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FROM OLDHAM'S COUPLING TO AIR CONDITIONERS

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Abstract: Oldham's coupling is a really interesting device. It is possible to understand the method with a knowledge of only junior high-school geometry. Compressors in air conditioners also use this idea, where involute curves are used for the teeth of the cogs in the compressor. Mathematics doesn't just reside in textbooks; it's alive in our daily lives.

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1. In a Certain Museum...

Mathematics is not just to be found in textbooks, it exists in our daily lives. It does not just have abstract forms, but is also tangible and constant. Once when I was visiting Kyoto University Museum and wondering whether there were any interesting educational materials, my eyes fell upon a model of a device known as Oldham's coupling and it caught my attention. Oldham's coupling is a design that was imported into Japan through Germany at the time of modernization in the 19th and 20th centuries. Oldham is apparently the name of the man who

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devised it. I was able to touch the object, and I was captivated by its peculiar and wondrous motion.

It has two parallel axes, which are slightly offset. How can the rotation of the left axis be correctly transmitted to the rotation of right axis? The idea of using three cogs may occur to an amateur. There would be two cogs with the same number of teeth, and one for changing the direction of motion. A solution is thus possible with a total of three cogs. However, when the axial distance is too small, correspondingly tiny cogs are required, which is not realistic. There is the so-called ‘universal joint’ used in cars, and while it is possible to solve the problem in this way, the device ends up being rather complicated. It is also possible to apply a belt, but belts stretch, shrink, and wear down, so the rotation is not transmitted correctly. ’Oldham’s coupling’ which I will introduce here, is extremely mathematical, but it does not require high level mathematics. Rather, it is possible to understand the method with a knowledge of only junior high-school geometry.

2. Transmitting Rotation between Parallel Axes

The structure of Oldham’s coupling is shown in Figure 1, which is a reproduction based on the work of Hitoshi Morita (see Morita, 1974).[1]

It is composed of 3 discs, a, b and c. Applying a rotation to disc a or c causes disc b to rotate while sliding with respect to a and c. This mechanism is known as a turning block double-slider crank mechanism, and b is called a double-sliding crank. The discs a and c have grooves cut along their diameters, and as shown in Figure 1(2), b has two corresponding projections which are at right angles to each other, and fit into the grooves in both a and c. When a rotates through a given angle, b and c also rotate through the same angle so the angular velocities of a and c are equal.

The discs a and c on the left and right sides of Oldham’s coupling rotate in circles at the same velocity, but the rotation of disc b is neither circular nor ellipsoidal, and its motion is peculiar. It moves according to intermediate parameters, like a cycloid or trochoid. Let’s confirm this mathematically below.

The 3 discs are modeled as shown in Figure 2. The point on the end of the groove in disc a is denoted P , and the point at the end of the groove in disc c is denoted Q. P rotates with a uniform circular motion about center O_1 with radius r. Q rotates with a uniform circular motion with radius r about O_2, which is only separated from O_1 by a distance d. The coordinates of $P(x, y)$ and $Q(x, y)$ are as follows.

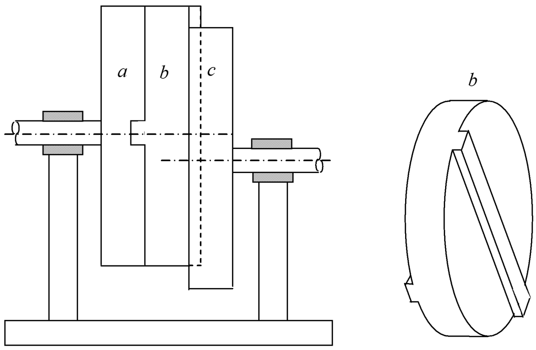

Figure 1: Turning block double-slider mechanism

$$P(r\cos(\frac{\pi}{2}-\theta), r\sin(\frac{\pi}{2}-\theta))$$
$$Q(r\cos(-\theta)+d, r\sin(-\theta))$$

The coordinates of the intersection point between the two perpendicular grooves is denoted $R(x, y)$, and this is the center, O_3, of disc b. Calculation yields

$$R(\frac{d}{2}(1-\cos 2\theta), \frac{d}{2}\sin 2\theta).$$

Point R moves in a uniform circular motion with center $(\frac{d}{2}, 0)$ and radius $\frac{d}{2}$. Also, note that the angular velocity of this point R is twice that of the discs a and c.

The point $S(x, y)$ on the circumference of disc b is therefore as follows.

$$S(x, y) = R(x, y) + P(x, y)$$

Writing the angular velocity as ω, the time as t, and the angle as θ, we have $\theta = \omega t$, and $P(x, y)$ moves with angular velocity ω. The center $R(x, y)$ moves with angular velocity 2ω, so it is clear that $S(x, y)$ does not have a constant angular velocity. It is possible to take the derivative in the x direction and in the y direction with respect to t, but it is not possible to express them with a simple equation in the same way as the cycloid (an equation like $f(x, y) = 0$). They cannot be expressed clearly using formulae, but taking a small value of δt, and numerically calculating the rate of change of $S(x, y)$, denoted $S'(\frac{dx}{dt}, \frac{dy}{dt})$, reveals that the motion does not have uniform angular velocity.

Furthermore, the orthogonality of the grooves is not an essential condition. It is only necessary for there to be an angle between them, and if this is the case then rotational motion can be correctly transmitted between the two axes. We won't delve deeply into this point here.

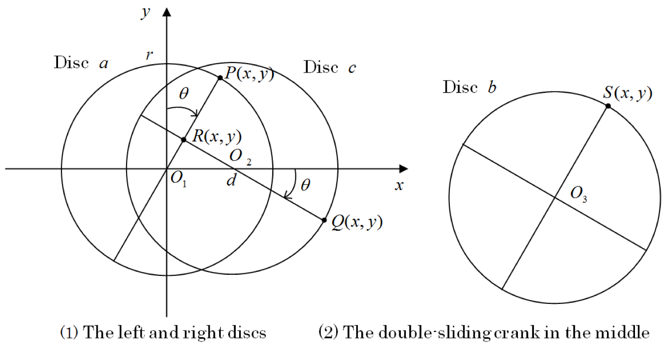

Figure 2: Coordinate system for Oldham's coupling

The way the 3 discs move when θ changes from 0 to $\frac{\pi}{2}$, is summarized in Figure 3. In order to make the nature of the change clear, a point on the discs has been marked with a circle. The discs a and b are shown with the point at the edge of the groove, and disc c is only offset by $\frac{\pi}{2}$. The discs on the left and right, a and c , have their centers fixed so they move with simple circular motions. The centre of the disc in the middle, b, is always moving, so its motion is complicated. The trajectory is neither a circular nor an elliptical orbit. Also, the speed is not constant. Initially, it begins from the position of disc a ($\theta = 0$), and advances while its velocity gradually increases. Finally it overlaps the position of disc c ($\theta = \frac{\pi}{2}$). The center of the curve enclosing the locus of motion for disc b is $(\frac{d}{2}, 0)$, and it is a circle with radius $r + \frac{d}{2}$. Disc b therefore never protrudes outside this circle.

Having expressed the coordinates mathematically, it is possible to derive the formula and find the change of speed, but the equation is awkward so I have omitted it here. Even so, the complicated motion of disc b is surprising.

We were able to confirm the motion of Oldham's coupling mathematically, but does it really move so nicely? I started to want to make a model. First

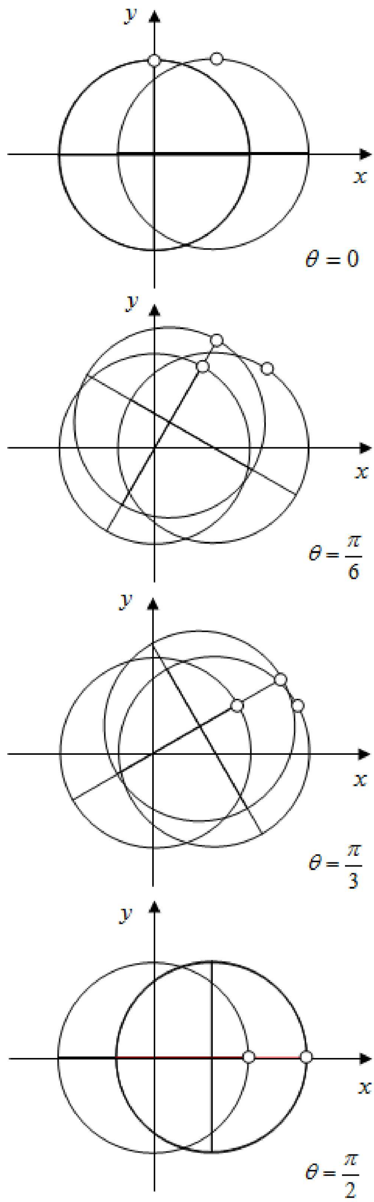

Figure 3: Transition diagram ($0 \leq \theta \leq \dfrac{\pi}{2}$)

I made a model using thick drawing paper, but it didn't move very well. I therefore decided to make a model out of wood. The materials needed for construction were on sale in certain DIY shops, and the model I created and assembled is shown in Figure 4. The cost of materials was about 2000 yen. The concave and convex parts which involve cutting grooves, and the making of protrusions call for precision, so I had an employee at the shop cut them for me. The Oldham's coupling that I saw in the museum was made from metal, but a wooden model was sufficient to reproduce the functionality.

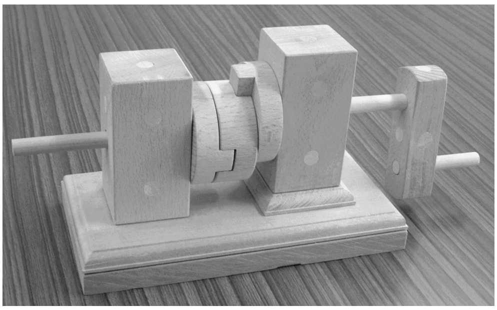

Figure 4: Home-made model of Oldham's coupling

I later had another meeting with Oldham's coupling, when it was introduced at a gathering. Kunio Sugahara from Osaka Kyoiku University taught me that it was possible to make a model out of paper if straws with different diameters are used. Knowing that mathematical ideas are reflected in industrial machinery, I was somehow delighted to have majored in mathematics. If the motion shown in Figure 4 is difficult to follow, you can search for it on the internet and watch an MPEG format animation of Oldham's coupling. I found such animations on two or three sites. It seems that Oldham's coupling is being taught at colleges and in lectures on mechanical engineering.

3. Compressors in Air Conditioners

When I first presented this article in 'Mathematics Seminar', I received letters from many readers (see Nishiyama, 2004).[2] One of these letters revealed to me that "there is an interesting use of Oldham's coupling". This was referring to the use of Oldham's coupling in air conditioners. The principle behind the cooling performed by air conditioners is the compression and subsequent absorption of heat through the expansion of the gas. Air conditioners thus require compressors. These compressors were originally piston systems, but the technology advanced to rotary systems which utilize Reuleaux triangles, and apparently there have been further advances so that present models utilize so-called scroll systems. These scroll systems are not exactly Oldham's couplings, but they use the principle of the double-sliding crank. Scrolling systems produce little vibration or noise, and are also used in car air conditioners.

Why didn't we realize that mathematical ideas are used inside air conditioners? It's because the compressors are precision devices so they are set firmly by casting, and we cannot look inside them. Figure 5 shows that scroll systems are composed of a fixed scroll (in gray) and a mobile scroll (in black) which rotates around the fixed scroll in close contact with it. Gas drawn into the input port is compressed into the center and expelled through the discharge port within around 3 rotations of the mobile scroll.

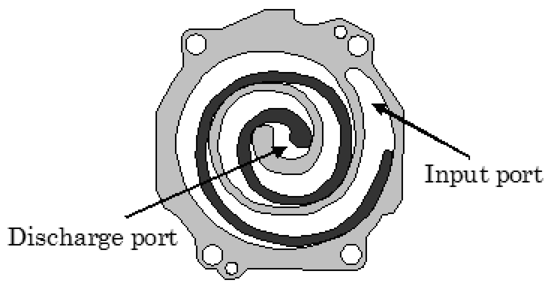

Figure 5: Compressor (Scrolling system)

Recently, so-called business museums have been growing popular, and there are many sites where businesses can introduce their products and permission is granted for general inspection at no charge. I visited the showroom of an air conditioner maker called *Daikin* in Osaka and was shown a model of the scroll system. The motion of this device is also interesting so I recorded an MPEG video using a digital camera. I was doubly surprised to see the link with Oldham's coupling that I had seen at Kyoto University Museum, and also to realize that this device was in use inside air conditioners.

4. The Involute Curves in Cogs

The curves used in the fixed and mobile scrolls are shown in Figure 6. They are known as 'involute' curves, and they are similar to well-known spiral curves but are a little different. The construction of the scroll system shown in Figure 5 can be well understood as a pair of involute curves aligned with an offset of 180° and set in motion in the manner of Oldham's coupling.

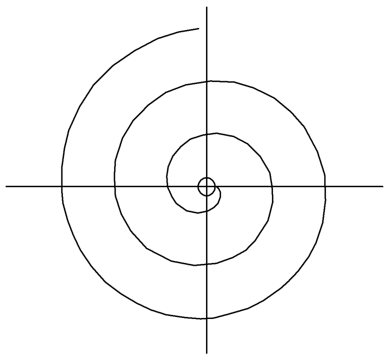

Figure 6: The involute curve

Involute curves were originally devised as a technical renovation of the shape of the teeth on cogs. They are the curve described by the tip of a thread while it is unravelled after having been wound around a circle (see Figure 7). When P is a point on the circumference of a circle O, the points $Q(x, y)$ at a distance $a\theta$ along the tangent from the point P are the coordinates of an involute curve. Expressing this as an equation yields

$$x = a\cos\theta + a\theta\sin\theta = a(\cos\theta + \theta\sin\theta),$$

$$y = a\sin\theta - a\theta\cos\theta = a(\sin\theta - \theta\cos\theta).$$

Cogs transmit motion between two rotating bodies, and the form of their teeth is particularly important. The shape has developed from cycloid or pin forms to the involute form, which is the most suitable form for cogs. The reason is illustrated in Figure 8. The teeth on cog O_1 are AA', and the teeth on cog O_2 are BB'. Suppose they are touching at point P. The contact point P moves from P_0 to P_3 along the mutual tangent line of the two cogs.

When O_1 rotates, the contact point advances to P_2 and the rotation is transmitted to O_2. At this point the form of the teeth is $P_{12}P_2$ and $P_{22}P_2$.

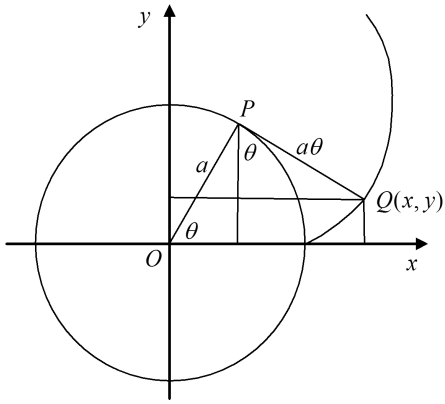

Figure 7: Cogs and involute curves

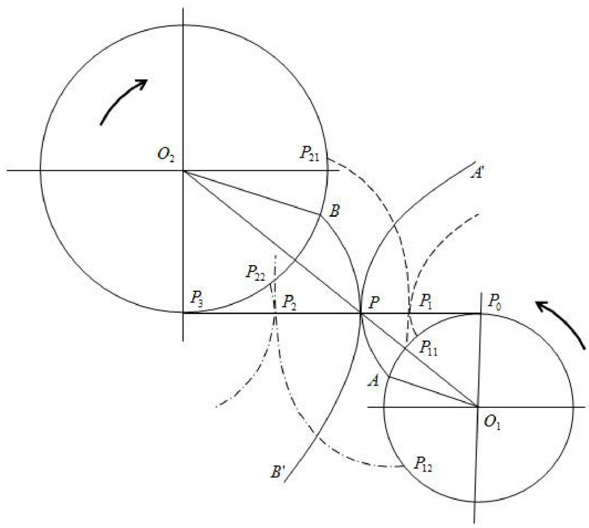

Figure 8: The meshing of involute-curve cog-teeth

Rotating O_1 backwards moves the contact point to P_1, and at that point the form of the teeth is $P_{11}P_1$ and $P_{21}P_1$. The rotational motion of both cogs is transmitted through the contact point. This contact point moves along the mutual tangent line. In order for the contact point to move along a straight line, the form of the teeth must be an involute curve. This is truly mathematical isn't it? Indeed, mathematics doesn't just reside in textbooks, it is alive in our daily lives.

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[1] H. Morita, *Kikogaku* [Mechanics], Tokyo: Jikkyo Press (1974), 158-159.

[2] Y. Nishiyama, Hakubutsukan de mita Oldham Tsugite [A discovery in a museum: Oldham's coupling], *Sugaku Semina* [Mathematics Seminar], **43**, No. 2 (2004), 7-9.