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# MEASURING AREAS: FROM POLYGONS TO LAND MAPS

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**Abstract:** This article explains how to measure area using both simple and complicated measurement methods. Elementary school children obtain area by counting triangles; junior-high school students use Cartesian coordinates; and high-school students study Heron's formula. In this article additional methods such as the trapezoid formula and Amsler's linear planimeter are presented.

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## 1. The Area of a Polygon

What kind of problem is this? Suppose we have an arbitrary polygon, where 'arbitrary' means that the number of vertices can be anything from 3 or more, and the shape does not have to be convex, but can also include concave polygons. Let us assume that the number of vertices and the coordinates of each vertex are known.

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$$P_1(x_1, y_1), P_2(x_2, y_2), \cdots, P_n(x_n, y_n)$$

Now let's try to obtain the area of such a polygon. Any method will do. But we'd like to have a simple algorithm that provides an accurate answer quickly, and ideally, is capable of handling an 'arbitrary' polygon. For example, how about the polygon shown in Figure 1, which has 8 vertices and is concave?

The most elementary method must surely be as follows. The 5th year elementary school arithmetic syllabus includes the concept that "the areas of squares, pentagons, hexagons and so on, can be obtained by dividing them up into a number of triangles". The octagon in Figure 1 can thus be divided up into 6 triangles (Figure 2). Then the area of each triangle can be obtained using the formula "base × height ÷ 2", and the total sum calculated. The line segment indicating the height of each triangle can be drawn nicely with a pair of set-squares, and measured with a ruler. The result will probably not be an exact number and will include some decimals, but it is certainly acceptable for elementary school pupils if they can follow this method, since they can add and multiply decimals.

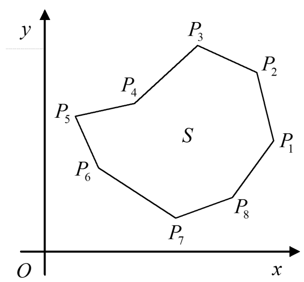


Figure 1: Find the area!

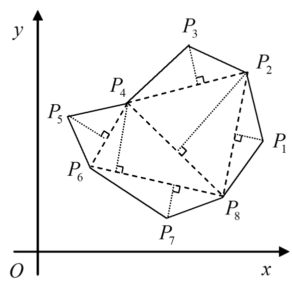


Figure 2: Partitioning into triangles

This most simple method is in fact used for measuring land even today, as the 'triangulation survey' method. However, this is a little too simple so we have another method we can apply. When students reach junior high-school they are expected to work on a larger number of problems on graph paper and they develop an awareness of Cartesian coordinates, so they can draw axes and auxiliary lines beside the axes like those shown in Figure 3. As can be seen at a glance, the area of this polygon can be obtained by subtracting the areas of 8 triangles and 4 rectangles from the area of the outer rectangle. This method does not require measurements with a ruler, and can be obtained simply from the vertex coordinates.

## 2. Heron's Formula

The method described above does not incorporate a notion of generality regarding the 'arbitrary' aspect of an arbitrary polygon. For each different case it's necessary to try and draw auxiliary lines suited to the given diagram. Let's see if we can add generality to this calculation algorithm. Partition the polygon into 6 triangles by drawing auxiliary lines starting with vertex $P_1$ , and passing through each vertex one-by-one from $P_3$ to $P_7$ (Figure 4).

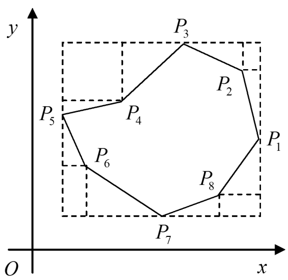


Figure 3: Parallel auxiliary lines

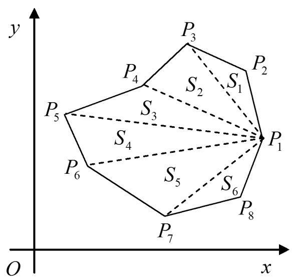


Figure 4: Generalized partitioning

In general, an $n$ sided convex polygon will be partitioned into $n-2$ triangles. When it comes to finding the area of the triangles, there is an equation known as Heron's formula. Although Heron's formula has currently been dropped from the high-school mathematics teaching guidelines, it is remarkably powerful. In short, taking the lengths of the three edges of a triangle as $a, b, c$, the area is

$$S = \sqrt{s(s-a)(s-b)(s-c)} \qquad (1)$$

(where $s = \frac{1}{2}(a+b+c)$).

The lengths of the edges can be obtained without measuring them using a ruler, by means of Pythagoras' theorem. Since the coordinates of each vertex are known, the distance $\overline{P_iP_j}$ between vertices $P_i(x_i, y_i)$ and $P_j(x_j, y_j)$ is

$$\overline{P_iP_j} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \qquad (2)$$

This is an effective method for adding the generality needed to handle $n$ sided polygons. It's not necessary to construct each diagram individually. It is however a little bit tricky in cases when the diagram is too complicated to construct, and the presence of the square root makes paper-and-pencil calculation difficult.

## 3. Applying the Trapezoid Formula

So far, I have explained three methods, but isn't there a method for accurately obtaining the result, which is also simple and quick? The method I will now explain is not taught in high-school mathematics, though it is a rather elegant solution which uses the trapezoid formula. Regardless of whether or not the number of vertices is increased, or whether the polygon is concave, the result can be obtained very quickly using this method.

The principle is simple. A vertical line is dropped down to the $x$ axis from each vertex. These 'legs' are denoted by $H_1, \cdots, H_8$ (Figure 5). The adjacent vertices and their vertical lines form trapezoids. For example, for the vertices $P_1$ and $P_2$, and their perpendicular lines $H_1$ and $H_2$, a trapezoid $P_1H_1P_2H_2$ is formed with upper-base $P_1H_1$, lower-base $P_2H_2$, and height $H_1H_2$. These trapezoids face sideways, and there are 8 of them in total, *i.e.*, there is one trapezoid for each vertex.

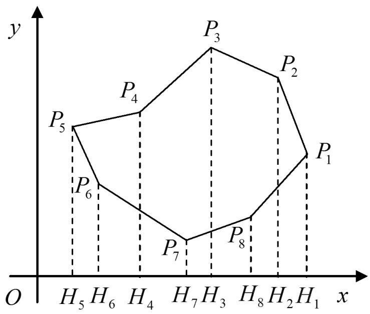


Figure 5: Dropping a vertical line down from each vertex

The vertices are split into two groups, from $P_1$ to $P_5$, and from $P_5$ to $P_1$, and are shown in the diagrams in Figures 6 and 7. The areas of each of these trapezoids can be obtained with "(upper-base + lower-base) × height ÷2 ".

Let's try expressing the areas $S_1$ to $S_8$ using coordinate values. The areas $S_1$ to $S_4$ in Figure 6 are as follows.

$$S_1 = (y_1 + y_2) \times (x_1 - x_2) \div 2,$$

$$S_2 = (y_2 + y_3) \times (x_2 - x_3) \div 2,$$

$$S_3 = (y_3 + y_4) \times (x_3 - x_4) \div 2,$$

$$S_4 = (y_4 + y_5) \times (x_4 - x_5) \div 2. \quad (3)$$

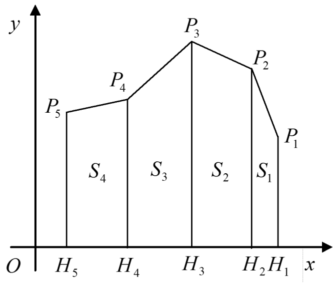


Figure 6: The positive area

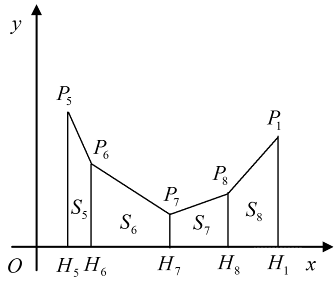


Figure 7: The negative area

The areas $S_5$ to $S_8$ in Figure 7 are as follows.

$$S_5 = (y_5 + y_6) \times (x_6 - x_5) \div 2,$$

$$S_6 = (y_6 + y_7) \times (x_7 - x_6) \div 2,$$

$$S_7 = (y_7 + y_8) \times (x_8 - x_7) \div 2,$$

$$S_8 = (y_8 + y_1) \times (x_1 - x_8) \div 2. \quad (4)$$

As can be understood from Figures 6 and 7, the area of the polygon is

$$S = (S_1 + S_2 + S_3 + S_4) - (S_5 + S_6 + S_7 + S_8). \quad (5)$$

Paying attention to the positively marked area in Figure 6, and the negatively marked area in Figure 7, the area of the polygon itself, $S$ , can be composed as follows.

$$\begin{aligned} S = \frac{1}{2}\{&(y_1 + y_2) \times (x_1 - x_2) + (y_2 + y_3) \times (x_2 - x_3) \\ &+(y_3 + y_4) \times (x_3 - x_4) + (y_4 + y_5) \times (x_4 - x_5) \\ &+(y_5 + y_6) \times (x_5 - x_6) + (y_6 + y_7) \times (x_6 - x_7) \\ &+(y_7 + y_8) \times (x_7 - x_8) + (y_8 + y_1) \times (x_8 - x_1)\} \quad (6) \end{aligned}$$

In this equation, the reason why all the terms on the right hand side are added is because the heights themselves are expressed with a sign, *i.e.*, from $S_1$ to $S_4$ , the heights are positive, but from $S_5$ to $S_8$ the heights are negative, and the corresponding areas themselves are also positive and negative.

In general, the area of an $n$ sided polygon is

$$S = \frac{1}{2}\{(y_1 + y_2)(x_1 - x_2) + (y_2 + y_3)(x_2 - x_3) + \cdots + (y_n + y_1)(x_n - x_1)\} \quad (7)$$

Since the subscripts on the parameters cycle, the program for finding the area is simple. Check for yourself how this works out in Visual Basic or C.

High-school mathematics textbooks contain the following explanation regarding integration. The area enclosed by two curves $y = f(x)$ and $y = g(x)$ is

$$S = \int_a^b \{f(x) - g(x)\}dx. \qquad (8)$$

In the case of the area of a polygon, this is decomposed into

$$S = \int_a^b f(x)dx - \int_a^b g(x)dx, \qquad (8)'$$

and each integral is handled using the trapezoid rule.

## 4. Areas on Maps

Areas on maps are not as easy to calculate as those of polygons. Terrain is expressed not with polygonal lines, but complicated curves. Polygonal lines with many vertices can be used to approximate curved lines but the result is not realistic. One frequently used method is to overlay square sections, and count the number of sections in order to find the area. It is also possible to cut out a paper shape of the terrain using a uniform material, measure the weight on a scale and thus obtain its area.

However, I'd like to introduce a more convenient method. It involves an instrument for measuring area known as a planimeter. The planimeter, which was devised in 1856 by the Swiss mathematician Jacob Amsler (1823-1912), is often used by architectural offices even now and it has a perfectly mathematical basis.

The basic construction of the planimeter is shown in Figure 8. The rods $BA$ and $BO$ are attached in such a way that they can freely rotate around a point. Given a closed curve denoted $\Gamma$ which encloses the area we want to measure, an observation point $O$ outside the curve is fixed. Next, the survey point $A$ follows the curve $\Gamma$ in a clockwise circle. The number of rotations of the attached measurement wheel is read off, and the area can be then be obtained by multiplying this with the length of the pole. Although it may seem like some kind of magic that the area of the closed curve can be found with a single loop, it is in fact similar to the calculation of polygonal areas using trapezoids.

Figure 9 contains a diagram showing that the resulting area can be obtained when the point $A$ follows a loop along $\Gamma$, and the area swept by the rod $AB$ is deductedD

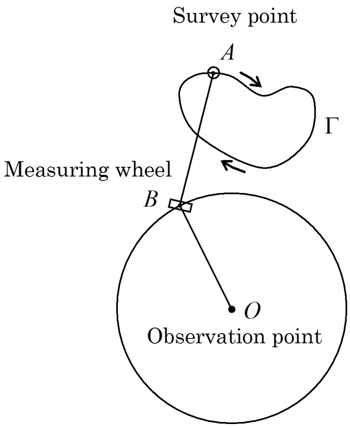


Figure 8: Reading off the number of rotations of the planimeter's measuring wheel

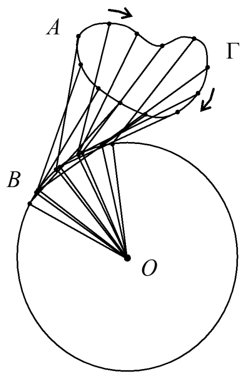


Figure 9: Pruning the area through which the rod $AB$ extends

### 5. The Principle Behind the Planimeter

The planimeter is a rather interesting instrument, so I'd like to briefly touch upon the principle behind it. Suppose that when the survey point $A$ moves to $A'$ the measuring wheel moves from $B$ to $B'$. In general, a movement $AA'$ can be separated into a translation $\overline{AA"}$ and a rotation $\widehat{A"A'}$. The area $dA$ covered when the rod $AB$ with length $l$ translates by an amount $ds$ and rotates by an amount $d\theta$ is

$$dA = lds + l^2 d\theta/2. \qquad (9)$$

Supposing the measuring wheel tracking $B$ rotates by $dn$, then

$$dn = ds. \qquad (10)$$

Substituting (10) into (9),

$$dA = ldn + l^2 d\theta/2. \qquad (9)'$$

At this point I'd like to add a little explanation regarding Equation (10). Figure 10 shows the relationship between $\angle ABO$ and the measuring wheel. Using $\alpha$ to denote the angle formed between the direction in which the measuring wheel rotates and the direction that the rod $AB$ advances, the amount of rotation $dn$ obeys the following relationship.

$$dn = \overset{\frown}{BB'} \cos\alpha \qquad (11)$$

Thus in the case of Figure 10(1), $\alpha = 0°$ and the measuring wheel has made a complete rotation. In case (3) $\alpha = 90°$ so the measuring wheel has not rotated at all. Case (2) is between cases (1) and (3), so the measuring wheel is sliding while it rotates, and has only rotated by $\cos\alpha$.

Now, the area is enclosed by one complete cycle of the closed curve so by integrating Equation (9)' we obtain the following formula.

$$\begin{aligned} S &= \oint dA = \oint l dn + \oint \frac{l^2}{2} d\theta \\ &= l \times n \qquad (12) \end{aligned}$$

$n$ is the number of complete rotations. Also, the integral related to the rotational displacement $d\theta$ is zero for a full cycle, so

$$\oint d\theta = 0, \qquad (13)$$

and the 2nd term disappears. The number of rotations of the measuring wheel is thus proportional to the area.

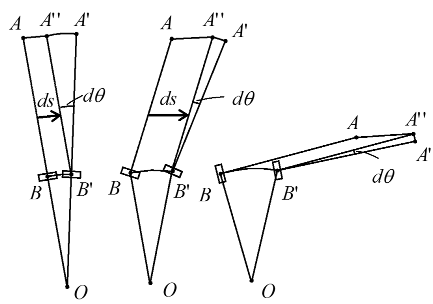


Figure 10: The relationship between angle ABO and the measuring wheel

The planimeter is an interesting instrument and I wanted to investigate it, but buying one can cost tens of thousands of yen. Luckily I managed to pick up an old planimeter much more cheaply through an internet auction. The device shown in Figure 11 is known as a 'polar planimeter', and it operates like that shown in Figure 8. The fixed observation point $O$ is placed outside the map. Then the survey point $A$ is looped around Japan's Lake Biwa in a clockwise fashion, the number of rotations made by the measuring wheel attached to the rod $l$ is then read off, multiplied by the length of the rod , and keeping the map's scale of reduction in mind, the area can thus be obtained. The area of Lake Biwa was obtained with surprising accuracy.

It is also a surprise that this equation is linear. The device shown in Figure 12 does not have an observation point. The area is measured as the planimeter traverses the map surface. The survey point has become a lens looking down from above, and the measuring wheel is attached to the rod. There are wheels attached to both the left and right ends of the rod, which can only move backwards and forwards in a region with a fixed width. This formula is often used today. There are other different formulae for planimeters, but all of them embody mathematical principles.

Figure 11: Polar planimeter

Figure 12: Linear planimeter