 10 give the following two basic constants (written in terms of **q**-iterated commutators (c.f. (14)):

$$\begin{aligned}
D_{l_1 l_2 l_3 l_4} =& \xi_{l_1 l_2 l_3 l_4} Y_{l_1 l_2 l_3 l_4} + q_{l_4 l_3} q_{\{l_1,l_2,l_4\}} \xi_{l_1 l_2 l_4 l_3} Y_{l_1 l_2 l_4 l_3} \\
&+ q_{l_3 l_2} q_{\{l_1,l_3\}} \xi_{l_1 l_3 l_2 l_4} Y_{l_1 l_3 l_2 l_4} + q_{l_3 l_2} q_{l_4 l_2} q_{\{l_1,l_3,l_4\}} \xi_{l_1 l_3 l_4 l_2} Y_{l_1 l_3 l_4 l_2} \\
&+ q_{l_4 l_2} q_{l_4 l_3} q_{\{l_1,l_4\}} \xi_{l_1 l_4 l_2 l_3} Y_{l_1 l_4 l_2 l_3} \\
&+ q_{l_3 l_2} q_{l_4 l_2} q_{l_4 l_3} q_{\{l_1,l_4\}} q_{\{l_1,l_3,l_4\}} \xi_{l_1 l_4 l_3 l_2} Y_{l_1 l_4 l_3 l_2}; \\
D_{l_1 l_2 l_4 l_3} =& q_{l_3 l_4} q_{\{l_1,l_2,l_3\}} \xi_{l_1 l_2 l_3 l_4} Y_{l_1 l_2 l_3 l_4} + \xi_{l_1 l_2 l_4 l_3} Y_{l_1 l_2 l_4 l_3} \\
&+ q_{l_3 l_2} q_{l_3 l_4} q_{\{l_1,l_3\}} \xi_{l_1 l_3 l_2 l_4} Y_{l_1 l_3 l_2 l_4} \\
&+ q_{l_3 l_2} q_{l_3 l_4} q_{l_4 l_2} q_{\{l_1,l_3\}} q_{\{l_1,l_3,l_4\}} \xi_{l_1 l_3 l_4 l_2} Y_{l_1 l_3 l_4 l_2} \\
&+ q_{l_4 l_2} q_{\{l_1,l_4\}} \xi_{l_1 l_4 l_2 l_3} Y_{l_1 l_4 l_2 l_3} + q_{l_3 l_2} q_{l_4 l_2} q_{\{l_1,l_3,l_4\}} \xi_{l_1 l_4 l_3 l_2} Y_{l_1 l_4 l_3 l_2}.
\end{aligned}$$

where $\xi_{i_1 i_2 i_3 i_4} := (1 - q_{\{i_1,i_3\}})(1 - q_{\{i_1,i_4\}})(1 - q_{\{i_1,i_2,i_4\}})(1 - q_{\{i_1,i_3,i_4\}})$.

Note that the last two basic constants are proportional respectively to corresponding the basic constants $C_{l_1 l_2 l_3 l_4}$, $C_{l_1 l_2 l_4 l_3}$.

### 4.2. Degenerate Cases

Here we consider $Q$-cocycle conditions $\beta_Q = 0$ for some special $Q$ of the following types:

**Case 1** $1 + q_{ii} + \cdots + q_{ii}^{m-1} \,(= [m]_{q_{ii}}) = 0$, if $Q = i^m$ (c.f. (9)).

**Case 2** $q_{ii}^{m-1} q_{\{i,j\}} = 1$, if $Q = i^m j$.

**Case 3** $q_{ii} q_{jj} q_{\{i,j\}}^2 = -1$, if $Q = i^2 j^2$.

**Case 4** $q_{ii}^2 q_{\{i,j\}}^2 q_{\{i,k\}}^2 q_{\{j,k\}} = 1$, if $Q = i^2 jk$.

### 4.2.1. Basic Constants in the Weight Subspaces $\mathcal{B}_{l_1^n}$, $l_1 \in \mathcal{N}$, $n \geq 2$

Let $Q = l_1^n$. Then the matrix $\mathrm{B}_{l_1^n}$ of $\partial^Q$ in the monomial basis $\mathbf{B}_{l_1^n} = \{e_{l_1 \ldots l_1}\}$ has determinant $\det \mathrm{B}_{l_1^n} = [n]_{q_{l_1 l_1}}$, so the $Q$-cocycle condition is: $c_{l_1^n} = \left\{[n]_{q_{l_1 l_1}} = 0\right\}$. If $c_{l_1^n}$ holds, then $\mathcal{C}_{l_1^n} = \mathbb{C}\left\{e_{l_1^n}\right\}$ i.e $\dim \mathcal{C}_{l_1^n} = 1$, otherwise $\mathcal{C}_{l_1^n} = \{0\}$.

### 4.2.2. Basic Constants in the Weight Subspaces $\mathcal{B}_{l_1^k l_2}$, $\mathcal{B}_{l_1 l_2^k}$, $k \geq 2$

We shall elaborate only the case $Q = l_1^k l_2$, $l_1 < l_2$, $k \geq 2$. The matrix $\mathrm{B}_{l_1^k l_2}$ of $\partial^Q$ w.r.t the monomial basis $\mathbf{B}_{l_1^k l_2} = \{e_{l_1 \ldots l_1 l_2}, e_{l_1 \ldots l_2 l_1}, \ldots, e_{l_2 l_1 \ldots l_1}\}$ has determinant

$$\det \mathrm{B}_{l_1^k l_2} = [k]_{q_{l_1 l_1}}! \cdot \prod_{i=1}^{k} \left(1 - q_{l_1 l_1}^{i-1} q_{\{l_1, l_2\}}\right), \tag{30}$$

(c.f. formula (13) in [1, Section 6]), so the $Q$-cocycle condition is given by $c_{l_1^k l_2} = \Big\{q_{l_1 l_1}^{k-1} q_{\{l_1, l_2\}} = 1,\ q_{l_1 l_1}^{i-1} q_{\{l_1, l_2\}} \neq 1,\ 1 \leq i \leq k-1,\ [j]_{q_{l_1 l_1}} \neq 0,$
$2 \leq j \leq k\Big\}$. If the condition $c_{l_1^k l_2}$ holds, then it is easy to check that the space $\mathcal{C}_{l_1^k l_2}$ is one-dimensional with the bases given by the iterated $\boldsymbol{q}$-commutator $Y_{l_2 l_1^k}$. To ilustrate this we take $k = 2$. Then $Q = l_1^2 l_2$ and the matrix $\mathrm{B}_{l_1 l_1 l_2}$ of $\partial^Q$ w.r.t basis $\mathbf{B}_{l_1 l_1 l_2} = \{e_{l_1 l_1 l_2}, e_{l_1 l_2 l_1}, e_{l_2 l_1 l_1}\}$ is given by

$$\mathrm{B}_{l_1 l_1 l_2} = \begin{pmatrix} 1 + q_{l_1 l_1} & q_{l_1 l_1} q_{l_1 l_2} & 0 \\ 0 & 1 & q_{l_1 l_2}(1 + q_{l_1 l_1}) \\ q_{l_2 l_1}^2 & q_{l_2 l_1} & 1 \end{pmatrix}$$

and its determinant is equal to $\det \mathrm{B}_{l_1 l_1 l_2} = (1 + q_{l_1 l_1})(1 - q_{\{l_1, l_2\}})(1 - q_{l_1 l_1} q_{\{l_1, l_2\}})$. The nullspace of the matrix $\mathrm{B}_{l_1 l_1 l_2}$ one obtains by solving the following system of equations:

$$\begin{aligned} (1 + q_{l_1 l_1})\, \alpha_{112} + q_{l_1 l_1} q_{l_1 l_2}\, \alpha_{121} &= 0 \\ \alpha_{121} + q_{l_1 l_2}(1 + q_{l_1 l_1})\, \alpha_{211} &= 0 \end{aligned}$$

$$q_{l_2l_1}^2\,\alpha_{112}+q_{l_2l_1}\,\alpha_{121}+\alpha_{211}=0$$

If $q_{l_1l_1}\neq -1$ we can take $\alpha_{211}$ as a free variable, then we get $\alpha_{112}=q_{l_1l_1}q_{l_1l_2}^2\,\alpha_{211}$, $\alpha_{121}=-q_{l_1l_2}(1+q_{l_1l_1})\,\alpha_{211}$. Hence

$$\begin{aligned}q_{l_1l_1}q_{l_1l_2}^2\,e_{112}-q_{l_1l_2}(1+q_{l_1l_1})\,e_{121}+e_{211}&=\Big[[e_{l_2},e_{l_1}]_{q_{l_1l_2}},e_{l_1}\Big]_{q_{l_1l_2}q_{l_1l_1}}\\&=[Y_{l_2l_1},e_{l_1}]_{q_{l_1l_2}q_{l_1l_1}}=Y_{l_2l_1l_1}\ (=Y_{l_2l_1^2})\end{aligned}$$

(c.f. (14)) is a basic constant when $q_{l_1l_1}q_{\{l_1,l_2\}}=1$ or $q_{\{l_1,l_2\}}=1$. On the other hand, in the case $q_{l_1l_1}=-1$ we obtain $\alpha_{121}=0$, so here we can take $\alpha_{112}$ as a free variable. Then we get $\alpha_{211}=-q_{l_2l_1}^2\,\alpha_{112}$. Hence $e_{112}-q_{l_2l_1}^2\,e_{211}=[e_{l_1l_1},e_{l_2}]_{q_{l_2l_1}^2}=X_{l_1l_1l_2}$ (c.f. (13)) is a basic constant if $q_{l_1l_1}=-1$. Note that under the $Q$-cocycle condition

$$c_{l_1^2l_2}=\big\{q_{l_1l_1}q_{\{l_1,l_2\}}=1,\ q_{\{l_1,l_2\}}\neq 1,\ q_{l_1l_1}\neq -1\big\}$$

the space $\mathcal{C}_{l_1^2l_2}$ (of $Q$-constants) is one-dimensional, where the iterated $\boldsymbol{q}$-commutator $Y_{l_2l_1^2}$ is a basic constant. Similarly, we can show that the space $\mathcal{C}_{l_1^kl_2}$, $k\geq 2$ is one-dimensional with a basic constant $Y_{l_2l_1^k}$.

Now in special cases $k=2,3$ we elaborate $(Q;T)$-constants.

(1) In case $Q=l_1^2l_2$ we have two subcases $T=l_1^2$ and $T=l_1l_2$, ($|T|=2$).

a) Let $T=l_1^2$. Then, by using the additional condition $q_{l_1l_1}=-1$, the basic constant $Y_{l_2l_1^2}$ can be written as the simple $\boldsymbol{q}$-commutator

$$X_{l_1l_1l_2}=[e_{l_1l_1},e_{l_2}]_{q_{l_2l_1}^2}.$$

(Compare above given commutators $Y_{l_2l_1l_1}$, $X_{l_1l_1l_2}$. Note that $q_{l_1l_1}q_{\{l_1,l_2\}}=1$ and $q_{l_1l_1}=-1$ imply $q_{\{l_1,l_2\}}=-1$, so we can take $-q_{l_2l_1}^2=1/q_{l_1l_1}q_{l_1l_2}^2$).

b) In the case $T=l_1l_2$, where we use the additional condition $q_{\{l_1,l_2\}}=1$, the basic constant $Y_{l_2l_1^2}$ simplifies to $[Y_{l_2l_1},e_{l_1}]_{q_{l_1l_2}}$.

(Note that $q_{l_1l_1}q_{\{l_1,l_2\}}=1$ and $q_{\{l_1,l_2\}}=1$ imply $q_{l_1l_1}=1$).

(2) In case $Q=l_1^3l_2$ we have four subcases $T=l_1^3$, $T=l_1^2$, $T=l_1^2l_2$, $T=l_1l_2$.

a) Let $T = l_1^3$. Then, by using the additional condition $1+q_{l_1l_1}+q_{l_1l_1}^2 = 0$, the basic constant $Y_{l_2l_1^3} = e_{2111} - q_{l_1l_2}(1+q_{l_1l_1}+q_{l_1l_1}^2)\, e_{1211} + q_{l_1l_1} q_{l_1l_2}^2 (1+q_{l_1l_1}+q_{l_1l_1}^2)\, e_{1121} - q_{l_1l_1}^3 q_{l_1l_2}^3\, e_{1112}$ can be written as the simple $\boldsymbol{q}$-commutator $X_{l_1l_1l_1l_2} = [e_{l_1l_1l_1}, e_{l_2}]_{q_{l_2l_1}^3}$.

(Note that $1 + q_{l_1l_1} + q_{l_1l_1}^2 = 0$ implies $q_{l_1l_1}^3 = 1$, so we can take $q_{l_2l_1}^3 = 1/q_{l_1l_1}^3 q_{l_1l_2}^3$).

b) Let $T = l_1^2$. Then, by using the additional condition $q_{l_1l_1} = -1$, the basic constant $Y_{l_2l_1^3}$ simplifies to $[Y_{l_2l_1}, e_{l_1l_1}]_{q_{l_1l_2}^2}$.

c) In the case $T = l_1^2 l_2$ with the additional condition $q_{l_1l_1} q_{\{l_1,l_2\}} = 1$, the basic constant $Y_{l_2l_1^2}$ simplifies to $\Big[[Y_{l_2l_1}, e_{l_1}]_{q_{l_1l_2}}, e_{l_1}\Big]_{q_{l_1l_2}}$.

d) In the case $T = l_1 l_2$, where we use the additional condition $q_{\{l_1,l_2\}} = 1$, the basic constant $Y_{l_2l_1^2}$ simplifies to $[Y_{l_2l_1l_1}, e_{l_1}]_{q_{l_1l_2}}$.

Similarly, we compute the basic constants in the weight subspaces $\mathcal{B}_{l_1 l_2^k}$, $k \geq 2$.

#### 4.2.3. Basic Constants in the Weight Subspaces $\mathcal{B}_{\mathsf{l}_1^2\mathsf{l}_2^2}$, $l_1, l_2 \in \mathcal{N}$, $l_1 \neq l_2$

The matrix $\mathrm{B}_{l_1^2l_2^2}$ of $\partial^Q$ w.r.t $\mathbf{B}_{l_1^2l_2^2} = \{e_{l_1l_1l_2l_2}, e_{l_1l_2l_1l_2}, e_{l_1l_2l_2l_1}, e_{l_2l_1l_1l_2}, e_{l_2l_1l_2l_1}, e_{l_2l_2l_1l_1}\}$ has determinant

$$\det \mathrm{B}_{l_1^2l_2^2} = (1+q_{l_1l_1})(1+q_{l_2l_2})\left(1-q_{\{l_1,l_2\}}\right)^2 \left(1-q_{l_1l_1}q_{\{l_1,l_2\}}\right) \left(1-q_{l_2l_2}q_{\{l_1,l_2\}}\right)\left(1+q_{l_1l_1}q_{l_2l_2}q_{\{l_1,l_2\}}^2\right).$$

Under the $Q$-cocycle condition $c_{l_1^2l_2^2} = \Big\{q_{l_1l_1}q_{l_2l_2}q_{\{l_1,l_2\}}^2 = -1, q_{\{l_1,l_2\}} \neq 1, q_{l_jl_j}q_{\{l_1,l_2\}} \neq 1, 1+q_{l_jl_j} \neq 0, j=1,2\Big\}$ we obtain a basic constant

$$\begin{aligned} C_{l_1^2l_2^2} =& q_{l_1l_2}q_{l_1l_1}q_{l_2l_2}\left(1-q_{\{l_1,l_2\}}\right) X^{l_1l_1l_2l_2} - (1+q_{l_1l_1})(1+q_{l_2l_2}) X^{l_1l_2l_1l_2} \\ &+ q_{l_2l_1}q_{l_2l_2}(1+q_{l_1l_1})\left(1-q_{l_1l_1}q_{\{l_1,l_2\}}\right) e_{l_1l_2l_2l_1} \\ &+ q_{l_2l_1}q_{l_1l_1}(1+q_{l_2l_2})\left(1-q_{l_2l_2}q_{\{l_1,l_2\}}\right) e_{l_2l_1l_1l_2} \end{aligned}$$

and the space of $Q$-constants is one-dimensional. Now we elaborate $(Q;T)$-constants.

$(i)$ Let $T=l_i^2$, $i\in\{1,2\}$. Then, by using the additional condition $q_{l_il_i}=-1$, the basic constant $C_{l_1^2l_2^2}$ simplifies to the iterated $\boldsymbol{q}$-commutator $Y_{l_il_il_jl_j}$, $\{i,j\}=\{1,2\}$.

$(ii)$ Let $T=l_1l_2$. Then, by using the additional condition $q_{\{l_1,l_2\}}=1$, the basic constant $C_{l_1^2l_2^2}$ simplifies to the following product $Y_{l_2l_1}\cdot Y_{l_2l_1}$ or to $\left[[e_{l_1},Y_{l_1l_2}]^+_{q_{l_2l_1}q_{l_2l_2}},e_{l_2}\right]_{q^2_{l_2l_1}q_{l_2l_2}}$, where $[x,y]^+_q=xy+qyx$ denote the well known $q$-anticommutator.

$(iii)$ Let $T=l_i^2l_j$ $\{i,j\}=\{1,2\}$. Then, by using the additional condition $q_{l_il_i}q_{\{l_1,l_2\}}=1$, the basic constant $C_{l_1^2l_2^2}$ can be written as the iterated $\boldsymbol{q}$-commutator $Y_{l_jl_il_il_j}$.

### 4.2.4. Basic Constants in the Weight Subspaces $\mathcal{B}_{l_1^2l_2l_3}$, $l_1,l_2,l_3\in\mathcal{N}$, $l_1\neq l_2\neq l_3\neq l_1$

In this case the determinant of the matrix $\mathrm{B}_Q=\mathrm{B}_{l_1^2l_2l_3}$ is given by

$$\begin{aligned}\det \mathrm{B}_{l_1^2l_2l_3}=&\left(1+q_{l_1l_1}\right)^2\left(1-q_{\{l_1,l_2\}}\right)^2\left(1-q_{\{l_1,l_3\}}\right)^2\left(1-q_{\{l_2,l_3\}}\right)\\&\left(1-q_{l_1l_1}q_{\{l_1,l_2\}}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_3\}}\right)\left(1-q_{\{l_1,l_2,l_3\}}\right)\\&\left(1-q^2_{l_1l_1}q^2_{\{l_1,l_2\}}q^2_{\{l_1,l_3\}}q_{\{l_2,l_3\}}\right).\end{aligned}$$

Under the $Q$-cocycle condition $c_{l_1^2l_2l_3}$ we obtain one-dimensional space of $Q$-constants $\mathcal{C}_{l_1^2l_2l_3}=\mathbb{C}\left\{C_{l_1^2l_2l_3}\right\}$ with the basis element

$$\begin{aligned}C_{l_1^2l_2l_3}=&q_{l_1l_1}q^2_{l_1l_2}\left(1-q_{\{l_1,l_3\}}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_3\}}\right)X^{l_1l_1l_2l_3}\\&+q_{l_1l_1}q^2_{l_3l_1}\left(1-q_{\{l_1,l_2\}}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_2\}}\right)X^{l_2l_3l_1l_1}\\&-q_{l_1l_2}\left(1+q_{l_1l_1}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_3\}}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_2\}}q_{\{l_1,l_3\}}\right)X^{l_1l_2l_1l_3}\\&-q_{l_3l_1}\left(1+q_{l_1l_1}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_2\}}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_2\}}q_{\{l_1,l_3\}}\right)X^{l_2l_1l_3l_1}\\&+q_{l_1l_2}q_{l_3l_1}\left(1+q_{l_1l_1}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_3\}}\right)\left(1-q_{l_1l_1}q_{\{l_1,l_2\}}\right)X^{l_1l_2l_3l_1}\\&+\left(1-q_{l_1l_1}q_{\{l_1,l_2\}}q_{\{l_1,l_3\}}\right)\left(1-q^2_{l_1l_1}q_{\{l_1,l_2\}}q_{\{l_1,l_3\}}\right)X^{l_2l_1l_1l_3}.\end{aligned}$$

Now we elaborate $(Q;T)$-constants.

$(a)$ Let $T=l_1l_2l_3$. By using the additional condition $q_{\{l_1,l_2,l_3\}}=1$, the basic constant $C_{l_1^2l_2l_3}$ can be written as the iterated $\boldsymbol{q}$-commutator

$$[C_{l_1l_2l_3},e_{l_1}]_{q_{l_1l_1}q_{l_1l_2}q_{l_1l_3}}$$

(where $C_{l_1l_2l_3}$ is given by (29)).

($b$) Let $T = l_1^2 l_j$, $j \in \{2,3\}$. Then, by using the additional condition $q_{l_1l_1}q_{\{l_1,l_j\}} = 1$, the basic constant $C_{l_1^2 l_2 l_3}$ simplifies to $Y_{l_j l_1 l_1 l_k}$, $\{j,k\} = \{2,3\}$.

($c$) Let $T = l_2 l_3$. By using the additional condition $q_{\{l_2,l_3\}} = 1$, the basic constant $C_{l_1^2 l_2 l_3}$ simplifies to $Y_{l_2 l_3 l_1 l_1}$.

($d$) Let $T = l_1 l_j$, $j \in \{2,3\}$. Then, by using the additional condition $q_{\{l_1,l_j\}} = 1$, the basic constant $C_{l_1^2 l_2 l_3}$ still makes two independent constants $Y_{l_j l_1 l_1 l_k}$ and $Y_{l_j l_1 l_k l_1}$, $\{j,k\} = \{2,3\}$.

($e$) Let $T = l_1^2$. Then, by using the additional condition $q_{l_1l_1} = -1$, the basic constant $C_{l_1^2 l_2 l_3}$ still makes two independent constants $Y_{l_1 l_1 l_2 l_3}$ and $Y_{l_1 l_1 l_3 l_2}$.

## 5. The Relationship between Basic Constants in Generic and Degenerated Subspaces of the Algebra $\mathcal{B}$

In this section, by working under top cocycle condition, we will compute the dimension of the space $\mathcal{C}_Q$ of all constants in the weight subspace $\mathcal{B}_Q$ of $\mathcal{B}$. To achieve this we shall make use of some notations from [3] and some considerations from [6] (c.f. Lemma 1.9.1).

Let $Q = \{l_1 \leq \cdots \leq l_n\} = \left\{k_1^{n_1}, \ldots, k_m^{n_m}, \ldots, k_p^{n_p}\right\}$ be a multiset of cardinality $n\,(= n_1 + \cdots + n_p)$. Then we define the submultisets $Q_{k_m}$, $(1 \leq m \leq p)$ by removing one copy of $k_m$ from $Q$ i.e $Q_{k_m} = Q \backslash \{k_m\} = \left\{k_1^{n_1}, \ldots, k_m^{n_m - 1}, \ldots, k_p^{n_p}\right\}$. Further let $\widehat{Q}_{k_m}$ denotes the set of all multiset permutations of the multiset $Q_{k_m}$.

Let us now assume that $a \colon \widehat{Q} \to \mathbb{C}\backslash\{0\}$ and $b_{k_m} : \widehat{Q}_{k_m} \to \mathbb{C}\backslash\{0\}$, $1 \leq m \leq p$ are functions (analagous to those in [3]) defined by:

$$a\,(j_1 \ldots j_n) = q_{j_n j_1} \cdots q_{j_n j_{n-1}}, \qquad j_1 \ldots j_n \in \widehat{Q}, \tag{31}$$

$$b_{k_m}\left(j_1 \ldots \widehat{k_m} \ldots j_n\right) = q_{k_m j_n}\, a\left(k_m\, j_1 \ldots \widehat{k_m} \ldots j_n\right), \tag{32}$$

$j_1 \ldots \widehat{k_m} \ldots j_n \in \widehat{Q}_{k_m}$ which are called (in [3]) commutation factors. As in Remark 4 we can rewrite (32) as follows

$$b_{k_m}\left(j_1 \ldots \widehat{k_m} \ldots j_n\right) = q_{\{k_m, j_n\}} q_{j_n j_1} \cdots \widehat{q_{j_n k_m}} \cdots q_{j_n j_{n-1}}. \tag{33}$$

### 5.1. Singular Orbits and the Dimension of the Space $\mathcal{C}_{\mathbf{Q}}$

For each $1 \leq i \leq n$, let $\langle t_{i,1}\rangle = \{id, t_{i,1}, (t_{i,1})^2, \dots, (t_{i,1})^{i-1}\}$ be the cyclic subgroup of (the symmetric group) $S_n$ generated by the cycle $t_{i,1} = (1\,2\dots i) \in S_n$ i.e

$$t_{i,1} = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n \\ 2 & 3 & \cdots & i & 1 & i+1 & \cdots & n \end{pmatrix}.$$

Its set of inversions is given by $I(t_{i,1}) = \{(1,i), (2,i), \dots, (i-1,i)\}$. Let us denote by $t_{1,i}$ the inverse of $t_{i,1}$ (i.e $t_{1,i} = (t_{i,1})^{-1}$). Then for each $\underline{j} \in \widehat{Q}$, $1 \leq i \leq n$ we have

$$e_{t_{i,1}\cdot\underline{j}} = e_{j_{t_{1,i}(1)}\cdots j_{t_{1,i}(n)}} \ (= e_{j_{t_{1,i}(1)}}\cdots e_{j_{t_{1,i}(n)}}) \tag{34}$$

(c.f. [6, Sections 1.8]). The $\langle t_{i,1}\rangle$-orbit on $\mathcal{B}_Q$, generated by $e_{j_1\dots j_n}$, $j_1\dots j_n = \underline{j} \in \widehat{Q}$, we denote by

$$\mathcal{B}_Q^{(j_1 j_2\dots j_i)j_{i+1}\dots j_n} := span_{\mathbb{C}}\left\{e_{t_{i,1}^{\alpha}\cdot\underline{j}} \mid 0 \leq \alpha \leq i-1\right\}.$$

These orbits are in one by one correspondence to cyclic $t_{i,1}$-equivalence classes $(j_1 j_2\dots j_i)j_{i+1}\dots j_n$ of the sequences $\underline{j} \in \widehat{Q}$. Notice that $T_{i,1}\left(e_{t_{i,1}^{\alpha}\cdot\underline{j}}\right) = c_\alpha\, e_{t_{i,1}^{\alpha+1}\cdot\underline{j}}$, $0 \leq \alpha \leq i-1$ (see Remark 10), where

$$\begin{aligned}
c_0 &= q_{j_i j_1} q_{j_i j_2} q_{j_i j_3} \dots q_{j_i j_{i-1}} \ (= a\,(j_1 \dots j_i)),\\
c_1 &= q_{j_{i-1} j_i} q_{j_{i-1} j_1} q_{j_{i-1} j_2} \dots q_{j_{i-1} j_{i-2}},\\
c_2 &= q_{j_{i-2} j_{i-1}} q_{j_{i-2} j_i} q_{j_{i-2} j_1} \dots q_{j_{i-2} j_{i-3}},\\
&\ \vdots\\
c_{i-2} &= q_{j_2 j_3} q_{j_2 j_4} q_{j_2 j_5} \dots q_{j_2 j_1},\\
c_{i-1} &= q_{j_1 j_2} q_{j_1 j_3} q_{j_1 j_4} \dots q_{j_1 j_i}.
\end{aligned}$$

(Compare with $c_k$, $0 \leq k \leq b-a$ treated in [6]; here they are modified w.r.t the inverse of $t_{a,b}$ for $a=1$, $b=i$). Hence $T_{i,1}|\mathcal{B}_Q^{(j_1 j_2\dots j_i)j_{i+1}\dots j_n}$ is a cyclic operator such that

$$\det\left(I - T_{i,1}|\mathcal{B}_Q^{(j_1 j_2\dots j_i)j_{i+1}\dots j_n}\right) = 1 - \prod_{0\leq\alpha\leq i-1} c_\alpha.$$

Now it is easy to see that a $\langle t_{i,1}\rangle$-orbit on $\mathcal{B}_Q$, $|Q| = n$ is singular if

$$1 - \prod_{0\leq\alpha\leq i-1} c_\alpha = 0 \tag{35}$$

and it is long singular when $i = n$, where (35) reduces to

$$1 - \prod_{1 \le a \ne b \le n} q_{l_a l_b} = 0. \tag{36}$$

The product runs over all $n \cdot (n-1)$ pairs $(l_a l_b)$ of elements from the multiset $Q$.

Note that (35) represents the top cocycle condition (20). Similarly, in generic cases the appropriate top cocycle condition (27) is represented with (36), because all orbits are long in generic ones.

Assume now that $\langle t_{i,2} \rangle$, $2 \le i \le n$ be the cyclic subgroup of $S_1 \times S_{n-1}$ generated by the cycle $t_{i,2} = (2\,3 \ldots i) \in S_1 \times S_{n-1}$ i.e

$$t_{i,2} = \begin{pmatrix} 1 & 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & 3 & 4 & \cdots & i & 2 & i+1 & \cdots & n \end{pmatrix}.$$

The $\langle t_{2,1}^2 t_{i,2} \rangle$ - orbit on $\mathcal{B}_Q$ we denote by

$$\mathcal{B}_Q^{j_1(j_2 j_3 \ldots j_i) j_{i+1} \ldots j_n} := span_{\mathbb{C}} \left\{ e_{t_{i,2}^{\beta} \cdot \underline{j}} \mid 0 \le \beta \le i-2 \right\}.$$

These orbits are in one by one correspondence to cyclic $t_{i,2}$ - equivalence classes $j_1(j_2 j_3 \ldots j_i) j_{i+1} \ldots j_n$ of the sequences $\underline{j} \in \widehat{Q}$.

Then we have $T_{2,1}^2 T_{i,2} \left( e_{t_{i,2}^{\beta} \cdot \underline{j}} \right) = d_\beta \, e_{t_{i,2}^{\beta+1} \cdot \underline{j}}$, $0 \le \beta \le i-2$, where

$$\begin{aligned} d_0 &= q_{\{j_1, j_i\}} q_{j_i j_2} q_{j_i j_3} q_{j_i j_4} \cdots q_{j_i j_{i-1}}, \\ d_1 &= q_{\{j_1, j_{i-1}\}} q_{j_{i-1} j_i} q_{j_{i-1} j_2} q_{j_{i-1} j_3} \cdots q_{j_{i-1} j_{i-2}}, \\ d_2 &= q_{\{j_1, j_{i-2}\}} q_{j_{i-2} j_{i-1}} q_{j_{i-2} j_i} q_{j_{i-2} j_2} \cdots q_{j_{i-2} j_{i-3}}, \\ &\vdots \\ d_{i-3} &= q_{\{j_1, j_3\}} q_{j_3 j_4} q_{j_3 j_5} q_{j_3 j_6} \cdots q_{j_3 j_2}, \\ d_{i-2} &= q_{\{j_1, j_2\}} q_{j_2 j_3} q_{j_2 j_4} q_{j_2 j_5} \cdots q_{j_2 j_i}. \end{aligned}$$

(Compare with (33)). Here we obtain

$$\det \left( I - T_{2,1}^2 T_{i,2} | \mathcal{B}_Q^{j_1(j_2 j_3 \ldots j_i) j_{i+1} \ldots j_n} \right) = 1 - \prod_{0 \le \beta \le i-2} d_\beta.$$

Similarly as above a $\langle t_{2,1}^2 t_{i,2} \rangle$ - orbit on $\mathcal{B}_Q$ is singular if

$$1 - \prod_{0 \le \beta \le i-2} d_\beta = 0 \tag{37}$$

and it is long singular when (37) reduces to (36).

Hence we can conclude that a $\langle t_{i,1}\rangle$ - orbit resp. $\langle t_{2,1}^2 t_{i,2}\rangle$ - orbit on $\mathcal{B}_Q$ is short singular when l.h.s. of (35) resp. l.h.s. of (37) is nontrivial divisor of l.h.s. of (36).

Let $\mathrm{T}_{k,1}$ denotes the matrix of the operator $T_{k,1}$ resp. $T_{2,1}^2 T_{k,2}$ in the monomial basis $\mathbf{B}_Q$, where I is identity matrix of $T_{1,1}$. Then by using the considerations of Remark 10 we can conclude that under the top cocycle condition it is enough to strudy only the matrices $(\mathrm{I}-\mathrm{T}_{n,1})$, $\left(\mathrm{I}-\mathrm{T}_{2,1}^2\mathrm{T}_{n,2}\right)$. If these matrices were transformed into a block-diagonal matrices, then the number of blocks in a block-diagonal matrix corresponds to the number of distinct singular orbits on $\mathcal{B}_Q$. Let

$\chi_1 =$ the number of distinct singular $\langle t_{n,1}\rangle$ - orbits on $\mathcal{B}_Q$,

$\chi_2 =$ the number of distinct singular $\langle t_{2,1}^2 t_{n,2}\rangle$ - orbits on $\mathcal{B}_Q$.

Then by applying (25) we have that the dimension of $\mathcal{C}_Q$ can be calculate by the formula:

$$\dim \mathcal{C}_Q = \chi_2 - \chi_1. \tag{38}$$

Now we are going to apply the Frønsdal's approach in calculating the dimensions of $\mathcal{C}_Q$ depending on the top cocycle condition (c.f. [3, 3.2.5]). Notice that in that paper all distinct singular orbits on $\mathcal{B}_Q$ are examined, as well as on the weight subspaces $\mathcal{B}_{Q_{k_m}}$, $1 \le m \le p$. Here it is necessary that $\chi$ resp. $\chi_{k_m}$ denotes the number of distinct singular orbits on $\mathcal{B}_Q$ resp. on $\mathcal{B}_{Q_{k_m}}$, $1 \le m \le p$ under top cocycle condition. Then

$$\dim \mathcal{C}_Q = \sum_{1\le m\le p} \chi_{k_m} - \chi \tag{39}$$

where these numbers are

$$\chi = \frac{|\widehat{Q}|}{n} = \frac{(n-1)!}{n_1!\cdots n_p!}, \qquad \chi_{k_m} = \frac{n_m\cdot(n-2)!}{n_1!\cdots n_p!}, \qquad \dim \mathcal{C}_Q = \frac{(n-2)!}{n_1!\cdots n_p!}.$$

when all orbits are long singular. Particulary, if $Q$ is set (i.e $n_m = 1$, for all $m$), then all orbits are long, thus

$$\dim \mathcal{C}_Q = n\cdot(n-2)! - (n-1)! = (n-2)!.$$

In the general case determining the dimension of $\mathcal{C}_Q$ in degenerated cases is more complicated, because some singular orbits can be short. In the following examples we shall determine $\dim \mathcal{C}_Q$ for some multisets of cardinality $n$ depending on the numbers of distinct singular orbits on $\mathcal{B}_Q$ and on $\mathcal{B}_{Q_{k_m}}$. Hence here we will use the Frosdal's approach, where we first assume that $l_1, l_2, l_3 \in \mathcal{N}$, $l_1 \neq l_2 \neq l_3 \neq l_1$ and $n \ge 2$.

**Example 12.** Let $Q = l_1^n$. Then we have one short orbit on $\mathcal{B}_{l_1^n}$ but also on $\mathcal{B}_{l_1^{n-1}}$. The short orbit on $\mathcal{B}_{l_1^n}$ is singular when $1 - q_{l_1 l_1}^{n-1} = 0$ and the short orbit on $\mathcal{B}_{l_1^{n-1}}$ is singular when $1 - q_{l_1 l_1}^{n} = 0$. By applying the well known formula:

$$1 - q^k = (1-q)[k]_q, \qquad (\text{where} \ \ [k]_q = \sum_{i=0}^{k-1} q^i \ \ \text{and} \ \ k \geq 1), \tag{40}$$

on the factors $1 - q_{l_1 l_1}^{n-1} = (1 - q_{l_1 l_1})\, [n-1]_{q_{l_1 l_1}}$, $1 - q_{l_1 l_1}^{n} = (1 - q_{l_1 l_1})\, [n]_{q_{l_1 l_1}}$ is obtained:

- if $1 - q_{l_1 l_1} = 0$, then both orbits are singular ($\chi = \chi_{l_1} = 1$), so $\dim \mathcal{C}_{l_1^n} = 0$;
- if $[n-1]_{q_{l_1 l_1}} = 0$ then the orbit on $\mathcal{B}_{l_1^n}$ is singular, but the orbit on $\mathcal{B}_{l_1^{n-1}}$ is nonsingular ($\chi = 1, \chi_{l_1} = 0$). Hence $\dim \mathcal{C}_{l_1^n} = -1$;
- if $[n]_{q_{l_1 l_1}} = 0$ then the orbit on $\mathcal{B}_{l_1^n}$ is nonsingular, but the orbit on $\mathcal{B}_{l_1^{n-1}}$ is singular. Hence $\chi = 0$, $\chi_{l_1} = 1$ and $\dim \mathcal{C}_{l_1^n} = 1$.

Thus $\dim \mathcal{C}_{l_1^n} = 1$ when $[n]_{q_{l_1 l_1}} = 0$ (c.f. 4.2.1).

**Example 13.** Let $Q = l_1^{n-1} l_2$. Then we have one long orbit on $\mathcal{B}_{l_1^{n-1} l_2}$, but also on $\mathcal{B}_{l_1^{n-2} l_2}$ and one short orbit on $\mathcal{B}_{l_1^{n-1}}$.

The long orbits are singular when $1 - \left(q_{l_1 l_1}^{n-2} q_{\{l_1, l_2\}}\right)^{n-1} = 0$ or by applying (40) when

$$1 - q_{l_1 l_1}^{n-2} q_{\{l_1, l_2\}} = 0 \qquad or \qquad \sum_{0 \leq i \leq n-2} \left(q_{l_1 l_1}^{n-2} q_{\{l_1, l_2\}}\right)^i = 0.$$

The short orbit is singular when $1 - q_{l_1 l_1}^{n-2} q_{\{l_1, l_2\}} = 0$. So we can conclude $\dim \mathcal{C}_{l_1^{n-1} l_2} = 1$ when all orbits are singular i.e if $1 - q_{l_1 l_1}^{n-2} q_{\{l_1, l_2\}} = 0$ (compare with 4.2.2).

On the other hand $\dim \mathcal{C}_{l_1^{n-1} l_2} = 0$ when the short orbit is nonsingular.

**Example 14.** Let $Q = l_1^{n-2} l_2^2$. Depending on parity of $n-2$ we distinguish two cases: (1) $n - 2 = 2k$ and (2) $n - 2 = 2k + 1$ for all $k \geq 0$. In the first case we have the multiset $Q = l_1^{2k} l_2^2$ of the cardinality $2k + 2$ ($k \geq 0$). Hence on $\mathcal{B}_{l_1^{2k} l_2^2}$ there are $k + 1$ orbits, one of them short. We have $k$ orbits on $\mathcal{B}_{l_1^{2k-1} l_2^2}$ and one orbit on $\mathcal{B}_{l_1^{2k} l_2}$, all long. The long orbits are singular when $1 - q_{l_1 l_1}^{k(2k-1)} q_{l_2 l_2} q_{\{l_1, l_2\}}^{2k} = 0$ or $1 + q_{l_1 l_1}^{k(2k-1)} q_{l_2 l_2} q_{\{l_1, l_2\}}^{2k} = 0$

and the short orbit is singular when $1 - q_{l_1 l_1}^{k(2k-1)} q_{l_2 l_2} q_{\{l_1,l_2\}}^{2k} = 0$. If all orbits are singular then by applying (39) we obtain $\dim \mathcal{C}_{l_1^{2k} l_2^2} = k+1-(k+1) = 0$. The space $\mathcal{C}_{l_1^{2k} l_2^2}$ is nonzero only in the case when the short orbit is nonsingular. Here we have $\dim \mathcal{C}_{l_1^{2k} l_2^2} = k+1-k = 1$ when the top cocycle condition $1 + q_{l_1 l_1}^{k(2k-1)} q_{l_2 l_2} q_{\{l_1,l_2\}}^{2k} = 0$ holds.

In the second case we have the multiset $Q = l_1^{2k+1} l_2^2$ of the cardinality $2k+3$ $(k \geq 0)$. Here we get: $k+1$ long orbits on $\mathcal{B}_{l_1^{2k+1} l_2^2}$, $k+1$ orbits, one of them short on $\mathcal{B}_{l_1^{2k} l_2^2}$ and one long orbit on $\mathcal{B}_{l_1^{2k+1} l_2}$. The long orbits are singular when $1 - q_{l_1 l_1}^{k(2k+1)} q_{l_2 l_2} q_{\{l_1,l_2\}}^{2k+1} = 0$ or $1 + q_{l_1 l_1}^{k(2k+1)} q_{l_2 l_2} q_{\{l_1,l_2\}}^{2k+1} = 0$ and the short orbit is singular when $1 - q_{l_1 l_1}^{k(2k+1)} q_{l_2 l_2} q_{\{l_1,l_2\}}^{2k+1} = 0$. $\dim \mathcal{C}_{l_1^{2k+1} l_2^2} = k+2-k-1 = 1$ when all orbits are singular. It can be easily to seen that the top cocycle condition is represent by $1 - q_{l_1 l_1}^{k(2k+1)} q_{l_2 l_2} q_{\{l_1,l_2\}}^{2k+1} = 0$ and the space $\mathcal{C}_{l_1^{2k+1} l_2^2}$ is zero when the short orbit is nonsingular. We can now conclude: $\dim \mathcal{C}_{l_1^{n-2} l_2^2} = 1$ when $1 + (-1)^{n-2} q_{l_1 l_1}^{\binom{n-2}{2}} q_{l_2 l_2} q_{\{l_1,l_2\}}^{n-2} = 0$ (compare with 4.2.3). Here we have used:

$$k(2k-1) = \frac{2k(2k-1)}{2} = \binom{2k}{2};\ k(2k+1) = \frac{(2k+1)(2k)}{2} = \binom{2k+1}{2}.$$

**Example 15.** In the case $Q = l_1^{n-2} l_2 l_3$ all orbits are long. They are singular when $1 - q_{l_1 l_1}^{(n-2)(n-3)} q_{\{l_1,l_2\}}^{n-2} q_{\{l_1,l_3\}}^{n-2} q_{\{l_2,l_3\}} = 0$. We have $\chi = n-1$, $\chi_{l_1} = n-2$, $\chi_{l_2} = \chi_{l_3} = 1$, hence $\dim \mathcal{C}_{l_1^{n-2} l_2 l_3} = 1$. Compare with 4.2.4.

Note that (36) represents 'the generic top cocycle condition'. On the other hand, by a certain specialization procedure from (36) we can obtain the appropriate 'degenerate top cocycle condition' or the values of parameters $q_{ij}$'s for which the space of all constants is zero (c.f. examples 12–15). Therefore, this leads us to the conclusion that 'the degenerate top cocycle condition' can be constructed from some 'generic top cocycle condition'. Thus the basic constants in degenerated $\mathcal{B}_Q$'s can be constructed from those in generic ones.

In accordance with that we can deduce that the fundamental problem for finding the space of all constants in algebra $\mathcal{B}$ can be reduced to the problem of determining the space of all constants belonging to generic weight subspace $\mathcal{B}_Q$ depending only on the top cocycle condition.

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