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EXTENDING THE HALL-PORSCHING BOUNDS FOR THE PERRON ROOT

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Abstract: Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be nonnegative with Perron root r and row sums $s_1, \ldots, s_n$, and denote $S = \max_i s_i$, $s = \min_i s_i$. We improve the Frobenius bounds $s \le r \le S$ by applying them to $\mathbf{DAD}^{-1}$, where $\mathbf{D}$ is obtained from the identity matrix $\mathbf{I}$ by replacing its certain diagonal entries with a suitably chosen positive number. As a special case, in changing only one entry, we obtain the Hall–Porsching bounds.

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1. Introduction

Let $\mathbf{A} = (a_{ik}) \in \mathbb{R}^{n\times n}$, $n \ge 2$, be nonnegative with Perron root r and row sums $s_1, \ldots, s_n$, and denote $S = \max_i s_i$, $s = \min_i s_i$. (The column sums can be considered analogously.) There are many improvements of the Frobenius bounds

$$s \le r \le S. \tag{1}$$

For ”classical” improvements, see [4, Section 3.1], [5, Section 2.1].

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Hall and Porsching [3, Theorem 1] proved that, for all $j = 1, \dots, n$,

$$\frac{1}{2}\min_{i\neq j}\left\{s_i - a_{ij} + a_{jj} + \left[(s_i - a_{ij} - a_{jj})^2 + 4a_{ij}(s_j - a_{jj})\right]^{\frac{1}{2}}\right\} \le r$$
$$\le \frac{1}{2}\max_{i\neq j}\left\{s_i - a_{ij} + a_{jj} + \left[(s_i - a_{ij} - a_{jj})^2 + 4a_{ij}(s_j - a_{jj})\right]^{\frac{1}{2}}\right\}. \qquad (2)$$

They assumed that $\mathbf{A}$ is irreducible, but (2) holds also in the reducible case by continuity. They also showed [3, Theorem 2] that these bounds, called "HP bounds" in the sequel, improve those found by Brauer [1, Theorem 2].

Brauer's method based on applying the Frobenius bounds (1) to $\mathbf{DAD}^{-1}$, where $\mathbf{D}$ is obtained from the identity matrix $\mathbf{I}$ by replacing its certain diagonal entry with a suitably chosen positive real number x. However (apparently to simplify the results) he disposed the individual a_{ij}'s by suitable estimations. Hall's and Porsching's method was different. In

$$r = \max_{\mathbf{0}<\mathbf{z}\in\mathbb{R}^n} \max\{\lambda \in \mathbb{R} \mid \lambda\mathbf{z} \le \mathbf{Az}\} = \min_{\mathbf{0}<\mathbf{z}\in\mathbb{R}^n} \min\{\lambda \in \mathbb{R} \mid \lambda\mathbf{z} \ge \mathbf{Az}\}$$

(see [2, p. 64]), they considered only such vectors $\mathbf{z}$ that are obtained from the vector $(1, \dots, 1)$ by replacing its j'th entry (j fixed) with x ($0 < x \le 1$), and optimized over x.

We generalize Brauer's method. Let $1 \le h < n$, and change h diagonal entries of $\mathbf{I}$ into $x (> 0)$, so obtaining $\mathbf{D}$. Since we will do all the computations under full information, it can be expected that in the case $h = 1$ or $h = n-1$ we will improve Brauer's bounds. Indeed, we will then obtain the Hall–Porsching bounds. So, we will provide an alternative proof and an extension of (2).

2. The General Case

Let us first assume that $\mathbf{A}$ is positive. Let $1 \le h < n$. Consider a partition of $N = \{1, \dots, n\}$ into two sets $I = \{i_1, \dots, i_h\}$ and $J = \{j_1, \dots, j_{n-h}\}$. (That is, $I, J \ne \emptyset$, $I \cap J = \emptyset$, $I \cup J = N$.) For $x \in \mathbb{R}$ with $x > 0$, define $\mathbf{D} = \operatorname{diag}(d_i)$ by $d_{i_1} = \cdots = d_{i_h} = x$ and $d_{j_1} = \cdots = d_{j_{n-h}} = 1$ otherwise. Also the matrix $\mathbf{B} = \mathbf{DAD}^{-1}$ has Perron root r. If $i \in I$, then the i'th row sum of $\mathbf{B}$ is

$$\sigma_i = \sum_{k\in I} a_{ik} + \Big(\sum_{k\in J} a_{ik}\Big)x.$$

If $j \in J$, the j'th row sum of $\mathbf{B}$ is

$$\tau_j = \Big(\sum_{k\in I} a_{jk}\Big)x^{-1} + \sum_{k\in J} a_{jk}.$$

We study first the lower bound. We require that the set $\{\sigma_i \,|\, i \in I\}$ contains the smallest row sum of $\mathbf{B}$. This happens if, for all $j \in J$, there exists $i \in I$ such that

$$\sum_{k\in I} a_{ik} + \Big(\sum_{k\in J} a_{ik}\Big)x \leq \Big(\sum_{k\in I} a_{jk}\Big)x^{-1} + \sum_{k\in J} a_{jk}. \tag{3}$$

Then, applying (1) to $\mathbf{B}$,

$$\min_{i\in I} \sigma_i \leq r. \tag{4}$$

Fix $j \in J$. Write (3) as

$$\Big(\sum_{k\in J} a_{ik}\Big)x^2 + \Big(\sum_{k\in I} a_{ik} - \sum_{k\in J} a_{jk}\Big)x - \sum_{k\in I} a_{jk} \leq 0; \tag{5}$$

then

$$(0 <)x \leq e_{ij} = \frac{1}{2}\Big(\sum_{k\in J} a_{ik}\Big)^{-1}\Big\{\sum_{k\in J} a_{jk} - \sum_{k\in I} a_{ik} + \Big[\Big(\sum_{k\in J} a_{jk} - \sum_{k\in I} a_{ik}\Big)^2 + 4\Big(\sum_{k\in J} a_{ik}\Big)\Big(\sum_{k\in I} a_{jk}\Big)\Big]^{\frac{1}{2}}\Big\}.$$

Optimally

$$x = e_j = \max_{i\in I} e_{ij}$$

for this particular j and

$$x = e = \min_{j\in J} e_j = \min_{j\in J}\max_{i\in I} e_{ij}$$

for all $j \in J$. Then (4) reads

$$r \geq \min_{i\in I}\Big[\sum_{k\in I} a_{ik} + \Big(\sum_{k\in J} a_{ik}\Big)e\Big] = \min_{i\in I}\Big(\sum_{k\in I} a_{ik} + \frac{1}{2}\min_{j\in J}\max_{i\in I}\Big\{\sum_{k\in J} a_{jk} - \sum_{k\in I} a_{ik} + \Big[\Big(\sum_{k\in J} a_{jk} - \sum_{k\in I} a_{ik}\Big)^2 + 4\Big(\sum_{k\in J} a_{ik}\Big)\Big(\sum_{k\in I} a_{jk}\Big)\Big]^{\frac{1}{2}}\Big\}\Big). \tag{6}$$

As to the upper bound, we proceed similarly. We state that the set $\{\tau_j \mid j \in J\}$ contains the largest row sum of $\mathbf{B}$. That is, for all $i \in I$, there exists $j \in J$ such that (3) holds. Then

$$r \leq \max_{j\in J} \tau_j. \tag{7}$$

Fix $i \in I$. Writing (5) as

$$\Big(\sum_{k\in I} a_{jk}\Big)x^{-2} + \Big(\sum_{k\in J} a_{jk} - \sum_{k\in I} a_{ik}\Big)x^{-1} - \sum_{k\in J} a_{ik} \geq 0,$$

we have

$$x^{-1} \geq f_{ij} = \frac{1}{2}\Big(\sum_{k\in I} a_{jk}\Big)^{-1}\Big\{\sum_{k\in I} a_{ik} - \sum_{k\in J} a_{jk} +$$
$$\Big[\Big(\sum_{k\in I} a_{ik} - \sum_{k\in J} a_{jk}\Big)^2 + 4\Big(\sum_{k\in J} a_{ik}\Big)\Big(\sum_{k\in I} a_{jk}\Big)\Big]^{\frac{1}{2}}\Big\}.$$

Optimally

$$x^{-1} = f_i = \min_{j\in J} f_{ij}$$

for this i and

$$x^{-1} = f = \max_{i\in I} f_i = \max_{i\in I}\min_{j\in J} f_{ij}$$

for all $i \in I$. Then (7) reads

$$r \leq \max_{j\in J}\Big[\Big(\sum_{k\in I} a_{jk}\Big)f + \sum_{k\in J} a_{jk}\Big] =$$
$$\max_{j\in J}\Bigg(\sum_{k\in J} a_{jk} + \frac{1}{2}\max_{i\in I}\min_{j\in J}\Big[\sum_{k\in I} a_{ik} - \sum_{k\in J} a_{jk} +$$
$$\Big[\Big(\sum_{k\in I} a_{ik} - \sum_{k\in J} a_{jk}\Big)^2 + 4\Big(\sum_{k\in J} a_{ik}\Big)\Big(\sum_{k\in I} a_{jk}\Big)\Big]^{\frac{1}{2}}\Big\}\Bigg). \tag{8}$$

By continuity, we can drop out the assumption on positivity. We have thus proved the following

Theorem 1. *Inequalities* (6) *and* (8) *hold.*

If $\mathbf{A}$ is positive and the s_i's are not all equal, then this theorem can always be applied to improve strictly the Frobenius bounds (1). Simply choose I and J so that $s_i < s_j$ for all $i \in I$, $j \in J$. Then all the σ_i's increase and τ_j's decrease strictly with x, and $e > 1$. If $\mathbf{A}$ has zero entries, then strict improvement does not nessarily happen (for a trivial counterexample, consider a diagonal matrix) but in most cases does.

3. The Special Cases $h = 1$ and $h = n - 1$

Let us consider the case $h = 1$. That is, we fix $i \in N$ and put $I = \{i\}$, $J = N \setminus \{i\}$. Then (6) simplifies into

$$r \geq a_{ii} + \frac{1}{2} \min_{\substack{1\leq j\leq n \\ j\neq i}} \Bigg\{ \sum_{\substack{k=1 \\ k\neq i}}^{n} a_{jk} - a_{ii} + \Bigg[\Big(\sum_{\substack{k=1 \\ k\neq i}}^{n} a_{jk} - a_{ii}\Big)^2 + 4\Big(\sum_{\substack{k=1 \\ k\neq i}}^{n} a_{ik}\Big)a_{ji}\Bigg]^{\frac{1}{2}} \Bigg\} = a_{ii} +$$

$$\frac{1}{2} \min_{\substack{1\leq j\leq n \\ j\neq i}} \Big\{ s_j - a_{ji} - a_{ii} + \big[(s_j - a_{ji} - a_{ii})^2 + 4a_{ji}(s_i - a_{ii})\big]^{\frac{1}{2}} \Big\} =$$

$$\frac{1}{2} \min_{\substack{1\leq j\leq n \\ j\neq i}} \Big\{ s_j - a_{ji} + a_{ii} + \big[(s_j - a_{ji} - a_{ii})^2 + 4a_{ji}(s_i - a_{ii})\big]^{\frac{1}{2}} \Big\} \tag{9}$$

for all $i = 1, \ldots, n$. Similarly, substitute $h = n - 1$ (i.e., fix $j \in N$ and put $J = \{j\}$, $I = N \setminus \{j\}$). Then (8) simplifies into

$$r \leq a_{jj} + \frac{1}{2} \max_{\substack{1\leq i\leq n \\ i\neq j}} \Bigg\{ \sum_{\substack{k=1 \\ k\neq j}}^{n} a_{ik} - a_{jj} + \Bigg[\Big(\sum_{\substack{k=1 \\ k\neq j}}^{n} a_{ik} - a_{jj}\Big)^2 + 4\Big(\sum_{\substack{k=1 \\ k\neq j}}^{n} a_{jk}\Big)a_{ij}\Bigg]^{\frac{1}{2}} \Bigg\} = a_{jj} +$$

$$\frac{1}{2} \max_{\substack{1\leq i\leq n \\ i\neq j}} \Big\{ s_i - a_{ij} - a_{jj} + \big[(s_i - a_{ij} - a_{jj})^2 + 4a_{ij}(s_j - a_{jj})\big]^{\frac{1}{2}} \Big\} =$$

$$\frac{1}{2} \max_{\substack{1\leq i\leq n \\ i\neq j}} \Big\{ s_i - a_{ij} + a_{jj} + \big[(s_i - a_{ij} - a_{jj})^2 + 4a_{ij}(s_j - a_{jj})\big]^{\frac{1}{2}} \Big\} \tag{10}$$

for all $j = 1, \dots, n$. The bounds (9) and (10) are just the HP bounds (2).

4. Examples

Example 1a. Let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 4 & 0 & 3 \\ 2 & 0 & 4 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix},$$

cited from [3, p. 163]. Then $r = 6.784$. The HP bounds give

$$6 \le r \le 7.123, \tag{11}$$

see [3]. Do we find better bounds applying $h = 2$? It is reasonable to put in I (respectively, in J) the two indices with smallest (largest) row sums. Since $s_1 = 10$, $s_2 = 7$, $s_3 = 6$, $s_4 = 5$, we set $I = \{3, 4\}$, $J = \{1, 2\}$. For $i = 3$ and $j = 1$, we have

$$\sum_{k \in J} a_{jk} = a_{11} + a_{12} = 3, \quad \sum_{k \in I} a_{ik} = a_{33} + a_{34} = 4,$$
$$\sum_{k \in J} a_{ik} = a_{31} + a_{32} = 2, \quad \sum_{k \in I} a_{jk} = a_{13} + a_{14} = 7,$$

and so

$$e_{31} = \frac{3 - 4 + \sqrt{(3-4)^2 + 4 \cdot 2 \cdot 7}}{2 \cdot 2} = \frac{\sqrt{57} - 1}{4} = 1.637.$$

Similarly,

$$e_{41} = \frac{\sqrt{14}}{2} = 1.871, \quad e_{32} = \frac{\sqrt{6}}{2} = 1.225, \quad e_{42} = \tfrac{3}{2}.$$

Further $e_1 = e_{41}$, $e_2 = e_{42}$, and so $e = e_{42}$. The minimum of

$$a_{33} + a_{34} + (a_{31} + a_{32})e = 4 + 2 \cdot \tfrac{3}{2} = 7$$

and

$$a_{43} + a_{44} + (a_{41} + a_{42})e = 3 + 2 \cdot \tfrac{3}{2} = 6$$

gives, by (6), the lower bound 6.

To find the upper bound, we have

$$f_{31} = \frac{4}{\sqrt{57}-1} = 0.611, \; f_{32} = \frac{2}{\sqrt{6}} = 0.816,$$
$$f_{41} = \frac{2}{\sqrt{14}} = 0.534, \; f_{42} = \tfrac{2}{3},$$

and further $f_3 = f_{31}$, $f_4 = f_{41}$, $f = f_{31}$. The maximum of

$$(a_{13}+a_{14})f + a_{11} + a_{12} = 7 \cdot \frac{4}{\sqrt{57}-1} + 3 = \tfrac{1}{2}(\sqrt{57}+7) = 7.275$$

and

$$(a_{23}+a_{24})f + a_{21} + a_{22} = 3 \cdot \frac{4}{\sqrt{57}-1} + 4 = \tfrac{1}{14}(59+3\sqrt{57}) = 5.832$$

gives, by (8), the upper bound 7.275. In all,

$$6 \le r \le 7.275. \tag{12}$$

The lower bound is equal to that in (11) but the upper a little worse.

Example 1b. Changing the order of the rows does not effect on r but effects on the bounds discussed here. We look what happens if we reverse the order of the rows of $\mathbf{A}$. So let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 4 & 0 \\ 0 & 4 & 0 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix}.$$

Now $s_1 = 5$, $s_2 = 6$, $s_3 = 7$, $s_4 = 10$. We apply first the HP bounds. Denote

$$\lambda_{ij} = \frac{1}{2}\Big\{ s_i - a_{ij} + a_{jj} + \big[(s_i - a_{ij} - a_{jj})^2 + 4a_{ij}(s_j - a_{jj})\big]^{\frac{1}{2}} \Big\};$$

then

$$\lambda_{21} = 2 + \sqrt{10} = 5.162, \; \lambda_{31} = 7, \; \lambda_{41} = 9,$$
$$\lambda_{12} = 2 + \sqrt{10} = 5.162, \; \lambda_{32} = \tfrac{1}{2}(3+\sqrt{105}) = 6.623,$$
$$\lambda_{42} = \tfrac{1}{2}(9+\sqrt{105}) = 9.623,$$
$$\lambda_{13} = \tfrac{1}{2}(3+\sqrt{65}) = 5.531, \; \lambda_{23} = 1 + \sqrt{15} = 4.873,$$
$$\lambda_{43} = \tfrac{1}{2}(7+\sqrt{143}) = 9.479,$$

$$\lambda_{14} = 4 + \sqrt{6} = 6.449,\ \lambda_{24} = 6,\ \lambda_{34} = 4 + 3\sqrt{2} = 8.243.$$

Hence, by (2),

$$6 \le r \le 8.243.$$

Compared with (11), the lower bound remains and the upper worsens.

Second, let us set $h = 2$, $I = \{1, 2\}$, $J = \{3, 4\}$. Then

$$\begin{gathered}
e_{13} = \tfrac{4}{3}, \quad e_{23} = \tfrac{1}{8}(\sqrt{65} + 1) = 1.133, \\
e_{14} = \tfrac{1}{6}(\sqrt{61} + 5) = 2.135, \quad e_{24} = \tfrac{1}{8}(\sqrt{73} + 5) = 1.693, \\
f_{13} = \tfrac{3}{4}, \quad f_{14} = \tfrac{1}{6}(\sqrt{61} - 5) = 0.468, \\
f_{23} = \tfrac{1}{8}(\sqrt{65} - 1) = 0.883, \quad f_{24} = \tfrac{1}{6}(\sqrt{73} - 5) = 0.591, \\
e_3 = e_{13},\ e_4 = e_{14},\ e = e_{13},\ f_1 = f_{14},\ f_2 = f_{24}, f = f_{24}, \\
a_{11} + a_{12} + (a_{13} + a_{14})e = 6, \\
a_{21} + a_{22} + (a_{23} + a_{24})e = 7\tfrac{1}{3}, \\
(a_{31} + a_{32})f + a_{33} + a_{34} = \tfrac{1}{3}(2\sqrt{73} - 1) = 5.363, \\
(a_{41} + a_{42})f + a_{43} + a_{44} = \tfrac{1}{2}(\sqrt{73} + 9) = 8.772.
\end{gathered}$$

Hence, by (6) and (8),

$$6 \le r \le 8.772.$$

Again, compared with (12), the lower bound remains and the upper worsens.

Example 2a. To give an example where the extension to $h = 2$ improves the HP bounds, consider

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix},$$

again cited from [3, p. 163]. Then $r = 3.508$. The HP bounds only repeat the Frobenius bounds

$$3 \le r \le 4. \tag{13}$$

But setting $h = 2$, $I = \{3, 4\}$, $J = \{1, 2\}$ yields

$$e_{31} = e_{41} = e_1 = \tfrac{1}{2}(\sqrt{5} + 1) = 1.618,$$

$$e_{32} = e_{42} = e_2 = e = \sqrt{2} = 1.414,$$
$$f_{31} = f_{41} = f_3 = f_4 = f = \tfrac{1}{2}(\sqrt{5} - 1) = 0.618,$$
$$f_{32} = f_{42} = \tfrac{1}{2}\sqrt{2} = 0.707,$$
$$a_{33} + a_{34} + (a_{31} + a_{32})e = a_{43} + a_{44} + (a_{41} + a_{42})e =$$
$$2 + \sqrt{2} = 3.414,$$
$$(a_{13} + a_{14})f + a_{11} + a_{12} = \tfrac{1}{2}(5 + \sqrt{5}) = 3.618,$$
$$(a_{23} + a_{24})f + a_{21} + a_{22} = \sqrt{5} + 1 = 3.236.$$

So we get better bounds

$$3.414 \le r \le 3.618 \tag{14}$$

by (6) and (8).

Example 2b. Again reversing the order of the rows, let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

In computing the HP bounds, the crucial λ_{ij}'s are $\lambda_{13} = \lambda_{23} = 2 + \sqrt{2} = 3.414$, $\lambda_{43} = 4$, improving (13) into

$$3.414 \le r \le 4.$$

Finally, let us set $h = 2$, $I = \{1, 2\}$, $J = \{3, 4\}$. Then $e = \frac{1}{2}\sqrt{6}$, $f = \frac{1}{4}(\sqrt{17} - 1)$, and (6) and (8) give respectively the lower bound $1 + \sqrt{6} = 3.449$ and the upper bound $\frac{1}{2}(3 + \sqrt{17}) = 3.562$. So

$$3.449 \le r \le 3.562,$$

which beats (14).

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