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ON SOME APPLICATIONS OF A STRONG LAW OF LARGE NUMBERS FOR A GENERALIZED GAUSS-KUZMIN OPERATOR

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Abstract: Some important applications of a strong law of large numbers for the associated Markov chain of the corresponding random system with complete connections associated with a generalized Gauss-Kuzmin operator are investigated.

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1. Introduction

For every $f \in Y$, where Y = C([0,1]), the Banach space of all complex-valued continuous functions on [0,1] under the supremum norm and for every $\alpha \geq 1$,

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the function $G_\alpha f$ introduced by W. Fluch [1] defined by

$$(G_\alpha f)(w) = \sum_{x \in N\div} \frac{\alpha^2}{(\alpha x + w)(\alpha x + \alpha - 1 + w)} \cdot f\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right), \quad (1.1)$$

for all $w \in [0,1]$, is called *a generalized Gauss -Kuzmin -operator* (see R.O. Kuzmin [17]).

Furthermore, for an arbitrary nonatomic probability measure μ on the σ-algebra $B_{[0,1]}$ of all Borel subsets of [0,1], we define

$$V_0(w) = \mu([0,w]),$$

$$V_{\mathrm{n}}(w) \equiv V_{\mathrm{n}}(w,\mu) = \int_0^w G_\alpha^n\ f\ (t)\, dt, \quad (1.2)$$

where $G_\alpha^n\ f = G_\alpha\left(G_\alpha^{n-1} f\right)$, for every $n \in \aleph^* = \{1, 2, ...\}$, $f \in Y$, $w \in [0,1]$. Assuming that the derivative V_0'exists everywhere in [0,1] and it is bounded (μ has bounded density), we have by induction thatV'_n exists and it is bounded for any n$\in \aleph^*$ with $V'_n\ (w) = (G_\alpha^n f)\ (w)$ such that

$$V'_n\ (w) = \sum_{x \in N\div} \frac{\alpha^2}{(\alpha x + w)(\alpha x + \alpha - 1 + w)} \cdot V'_{n-1}\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right), \quad (1.3)$$

for every $\alpha \geq 1$, $w \in [0,1]$. By considering further the functions g_{n}, for every $\alpha \geq 1$, $n \in \aleph^*$, defined by

$$g_{\mathrm{n}}(w) = \frac{V'_n\,(w)}{\rho_\alpha\,(w)}, \quad (1.4)$$

where $\rho_\alpha\ (\mathrm{w}) = \frac{\alpha}{\alpha + w}$, for every $w \in [0,1]$ and by replacing V'_n by $\rho_\alpha \cdot g_n$ in (1.3), we obtain the following relation

$$g_n\,(w) = \sum_{x \in N\div} \frac{\alpha\,(\alpha + w)}{(\alpha x + w)(\alpha x + \alpha + w)} \cdot g_{n-1}\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right), \quad (1.5)$$

for any $n \in \aleph^*$, $w \in [0,1]$. Equation (1.5) gives us the possibility to obtain that, for every$\alpha \geq 1$, the function

$$P_\alpha(w,x) = \frac{\alpha\,(\alpha + w)}{(\alpha x + w)(\alpha x + \alpha + w)}, \quad w \in [0,1]\,,\ x \in \aleph^*, \quad (1.6)$$

defines a transition probability function from $\left([0,1], B_{[0,1]}\right)$ to $(X, P\,(X))$, where $\mathrm{X} = \aleph^*$ and P(X) the power set of X.

As it was proved in Ch. Ganatsiou [2], equations (1.5), (1.6) lead to the consideration of a family of random systems with complete connections (RSCCs)

$$(W, W), (X, X), u_\alpha, P_\alpha, \quad \alpha \geq 1, \tag{1.7}$$

associated with the Gauss – Kuzmin operator G_α defined by (1.1), where

$$W = [0,1], \quad W = B_{[0,1]}, \quad X = \aleph^*,$$
$$X = P(X) \quad \text{(the power set)},$$
$$u_\alpha(w,x) = \frac{\alpha}{\alpha x + \alpha - 1 + w},$$
$$P_\alpha(w,x) = \frac{\alpha(\alpha + w)}{(\alpha x + w)(\alpha x + \alpha + w)}, \quad w \in W, \ x \in X.$$

By applying the existence Theorem 4.5 of appendix, it follows that, for an arbitrary fixed $w_0 \in W$, there exists a sequence of random variables $(\zeta_n)_n \in \aleph$, $\aleph = \{0,1,2\ldots\}$, with initial distribution concentrated at w_0, called *the associated Markov chain* of the family of RSCCs (1.7). Its corresponding *Markov transition operator* $U_\alpha$is defined by the following equation

$$(U_\alpha f)(w) = \sum_{x \in \aleph} \frac{\alpha(\alpha + w)}{(\alpha x + w)(\alpha x + \alpha + w)} \cdot f\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right) \tag{1.8}$$

for all complex-valued measurable bounded functions f on [0,1].

The present paper arises as an attempt to investigate some important applications of a strong law of large numbers for the associated Markov chain $(\zeta_n)_{n \in \aleph}$ of the corresponding parametric random system with complete connections (1.7) (see Ch. Ganatsiou [3]), for a special case, for every α >2, which extend certain asymptotic formulas related with the ordinary continued fraction expansion of any irrational number y in [0,1] (M. Iosifescu and S. Grigorescu [8], A. Ya. Khinchin [16]). Our approach is given in the context of the theory of dependence with complete connections (M. Iosifescu and S. Grigorescu [8]).

The paper is organized as follows. In Section 2, we present all the necessary results regarding the asymptotic behaviour of the parametric random system with complete connections (1.7), in order to make more comprehensible the presentation of the paper. In Section 3, we obtain two concrete asymptotic results, for a special case, for every α >2, by using a strong law of large numbers for the associated Markov chains of random systems with complete connections. Finally, in Section 4, we give an appendix comprising certain concepts of the theory of random systems with complete connections.

2. Auxiliary Results

The ergodic behaviour of the random system with complete connections (1.7) is expressed by the following (Ch. Ganatsiou [2])

Theorem 2.1. *The family of RSCCs (1.7) associated with the generalized Gauss – Kuzmin operator G_α is with contraction. Moreover, its associated Markov transition operator U_a defined by (1.8) is regular with respect to L([0,1]), the Banach space of all real – valued bounded Lipschitz functions on [0,1].*

By virtue of Theorem 2.1, it follows from (M. Iosifescu and S. Grigorescu [8], Theorem 3.4.5) that the family of RSCCs (1.7) is uniformly ergodic. Then Theorem 4.7 of appendix (see also M. Iosifescu and S. Grigorescu [8], Theorem 3.1.24) implies that, for every $\alpha \geq 1$, there exists a unique probability measure γ_α on W, which is stationary for the kernel Q_α, such that

$$\lim_{n\to\infty} U_\alpha^n f = \int_W f \, d\gamma_\alpha, \quad f \in L([0,1]). \tag{2.1}$$

This means that

$$\gamma_\alpha(B) = \int_W Q_\alpha(w,B)\, \gamma_\alpha(dw), \tag{2.2}$$

where

$$Q_\alpha(w,B) = \sum_{x\in B_{\mathbf{w}}} P_\alpha(w,x), \tag{2.3}$$

with

$$B_{\mathrm{w}} = f x \in \aleph^* / u_\alpha(w,x) \in B g,$$

for every $B \in W$, $w \in [0,1]$, is the transition probability function of the associated Markov chain $(\zeta_{\mathrm{n}})_{\mathrm{n}} \in_{\aleph}$. Moreover, as it was proved in Ch. Ganatsiou [6], the probability measure γ_α has the density

$$\rho_\alpha(w) = \frac{\alpha}{\alpha + w}, \text{ for every } w \in [0,1], \tag{2.4}$$

with constant $1/\alpha \log(1+\alpha^{-1})$, for the special case $\alpha u^{-1} + 1 - \alpha[u^{-1} + \alpha^{-1}] < 1$, for every $\alpha > 2$, $0 < u \leq 1$. Our approach is similar to that given in [5], [9], [11] and [12].

Remarks. (1) It is notable that for $\alpha = 1$ the RSCC associated with the generalized Gauss-Kuzmin operator is identical to that associated with the ordinary continued fraction expansion (see M. Iosifescu and S. Grogorescu [8]).

Moreover the corresponding limit probability measure associated with the family of RSCCs (1.7) for α =1 is identical to the limit probability measure associated with the above random system with complete connections for the ordinary continued fraction expansion, that is, identical to the Gauss's measure γ on $B_{[0,1]}$ defined by

$$\gamma(\mathrm{A}) = \frac{1}{\log 2}\int_A \frac{dt}{t+1'}, A \in B_{[0,1]}.$$

(2) It is an open problem the determination of an analogous limit probability measure for the case

$$\alpha u^{-1} + 1 - \alpha[u^{-1} + \alpha^{-1}] > 1, \quad 0 < u \leq 1.$$

3. Asymptotical Results

By virtue of Proposition 2.1.4 (M. Iosifescu and S. Theodorescu [7]), for any w∈[0,1], there exists a probability P_w, such that the associated Markov chain related with the family of RSCCs (1.7) is $(\zeta_n)_{n\,\in\aleph}$ introduced in Section 1. This gives us the possibility to state for an arbitrary real continuous function h defined on [0,1] a strong law of large numbers for the sequence $(h(\zeta_n))_{n\,\in\aleph}$ with respect to P_w (for more details the reader may consult Theorem 2.2.15 of M. Iosifescu and S. Theodorescu [7]) given by the following

Theorem 3.1. *For any w∈[0, 1], a>2 such that $\alpha u^{-1} + 1 - \alpha[u^{-1} + \alpha^{-1}] < 1$, $0 < u \leq 1$ and for any real- valued continuous function h defined on [0,1] we have*

$$\lim_{n\to\infty} n^{-1}\sum_{\kappa=1}^{n} h(\zeta_\kappa) = \int_0^1 h(y)\,\gamma_\alpha(dy), \quad P_w\text{- a.e.}$$

Hence by considering particular forms of the function h we may prove the following asymptotical results regarding the chain $(\zeta_n)_{n\,\in\aleph}$.

Theorem 3.2. *For any w∈[0,1] and a>2 such that $\alpha u^{-1} + 1 - \alpha[u^{-1} + \alpha^{-1}] < 1$, $0 < u \leq 1$, we have*

$$\lim_{n\to\infty} n^{-1}\sum_{\kappa=1}^{n} \zeta_\kappa = \frac{1}{\log(1+\alpha^{-1})} - \alpha, \quad P_w\text{- a.e.}$$

Proof. By applying Theorem 3.1 for h(y) = y, we obtain

$$\lim_{n\to\infty} n^{-1}\sum_{\kappa=1}^{n} \zeta_\kappa = \frac{1}{\alpha\cdot\log(1+\alpha^{-1})}\cdot\int_0^1 y\cdot\rho_\alpha(y)\,dy$$

$$
\begin{aligned}
&=\frac{1}{\alpha \cdot \log\left(1+\alpha^{-1}\right)} \cdot \int_0^1 y \cdot \frac{\alpha}{\alpha+y} dy \\
&=\frac{1}{\log\left(1+\alpha^{-1}\right)}\left(1-\alpha \cdot \log(\alpha+1)+\alpha \cdot \log \alpha\right) \\
&=\frac{1}{\log\left(1+\alpha^{-1}\right)}-\alpha, \quad P_{\mathrm{w}}\text{- a.e.}
\end{aligned}
$$

Theorem 3.3. *For any* $w \in (0,1]$ *and* $a>2$ *such that* $\alpha u^{-1} + 1 - \alpha[u^{-1}+\alpha^{-1}] < 1$, $0 < u \leq 1$ *we have*

$$\lim_{n\to\infty} n^{-1} \cdot \sum_{\kappa=1}^{n} \log \zeta_\kappa = -\frac{\pi^2}{12\alpha \cdot \log\left(1+\alpha^{-1}\right)}, \quad P_w\text{- a.e.}$$

Proof. By applying Theorem 3.1 for $h(y) = y^\varepsilon . \log y$, $\varepsilon > 0$, we obtain

$$\lim_{n\to\infty} n^{-1} \cdot \sum_{\kappa=1}^{n} \zeta_\kappa^\varepsilon \log \zeta_\kappa = \frac{1}{\alpha \cdot \log\left(1+\alpha^{-1}\right)} \cdot \int_{0+}^{1} y^\varepsilon \cdot \log y \rho_\alpha(y)\, dy,, \quad P_w\text{- a.e.}$$

Letting $\varepsilon \to 0$ we obtain

$$\lim_{n\to\infty} n^{-1} \cdot \sum_{\kappa=1}^{n} \log \zeta_\kappa = \frac{1}{\alpha \cdot \log\left(1+\alpha^{-1}\right)} \cdot \int_{0+}^{1} \log y \cdot \frac{\alpha}{\alpha+y} dy, \quad P_w\text{- a.e.} \quad (3.1)$$

By partial integration the right side of (3.1) becomes $-\frac{\pi^2}{12\alpha \cdot \log(1+\alpha^{-1})}$. So the proof is complete.

Remarks. (1) For $\alpha = 1$, Theorem 3.2 becomes

$$\lim_{n\to\infty} n^{-1} \sum_{\kappa=1}^{n} \zeta_\kappa = \frac{1}{\log 2} - 1, \quad \lambda\text{- a.e.}$$

where λ denotes the Lebesgue measure on W.

(2) For $\alpha = 1$, Theorem 3.3 becomes

$$\lim_{n\to\infty} n^{-1} \sum_{\kappa=1}^{n} \log \zeta_\kappa = -\frac{\pi^2}{12 \log 2}, \quad \lambda\text{- a.e.}$$

where λ denotes the Lebesgue measure on$W$. Here the measure γ_1 is identical to the classic Gauss's measure γ, that is,

$$\gamma_1(\mathrm{A}) = \frac{1}{\log 2} \int_A \frac{1}{1+w} dw, \quad \text{for every } A \in B_{[0,1]}$$

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Appendix

Definition 4.1. A quadruple $\{(\mathrm{W}, W), (\mathrm{X}, X), u\mathrm{P}\}$is named a *homogeneous random system with complete connections if*

(i) (W, W) and (X, X) are arbitrary measurable spaces;

(ii) u :W x X →W is a ($W \otimes X$, W) – measurable function;

(iii) P is a transition probability function from (W, W), to (X, X).

Next, we denote the element $(\mathrm{x}_1,\mathrm{x}_2,\ldots, \mathrm{x}_\mathrm{n}) \in \mathrm{X}^\mathrm{n}$ by $X^{(\mathrm{n})}$.

Definition 4.2. The functions $\mathrm{u}^{(\mathrm{n})} : \mathrm{W} \times \mathrm{X}^\mathrm{n} \to \mathrm{W}$, $\mathrm{n} \in \mathrm{N}^*$, are defined as follows

$$u^{(n+1)}\left(w, x^{(n+1)}\right) = \begin{cases} u(w,x), & \text{if } n = 0, \\ u\left(u^{(n)}\left(w, x^{(n)}\right), x_{n+1}\right), & \text{if } n \geq 1. \end{cases}$$

Convention. We snall write $w \cdot x^{(n)}$ instead of $\mathrm{u}^{(\mathrm{n})}$ (w, $\mathrm{x}^{(\mathrm{n})}$). Throughout the paper, we shall consider the discrete case for the space (X, X).

Definition 4.3. The transition probability functions P_r, $\mathrm{r} \in \aleph^*$, are defined by

$$P_{\mathrm{r}}(w, A) = \begin{cases} P(w, A), & \text{if } r = 1, \\ \sum\limits_{x_1 \in X} P(w, x_1) \sum\limits_{x_2 \in X} P(wx_1, x_2) \ldots \sum\limits_{x_{\mathrm{r}} \in X} P\left(wx^{(r-1)}, x_r\right) \cdot I_A\left(x^{(r)}\right), & \text{if } r > 1, \end{cases}$$

for any w $\in$ W, r $\in \aleph^*$ and A $\in X^{\mathrm{r}}$.

Definition 4.4. Assume that X^0 x A = A. Then we define

$$P_r^n(w, A) = P_{n+r-1}\left(w, X^{n-1}xA\right),$$

for any w $\in$ W, r $\in \aleph^*$ and A $\in X^{\mathrm{r}}$.

Theorem 4.5. *Let* $\{(W, W), (X, X), u, P\}$ *be a homogeneous random system with complete connections and suppose an arbitrary fixed* $w_0 \in W$. *Then there exist a probability space* $(\Omega, \mathrm{K}, P_{w_0})$ *and two sequences of random variables* $(\xi_{\mathrm{n}})_{\mathrm{n}\in\aleph*}(\xi_{\mathrm{n}}: \Omega \to \mathrm{X})$ *and* $(\zeta_{\mathrm{n}})_{\mathrm{n}\in\aleph}$ $(\zeta_{\mathrm{n}} : \Omega \to W)$ *such that:*

(i) (1) $P_{w_0}([\xi_n, \ldots, \xi_{n+r-1}]) = P_r^n(w_0, A)$;

(2) $P_{w_0}\left([\xi_{n+m}, \ldots, \xi_{n+m+r-1}] \in A/\xi^{(n)}\right) = P_r^m\left(w_0\xi^{(n)}, A\right), P_{w_0} - a.e.$;

(3) $P_{w_0}\left([\xi_{n+m}, \ldots, \xi_{n+m+r-1}] \in A/\xi^{(m)}, \zeta^{(n)}\right) = P_r^m(\zeta_n, A), P_{w_0} - a.e.$;

for any n, m r $\in \aleph^*$ *and A* $\in X^{\mathrm{r}}$, *where* $\xi^{(\mathrm{n})}$ *and* $\zeta^{(\mathrm{n})}$ *are, respectively, the random vectors* $(\xi_1, \xi_2, ..., \xi_{\mathrm{n}})$ *and* $(\zeta_1, \zeta_2, ..., \zeta_{\mathrm{n}})$.

(ii) $(\zeta_{\mathrm{n}})_{\mathrm{n}}$ *is W-valued homogeneous Markov chain with initial distribution concentrated at* w_0 *and transition operator U given by the equation*

$$Uf(w) = \sum_{x \in X} P(w, x) f(wx) \tag{A1}$$

for any f $\in$ *B(W, W). (Here B(W, W) is the Banach space of all bounded W -measurable real-valued functions defined on W).*

Remark. The sequence $(\zeta_{\mathrm{n}})_{\mathrm{n}\in\aleph}$ is called the *associated Markov chain.* For $\mathrm{f(w)} = \mathrm{I_A(w)}$, A$\in W$, we obtain by (A1) the transition probability function of the associated Markov chain given by the equality

$$Q(w, A) = \sum_{x \in X} P(w, x) I_B(wx) (= P(w, A_w)), \tag{A2}$$

for any w $\in$ W, A $\in W$ with $\mathrm{A_w} = \{\mathrm{x} \in \mathrm{X}/\mathrm{wx} \in \mathrm{A}\}$.

Definition 4.6. A random system with complete connections $\{(\mathrm{W}, \mathcal{W}),$ $(\mathrm{X}, \mathcal{X}),$ u, P$\}$is called *uniformly ergodic*, if for any r $\in \aleph^*$, there exists a probability P_r^∞ on $\mathcal{X}^{\,\mathrm{r}}$ such that lim $\varepsilon_{\mathrm{n}} = 0$ as n $\rightarrow \infty$, where

$$\varepsilon_n = \sup_{\substack{w \in W,\, r \in N^* \\ A \in \mathcal{X}^r}} \left| P_r^n \left(w, A\right) - P_r^\infty \left(A\right) \right| .$$

Theorem 4.7. *Let* $\{(W,\ \mathcal{W}),\ (X,\ \mathcal{X}),\ u,\ P\}$*be a random system with complete connections uniformly ergodic. Then there exists a unique limiting probability measure* Q^∞ *on* $\mathcal{W}$ *which satisfies the equation*

$$\int_W Q\left(w, A\right)\, Q^\infty\left(dw\right) = Q^\infty\left(B\right),$$

for any $B \in \mathcal{W}$*, w* $\in$ *W, where Q is the probability function defined by (A2).*

Remark. A random system with complete connections is a particular case of infinite order chain [2, 4, 10, 14]. Another way for investigating higher-order Markov chains is the construction of multiple Markov chains through collections of directed circuits and positive weights named as *higher-order circuit chains* [13, 15].