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GROUP DIVISIBLE DESIGNS WITH TWO ASSOCIATE CLASSES AND WITH TWO UNEQUAL GROUPS

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Abstract: A group divisible design $\mathsf{GDD}(m, n; 3, \lambda_1, \lambda_2)$ is an ordered triple $(V, G, \mathcal{B})$, where V is a $m + n$-set of symbols, G is a partition of V into 2 sets of sizes m, n, each set being called *group*, and $\mathcal{B}$ is a collection of 3-subsets (called *blocks*) of V, such that each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks. In this paper, we find necessary and sufficient conditions for the existence of a $\mathsf{GDD}(m, n; 3, \lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2$.

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1. Introduction

A *balanced incomplete block design* $\mathrm{BIBD}(v, b, r, k, \lambda)$ is a set S of v elements together with a collection $\mathcal{B}$ of b k-subsets of S, called *blocks*, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks. The number $|S| = v$ is called the *order* of the BIBD. Note that there is no condition on the size of the blocks in $\mathcal{B}$. If all blocks are of the same size k, then we have a *Steiner system* $\mathrm{S}(v, k)$. A PBD with index λ can be defined similarly; each pair of distinct elements occurs in λ blocks. If all blocks are same size, say k, then we get a balanced incomplete block design $\mathsf{BIBD}(v, b, r, k, \lambda)$.

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In other words, a $\mathsf{BIBD}(v, b, r, k, \lambda)$ is a set S of v elements together with a collection of b k-subsets of S, called blocks, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks (see [6], [7], [9]).

A *group divisible design* $\mathrm{GDD}(v = v_1+v_2+\cdots+v_g, g, k, \lambda_1, \lambda_2)$ is an ordered triple $(V, G, \mathcal{B})$, where V is a v-set of symbols, G is a partition of V into g sets of sizes $v_1, v_2, \ldots, v_g$, each set being called *group*, and $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of V, such that each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks (see [6], [7]). Elements occurring together in the same group are called *first associates*, and elements occurring in different groups we called *second associates*. We say that the GDD is defined on the set V. The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs [1]. The existence question for $k = 3$ has been solved by Sarvate, Fu and Rodger (see [6], [7]) when all groups are the same size.

In this paper, we continue to focus on blocks of size 3, solving the problem when the required designs having two groups of unequal size, namely, we consider the problem of determining necessary conditions for an existence of $\mathrm{GDD}(v = m + n, 2, 3, \lambda_1, \lambda_2)$ and prove that the conditions are sufficient. Since we are dealing on GDDs with two groups and block size 3, we will use $\mathrm{GDD}(m, n; \lambda_1, \lambda_2)$ for $\mathrm{GDD}(v = m + n, 2, 3, \lambda_1, \lambda_2)$ from now on, and we refer to the blocks as *triples*. We denote $(X, Y; \mathcal{B})$ for a $\mathrm{GDD}(m, n; \lambda_1, \lambda_2)$ if X and Y are m-set and n-set, respectively. Chaiyasana, Hurd, Punnim and Sarvate [2] have written a paper in this direction. In particular, they have completely solved the problem of determining all pairs of integers (n, λ) in which a $\mathrm{GDD}(1, n; 1, \lambda)$ exists. More work intends to solve the existence problem of a $\mathrm{GDD}(m, n; \lambda_1, \lambda_2)$ for possible m, n, λ_1 and λ_2. Lapchinda and Pabhapote [8] solved the problem when the designs have unequal sizes and $\lambda_1 - \lambda_2 = 1$. In [10], Pabhapote and Punnim solved the existence of a $\mathrm{GDD}(m, n; \lambda, 1)$. Recently, the existence of the design $\mathrm{GDD}(m, n; \lambda, 2)$ is completed solved when $\lambda \geq 2$, see [11]. Also Chaiyasana and Pabhapote [4] have completely solved the problem of determining all triples of integers (m, n, λ) in which a $\mathrm{GDD}(m, n; \lambda, 3), \lambda \geq 3$, exists. Moreover, Uiyyasathian and Pabhapote [12] found all triples of integers (m, n, λ) in which a $\mathrm{GDD}(m, n; \lambda, 4), \lambda \geq 4$, exists and in [3], Chaiyasana and Lapchinda investigate all triples of integers (m, n, λ) in which a $\mathrm{GDD}(m, n; \lambda, 5)$, $\lambda \geq 5$, exists. Analogously, in this paper, we continue to reveal all 4-tuples of integers $(m, n, \lambda_1, \lambda_2)$ in which a $\mathrm{GDD}(m, n; \lambda_1, \lambda_2)$ exists for $\lambda_1 \geq \lambda_2$. When $\lambda \leq \lambda_2$, a construction to prove the sufficiency seems to be much more complicated, which in fact remains an open problem in general case. We will see that necessary conditions on the existence of a $\mathrm{GDD}(m, n; \lambda_1, \lambda_2)$ can be easily obtained by describing it graphically as follows.

Let λK_v denote the graph on v vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A *G-decomposition* of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Thus the existence of a $\mathrm{GDD}(m, n; \lambda_1, \lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. The graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ is of order $m + n$ and size $\lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$. It contains m vertices of degree $\lambda_1(m-1) + \lambda_2 n$ and n vertices of degree $\lambda_1(n-1) + \lambda_2 m$. Thus the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ implies

1. $3 \mid \lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$, and

2. $2 \mid \lambda_1(m-1) + \lambda_2 n$ and $2 \mid \lambda_1(n-1) + \lambda_2 m$.

2. Preliminary Results

We will review some known results concerning triple designs that will be used in the sequel.

Theorem 2.1. (see [9]) *Let v be a positive integer. Then there exists a* $\mathsf{BIBD}(v, 3, 1)$ *if and only if* $v \equiv 1$ *or* $3 \pmod 6$.

A $\mathsf{BIBD}(v, 3, 1)$ is usually called *Steiner triple system* and is denoted by $\mathsf{STS}(v)$. Let $(V, \mathcal{B})$ be an $\mathsf{STS}(v)$ where V is a set of v elements. Then the number of blocks or triples is $b = |\mathcal{B}| = v(v-1)/6$.

The following results on existence of λ-fold triple systems are well known (see, e.g., [9]).

Theorem 2.2. *Let n be a positive integer. Then a* $\mathsf{BIBD}(n, 3, \lambda)$ *exists if and only if λ and n are in one of the following cases:*

1. $\lambda \equiv 0 \pmod 6$ *and* $n \neq 2$,

2. $\lambda \equiv 1$ *or* $5 \pmod 6$ *and* $n \equiv 1$ *or* $3 \pmod 6$,

3. $\lambda \equiv 2$ *or* $4 \pmod 6$ *and* $n \equiv 0$ *or* $1 \pmod 3$, *and*

4. $\lambda \equiv 3 \pmod 6$ *and n is odd.*

The results of [3], [4], [10], [11], and [12] will be useful for obtaining the main Theorem in the last section. Their results will be stated as follows.

Theorem 2.3. (see [10]) *Let m and n be positive integers not equal to 2. Then there exists a* GDD$(m, n, \lambda, 1)$ *if and only if*

1. $3 \mid \lambda[m(m-1)+n(n-1)]+2mn$, *and*
2. $2 \mid \lambda(m-1)+n$ *and* $2 \mid \lambda(n-1)+m$.

Theorem 2.4. (see [11]) *Let m and n be positive integers not equal to 2. Then there exists a* GDD$(m, n, \lambda, 2), \lambda \geq 2$ *if and only if*

1. $3 \mid \lambda[m(m-1)+n(n-1)]+mn$, *and*
2. $2 \mid \lambda(m-1)$ *and* $2 \mid \lambda(n-1)$.

Theorem 2.5. (see [4]) *Let m and n be positive integers not equal to 2. Then there exists a* GDD$(m, n, \lambda, 3), \lambda \geq 3$ *if and only if*

1. $3 \mid \lambda[m(m-1)+n(n-1)]$, *and*
2. $2 \mid \lambda(m-1)+n$ *and* $2 \mid \lambda(n-1)+m$.

Theorem 2.6. (see [12]) *Let m and n be positive integers not equal to 2. Then there exists a* GDD$(m, n, \lambda, 4), \lambda \geq 4$ *if and only if*

1. $3 \mid \lambda[m(m-1)+n(n-1)]+2mn$, *and*
2. $2 \mid \lambda(m-1)$ *and* $2 \mid \lambda(n-1)$.

Theorem 2.7. (see [3]) *Let m and n be positive integers not equal to 2. Then there exists a* GDD$(m, n, \lambda, 5), \lambda \geq 5$ *if and only if*

1. $3 \mid \lambda[m(m-1)+n(n-1)]+mn$, *and*
2. $2 \mid \lambda(m-1)+n$ *and* $2 \mid \lambda(n-1)+m$.

The following notations will be used throughout the paper for our constructions.

1. Let V be a v-set. $\mathsf{BIBD}(V, 3, \lambda)$ can be defined as

$$\mathsf{BIBD}(V, 3, \lambda) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is a } \mathsf{BIBD}(v, 3, \lambda)\}.$$

2. Let X and Y be disjoint sets of cardinality m and n, respectively.

   We define $\mathsf{GDD}(X, Y; \lambda_1, \lambda_2)$ as

$$\mathsf{GDD}(X, Y; \lambda_1, \lambda_2) = \{\mathcal{B} : (X, Y; \mathcal{B}) \text{ is a } \mathsf{GDD}(m, n; \lambda_1, \lambda_2)\}.$$

3. Let λ_1, λ_2 be positive integers. Then the *spectrum of* λ_1, λ_2, denoted by $\mathcal{S}(\lambda_1, \lambda_2)$, is defined by

$$\mathcal{S}(\lambda_1, \lambda_2) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : a\ \mathsf{GDD}(m, n; \lambda_1, \lambda_2) \text{ exists}\}.$$

4. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus " $\cup$ " in our construction will be used for the union of multisets.

3. GDD$(m, n; \lambda, 6)$

Let λ be a positive integer. We consider in this section the problem of determining all pairs of integers (m, n) in which a GDD$(m, n; \lambda, 6), \lambda \geq 6$ exists. Recall that the existence of GDD$(m, n, \lambda, 6)$ implies $3 \mid \lambda[m(m-1) + n(n-1)]$, $2 \mid \lambda(m-1)$ and $2 \mid \lambda(n-1)$.

By solving systems of linear congruences we obtain the following necessary conditions.

Lemma 3.1. *Let t be a non-negative integer:*

(a) *If $(m, n) \in S(6t + 1, 6)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k + 1, 6h + 1\}, \{6k + 1, 6h + 3\}, \{6k + 3, 6h + 3\}\}$.*

(b) *If $(m, n) \in S(6t + 2, 6)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k+1, 6h+1\}, \{6k+1, 6h+3\}, \{6k+1, 6h+4\}, \{6k+1, 6h+6\}, \{6k+3, 6h+3\}, \{6k+3, 6h+4\}, \{6k+3, 6h+6\}, \{6k+4, 6h+4\}, \{6k+4, 6h+6\}, \{6k+6, 6h+6\}\}$.*

(c) *If $(m, n) \in S(6t + 3, 6)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k+1, 6h+1\}, \{6k+1, 6h+3\}, \{6k+1, 6h+5\}, \{6k+3, 6h+3\}, \{6k+3, 6h+5\}, \{6k+5, 6h+5\}\}$.*

(d) *If $(m, n) \in S(6t + 4, 6)$, then the pairs are the same as those of case* (b).

(e) *If $(m, n) \in S(6t + 5, 6)$, then the pairs are the same as those of case* (a).

(f) *If* $(m,n) \in S(6t+6,6)$, *then* m *and* n *can be all positive integers.*

Let m and n be positive integers not equal to 2. Since $m+n \neq 2$, the existence of an BIBD$(m+n,3,6)$ is equivalence to the existence of a GDD$(m,n,6,6)$. Thus we obtain the following result.

Lemma 3.2. *Let* h *and* k *be non-negative integers. Then* $(m,n) \in S(6,6)$,

Let m and n be positive integers not equal to 2. Let X be an m-set and Y be an n-set. Suppose, for fixed positive integer i that GDD$(X,Y;6,6)$, BIBD$(X,3,i)$ and BIBD$(Y,3,i)$ are not empty. Then GDD$(X,Y;6+i,6)$ is not empty. By choosing an appropriate integer i together with results in Theorem 2.2, we obtain the following results.

Lemma 3.3. *Let* h *and* k *be non-negative integers. Then*

(a) $(6k+1,6h+1),(6k+1,6h+3),(6k+3,6h+3) \in S(7,6)$,

(b) $(6k+1,6h+1),(6k+1,6h+3),(6k+3,6h+3) \in S(11,6)$,

(c) $(6k+1,6h+1),(6k+1,6h+3),(6k+1,6h+4),(6k+1,6h+6),(6k+3,6h+3),(6k+3,6h+4),(6k+3,6h+6),(6k+4,6h+4),(6k+4,6h+6),(6k+6,6h+6) \in S(8,6)$,

(d) $(6k+1,6h+1),(6k+1,6h+3),(6k+1,6h+4),(6k+1,6h+6),(6k+3,6h+3),(6k+3,6h+4),(6k+3,6h+6),(6k+4,6h+4),(6k+4,6h+6),(6k+6,6h+6) \in S(10,6)$, *and*

(e) $(6k+1,6h+1),(6k+1,6h+3),(6k+1,6h+5),(6k+3,6h+3),(6k+3,6h+5),(6k+5,6h+5) \in S(9,6)$.

By previous Lemmas and Theorem 2.2(a), we obtain the following Theorem.

Theorem 3.4. *Let* m *and* n *be positive integers not equal to 2. Then there exists a* GDD$(m,n,\lambda,6), \lambda \geq 6$ *if and only if*

1. $3 \mid \lambda[m(m-1)+n(n-1)]$, *and*

2. $2 \mid \lambda(m-1)$ *and* $2 \mid \lambda(n-1)$.

4. GDD$(m, n, \lambda_1, \lambda_2)$

Let λ_1 and λ_2 be positive integers with $\lambda_1 \geq \lambda_2$. Let m and n be positive integers not equal to 2. We consider in this section the problem of determining all pairs of integers (m, n) in which a GDD$(m, n, \lambda_1, \lambda_2)$ exists. Combining results in previous sections, we now have an existence of a GDD$(m, n; r+i, i)$ for $r = 1, 2, \ldots, 5$ and $i = 1, 2, \ldots, 6$ whenever m and n are not equal to 2, so we can readily extend to any λ_1, λ_2 with $\lambda_1 \geq \lambda_2$ by the following Lemma.

Lemma 4.1. *Let m and n be positive integers not equal to 2. If there exists a* GDD$(m, n; r, s)$ *with $r \geq s$, then a* GDD$(m, n; 6u + r, 6v + s), 6u \geq 6v$ *exists*

Proof. Let X be an m-set and Y be an n-set. By assumtion we have GDD$(X, Y; r, s) \neq \emptyset$. Choose $\mathcal{B}_1 \in$ GDD$(X, Y; r, s)$. Since $m + n \neq 2$, by Theorem 2.2(a) there exists $\mathcal{B}_2 \in$ BIBD$(X \cup Y, 3, 6v)$. Also there exist $\mathcal{B}_3 \in$ BIBD$(X, 3, 6(u - v))$ and $\mathcal{B}_4 \in$ BIBD$(Y, 3, 6(u - v))$ since m and n are not equal to 2. It is easy to see that $(X, Y; \mathcal{B})$ forms a GDD$(m, n; 6u + r, 6v + s)$, where $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. □

Finally, we have the main result as in the following theorem.

Theorem 4.2. *Let m and n be positive integers not equal to 2. Then there exists a GDD$(m, n; \lambda_1, \lambda_2), \lambda_1 \geq \lambda_2$ if and only if*

1. $3 \mid \lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$, *and*
2. $2 \mid \lambda_1(m-1) + \lambda_2 n$ *and* $2 \mid \lambda_1(n-1) + \lambda_2 m$.

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References

[1] R.C. Bose, T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *Journal of the American Statistical Association*, **47** (1952), 151-184.

[2] A. Chaiyasena, S.P. Hurd, N. Punnim and D. G. Sarvate, Group Divisible Designs with Two Associattion Classes, to appear in *Journal of Combinatorial Mathematics and Combinatorial Computing.*

[3] A. Chaiyasena, W. Lapchinda, Group divisible designs with two associattion classes and $\lambda_2 = 5$, submitted.

[4] A. Chaiyasena, N. Pabhapote, Group divisible designs with two associattion classes and $\lambda_2 = 3$, *International Journal of Pure and Applied Mathematics*, **71**, No. 3 (2011), 455-463.

[5] S.I. El-Zanati, N. Punnim, C.A. Rodger, Gregarious GDDs with two associate classes, *Graphs and Combinatorics*, **26** (2010), 775 - 780.

[6] H.L. Fu, C.A. Rodger, Group divisible designs with two associate classes: $n = 2$ or $m = 2$, *Journal of Combinatorial Theory, Series A*, **83** (1998), 94-117.

[7] H.L. Fu, C.A. Rodger, D.G. Sarvate, The existence of group divisible designs with first and second associates, having block size 3, *Ars Combinatoria*, **54** (2000), 33-50.

[8] W. Lapchinda, N. Pabhapote, Group divisible designs with two associate classes and $\lambda_1 - \lambda_2 = 1$, *International Journal of Pure and Applied Mathematics*, **54**, No. 4 (2009), 601-608.

[9] C.C. Lindner, C.A. Rodger, *Design Theory*, CRC Press, Boca Raton (1997).

[10] N. Pabhapote, N. Punnim, Group divisible designs with two associate classes and $\lambda_2 = 1$, *International Journal of Mathematics and Mathematical Sciences*, **2011** (2011), Article ID 148580, 10 pages.

[11] C. Uiyyasathian, W. Lapchinda, Group divisible designs with two associate classes and $\lambda_2 = 2$, *International Journal of Pure and Applied Mathematics*, **55**, No. 4 (2009), 561-568.

[12] C. Uiyyasathian, N. Pabhapote, Group divisible designs with two associate classes and $\lambda_2 = 4$, *International Journal of Pure and Applied Mathematics*, **73**, No. 3 (2011), 289-298.