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AN IDEAL-BASED ZERO DIVISOR GRAPH OF PO-Γ-SEMIGROUPS

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Abstract: In this paper, we study the notion of ideal-based zero-divisor graph structure of po-Γ-semigroup M with respect to reflexive ideal I of M, denoted by $\Gamma_I(M)$, whose vertices are the set $\{x \in M\backslash I$: there exists $y \in M\backslash I$ such that $x\Gamma y \subseteq I\}$ with distinct vertices x and y are adjacent if and only if $x\Gamma y \subseteq I$. We investigate the interplay between the po-Γ-semigroups properties of M and the graph-theoretic properties of $\Gamma_I(M)$.

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1. Preliminaries

The concept of po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee in [8].

Let M and Γ be any two non-empty sets. M is called a Γ-semigroup if there exists a mapping $M \times \Gamma \times M \to M$, written as $(a,\ \gamma,\ b) \to a\gamma b$, satisfying the following identities $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in M$ and $\gamma,\ \mu \in \Gamma$.

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A po-Γ-semigroup (: partially ordered Γ-semigroup) is an ordered set M which is a Γ-semigroup such that $a \leq b \Rightarrow a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in M$ and $\gamma \in \Gamma$.

Following [6], an element $a \in M$ is called a zero element of M if $x\gamma a = a\gamma x = a$ and $a \leq x$ for all $x \in M$ and $\gamma \in \Gamma$ and it is denoted by 0. A non-empty subset L of M is called a left (resp. right) ideal of M if $M\Gamma L \subseteq L$ (resp. $L\Gamma M \subseteq L$) and for all $a \in L,\ b \in M,\ b \leq a$ implies $b \in L$. A non-empty subset I of M is called an ideal if I is a both left and right ideal of M. An ideal I of M is said to be reflexive if $a\gamma b \in I$ implies $b\gamma a \in I$ for $a, b \in M$ and $\gamma \in \Gamma$. A proper ideal P of M is said to be prime if for any ideals $A,\ B$ of M such that $A\Gamma B \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$. Following [6], an ideal P is called completely prime if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$. It is clear that if I is a reflexive ideal of M, then I is prime if and only if I is completely prime. For any non-empty subsets A, B of M, we write the set $(A : B) = \{m \in M : m\Gamma B \subseteq A\}$. We denote by $I(a)$ the ideal of M generated by a.

In [3], Beck introduced the concept of a zero-divisor graph of a commutative ring with identity, but this work was mostly concerned with coloring of rings. In [2], Anderson and Livingston associate a graph (simple) $\Gamma(R)$ to a commutative ring R with identity with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero divisor of R and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only $xy = 0$. They investigated the interplay between the ring-theoretic properties of R and the graph-theoretics properties of $\Gamma(R)$.

In [9], Redmond has generalized the notion of the zero-divisor graph. For a given ideal I of R, he defined an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R - I : xy \in I\ for\ some\ y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. The zero-divisor graph has been introduced and studied for various algebraic structures by several authors [5 and 7].

In this paper, we study the undirected graph $\Gamma_I(M)$ of po-Γ-semigroups for any reflexive ideal I of M.

Let M be a po-Γ-semigroup and J a reflexive ideal of M. Then the zero divisor graph of M with respect to the ideal J, denoted by $\Gamma_J(M)$, is the graph whose vertices are the set $\{x \in M\backslash J : x\Gamma y \subseteq J\ for\ some\ y \in M\backslash J\}$ with distinct vertices x and y adjacent if and only if $x\Gamma y \subseteq J$. If $J = 0$, then $\Gamma_J(M) = \Gamma(M)$ and J is a prime ideal of N if and only if $\Gamma_J(M) = \phi$. Throughout this paper M stands for a nonzero po-Γ-semigroup with zero element and I is a reflexive ideal of M. For distinct vertices x and y of a graph G, let $d(x, y)$ be the length of the shortest path from x to y. The diameter of a connected graph is the supremum of the distances between vertices. The core K of a graph G is the union of all cycles of G.

In this paper the notations of graph theory are from [4], the notations of po-Γ-semigroups are from [6]and [8].

2. Main Results

Theorem 2.1. *Let I be a reflexive ideal of M. Then $\Gamma_I(M)$ is connected and $diam(\Gamma_I(M)) \leq 3$.*

Proof. Let $x, y \in \Gamma_I(M)$. Then there exist $z \in M\backslash I$ and $w \in M\backslash I$ with $x\Gamma z \subseteq I$ and $w\Gamma y \subseteq I$. If $x\Gamma y \subseteq I$, then $x - y$ is a path of length 1. If $x\Gamma y \nsubseteq I$ and $z\Gamma w \subseteq I$, then $x - z - w - y$ is a path of length 3. If $x\Gamma y \nsubseteq I$ and $z\Gamma w \nsubseteq I$, then there exist $\gamma \in \Gamma$ such that $x - z\gamma w - y$ is a path of length 2.

Theorem 2.2. *Let I be a reflexive ideal of M and if $a - x - b$ is a path in $\Gamma_I(M)$, then either $I \cup \{x\}$ is an ideal of M or $a - x - b$ is contained in a cycle of length ≤ 4.*

Proof. Let $a - x - b$ be a path in $\Gamma_I(M)$. Then either $(I : a) \cap (I : b) = I \cup \{x\}$ or there is $c \in ((I : a) \cap (I : b))\backslash(I \cup \{x\})$ with $a\Gamma c \subseteq I$ and $b\Gamma c \subseteq I$. This shows that either $I \cup \{x\}$ is an ideal of M or $a - x - b$ is contained in a cycle of length ≤ 4. □

In view of above theorem, we have the following corollary.

Corollary 2.3. *Let $|\Gamma_I(M)| \geq 3$ and $I \cup \{x\}$ is not an ideal of M for any $x \in M\backslash I$. Then any edge in $\Gamma_I(M)$ is contained in a cycle of length ≤ 4, and therefore $\Gamma_I(M)$ is a union of triangles and squares.*

Lemma 2.4. *Let I be a completely reflexive ideal of M. Then a pentagon or hexagon can not be a $\Gamma_I(M)$.*

Proof. Suppose that $\Gamma_I(M)$ is $a - b - c - d - e - a$, a pentagon. Then by Theorem 2.2, for one of the vertices (say b), $I \cup \{b\}$ is an ideal of M. Then in the pentagon, $d\Gamma e \subseteq I$. Since $I \cup \{b\}$ is ideal, $b\gamma d = b = b\gamma_1 e$ for some $\gamma, \gamma_1 \in \Gamma$. But $b\gamma(d\gamma_1 e) \in I$, then $b = b\gamma_1 e = (b\gamma d)\gamma_1 e = b\gamma(d\gamma_1 e) \in I$, a contradiction. The proof for the hexagon is the same. □

Theorem 2.5. *Let I be a reflexive ideal of M. If $\Gamma_I(M)$ contains a cycle, then the core K of $\Gamma_I(M)$ is a union of triangles and rectangles. Moreover, any vertex in $\Gamma_I(M)$ is either a vertex of the core K of $\Gamma_I(M)$ or else is an end vertex of $\Gamma_I(M)$.*

Proof. Let $a \in K$ and assume that a is not in any triangle or rectangle in $\Gamma_I(M)$. Then a is part of a cycle $a - b - c - d - ... - a$ which implies $c\Gamma d \subseteq I$.

By Theorem 2.2, $I \cup \{a\}$ is an ideal of M. Then there exist $\gamma, \gamma_1 \in \Gamma$ such that $d\gamma a = a = a\gamma_1 c$ and $d\gamma(c\gamma_1 a) \in I$ which imply $a \in I$, a contradiction. So the core K of $\Gamma_I(M)$ is a union of triangles and rectangles.

For the 'moreover' statement, we can assume $|\Gamma_I(M)| \geq 3$. If x is a vertex in $\Gamma_I(M)$, then one of the following is true:

1. x is in the core;
2. x is an end vertex of $\Gamma_I(M)$;
3. $a - x - b$ is a path in $\Gamma_I(M)$, where a is an end vertex and $b \in K$;
4. $a - x - y - b$ or $a - y - x - b$ is a path in $\Gamma_I(N)$, where a is an end vertex and $b \in K$.

In the first two cases, we are done. Let us assume that $a - x - b$ is a path with $b \in K$. Then by Theorem 2.2, $I \cup \{x\}$ is an ideal of M and $x - b - c - d - b$ or $x - b - c - d - e - b$ is a path in $\Gamma_I(M)$ which implies $c\Gamma d \subseteq I$. Since $x \notin K$, there exists $\gamma \in \Gamma$ such that $x\gamma c = x$ and so x is a vertex in the cycle $x - b - c - d - x$, a contradiction.

Without loss of generality, assume $a - x - y - b$ is a path in $\Gamma_I(M)$. Since $b \in K$, there is some $c \in K$ such that $c \neq b$ and $b - c$ is part of a cycle. Then $a - x - y - b - c$ is a path in $\Gamma_I(M)$. But the distance from a to c is four, a contradiction unless $y - c$ or $x - c$ is an edge. However, if $y - c$ is an edge, then $y \in K$. By case 3, x is also in the core. If instead, $x - c$ is an edge, then $x - y - b - c - x$ is a cycle. Thus $x, y \in K$. Hence it must be the case that any vertex x of $\Gamma_I(M)$ is either an end or in the core. $\square$

Following [6] for any subset S of M, let $S^n = S \overbrace{\Gamma S ... \Gamma S}^{n-1 \; term \; \Gamma S}$ for any $n \in \mathbb{N}$ and $\sqrt{S} = \{m \in M \ : \ m^n \subseteq I \text{ for some } n \in \mathbb{N}\}$. Recall that a bipartite graph is one whose vertex set can be partitioned into two subsets so that no edge has both ends in any one subset. The complete bipartite graph is one in which each vertex is joined to every vertex that is not in the same subset.

Theorem 2.6. *Let I be a reflexive ideal of M. Then the following hold:*

(1) If P_1 and P_2 are prime ideals of M and $I = P_1 \cap P_2$, then $\Gamma_I(M)$ is a complete bipartite graph.

(2) If $I = \sqrt{I}$, then $\Gamma_I(M)$ is a complete bipartite graph if and only if there exist prime ideals P_1 and P_2 of M such that $I = P_1 \cap P_2$.

Proof. (1) Let $a, b \in M \backslash I$ with $a\Gamma b \subseteq I$. Then $I(a)\Gamma I(b) \subseteq P_1$ and $I(a)\Gamma I(b) \subseteq P_2$. Since P_1 and P_2 are prime, we have $a \in P_1$ or $b \in P_1$ and $a \in P_2$ or $b \in P_2$. Therefore, suppose $a \in P_1 \backslash P_2$ and $b \in P_2 \backslash P_1$. Thus, $\Gamma_I(M)$ is a complete bipartite graph with parts $P_1 \backslash P_2$ and $P_2 \backslash P_1$.

(2) Suppose that the parts of $\Gamma_I(M)$ are V_1 and V_2. Set $P_1 = V_1 \cup I$ and $P_2 = V_2 \cup I$. It is clear that $I = P_1 \cap P_2$. We now claim that P_1 is a prime ideal of M. Let $a \in P_1$, $m \in M$ and $\gamma \in \Gamma$.

Case (i) If $a \in I$, then $a\gamma m \in I \subseteq P_1$.

Case (ii) If $a \in V_1$, there exists $c \in V_2$ such that $c\Gamma a \subseteq I$. So $c\Gamma(m\gamma a) \subseteq I$. If $m\gamma a \notin I$, then $m\gamma a \in V_1$ which implies $m\gamma a \in P_1$. Otherwise $m\gamma a \in I$. Then $m\gamma a \in P_1$. Remaining part is trivial. So P_1 is an ideal of M.

Suppose $a\Gamma b \subseteq P_1$ and $a, b \notin P_1$. Since $P_1 = V_1 \cup I$, for any $\gamma \in \Gamma$ we have either $a\gamma b \in I$ or $a\gamma b \in V_1$, and so in any case there exists $c \in V_2$ such that $c\Gamma(a\Gamma b) \subseteq I$. So $a\Gamma(c\Gamma b) \subseteq I$. If $c\Gamma b \subseteq I$, then $b \in V_1$, a contradiction. Hence there exists $\gamma \in \Gamma$ such that $c\gamma b \notin I$ which implies $c\gamma b \in V_1$. Therefore for any $\gamma \in \Gamma$ again we have either $c\gamma b \in I$ or $c\Gamma(c\gamma b) \subseteq I$ which implies $(c\Gamma c)\Gamma b \subseteq I$. Since $I = \sqrt{I}$, we have there exists $\gamma_1 \in \Gamma$ such that $c\gamma_1 c \notin I$, so $c\gamma_1 c \in V_2$. So $b \in V_1$ a contradiction. Thus, P_1 is a prime ideal of M. □

The following example shows that the condition $I = \sqrt{I}$ on ideal I is not superficial in Theorem 2.6 (2).

Example 2.7. Let $M = \{\{a, b, c\}, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ and $\Gamma = \{\phi, \{a\}\}$. If $ABC = A \cap B \cap C$ and $A \leq C \Leftrightarrow A \subseteq C$ for all $A, C \in M$ and $B \in \Gamma$, then M is a po-Γ-semigroup with zero element $\{\phi\}$. If $I = \{\{\phi\}, \{b\}, \{c\}\}$, then $I \neq \sqrt{I}$ and $\Gamma_I(M)$ is a complete bipartite graph with vertex set $\{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, but I can not be written as the intersection of two prime ideals of M.

Hereafter we assume that M is a commutative po-Γ-semigroup (i.e., $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$). The scole of I in M, denoted $\sum(I)$, is defined to be the union of all the minimal ideals of I in M.

Lemma 2.8. *If J is a minimal ideal of I in M, then for any $x \in J \backslash I$, we have $J = I \cup (M\Gamma x]$ or $J = (I \cup \{x\}]$ with $(M\Gamma x] \subseteq I$.*

Proof. If $(M\Gamma x] \subseteq I$, then $I \subseteq (I \cup \{x\}] \subseteq J$ which implies $J = (I \cup \{x\}]$. Otherwise $(M\Gamma x] \nsubseteq I$. Then $I \subset I \cup (M\Gamma x] \subseteq J$ implies $J = I \cup (M\Gamma x]$. □

Lemma 2.9. *If $x \in \sum(I)$ and $(M\Gamma x] \nsubseteq I$, then $x \in (M\Gamma x]$.*

Proof. If $x \in \sum(I)$, then $x \in J$, where J is some minimal ideal of I in M. Since $(M\Gamma x] \subset J$ and $(M\Gamma x] \nsubseteq I$, we have $J = I \cup (M\Gamma x]$ which implies $x \in (M\Gamma x]$. □

Lemma 2.10. *If $I \subset (M\Gamma y] \subseteq (M\gamma x]$, then $(I : x) \subseteq (I : y)$.*

Proof. By Lemma 2.8, we have $y \in (M\Gamma y]$. If $m\Gamma x \subseteq I$, then since $(M\Gamma y] \subseteq$

$(M\Gamma x]$, $y \leq m_1\gamma_1 x$ for some $m_1 \in M$ and $\gamma_1 \in \Gamma$ which implies $m\Gamma y \subseteq I$. $\square$

Theorem 2.11. *If $M = \sum(I)$ and $(M\Gamma x] \nsubseteq I$ is a minimal ideal of I, then $x\Gamma x \nsubseteq I$.*

Proof. Let $M = \sum(I)$ and $(M\Gamma x] \nsubseteq I$ be a minimal ideal of I. Then there exists $m \in M$ and $\gamma \in \Gamma$ such that $m\gamma x \notin I$. By Lemma 2.9, we have $x \in (M\Gamma x]$.

Case (i) $I \subseteq (M\Gamma m\gamma x]$, then the minimality of $(M\Gamma x]$ implies either $(M\Gamma m\gamma x] = (M\Gamma x]$ or $(M\Gamma m\gamma x] = I$.

Suppose $(M\Gamma m\gamma x] = (M\Gamma x]$. Then $x \in (M\Gamma x] = (M\Gamma m\gamma x] \subseteq (m\Gamma M]$. Using $m \in M = \sum(I)$ and $m\gamma x \notin I$, the minimality of $(M\Gamma m]$ forces $(M\Gamma x] = (M\Gamma m]$. Then by Lemma 2.10, we have $(I : x) = (I : m)$. If $x\Gamma x \subseteq I$, then $m\gamma x \in I$, a contradiction.

If, on the other hand, $(M\Gamma m\gamma x] = I$, then $(I \cup m\gamma x] \subseteq (M\Gamma x]$. By minimality of $(M\Gamma x]$ implies $(M\Gamma x] = (I \cup m\gamma x]$ which implies $x \leq m\gamma x$ and $m\gamma x \leq m\gamma m\gamma x \in M\Gamma m\gamma x$. Then $m\gamma x \in (M\Gamma m\gamma x] = I$, a contradiction.

Case (ii) $I \nsubseteq (M\Gamma m\gamma x]$. Then the minimality of $(M\Gamma x]$, we have $I \cup (M\Gamma m\gamma x] = (M\Gamma x]$. Since $m \in \sum(I)$, we have $m \in I \cup (M\Gamma m]$ is a minimal deal of I in M and so $x \in (M\Gamma x] = I \cup (M\Gamma m\gamma x] \subseteq I \cup (M\Gamma m]$. By minimality of $I \cup (M\Gamma m]$, we have $(M\Gamma x] = (I \cup M\Gamma m]$ which implies $m \leq m_1\gamma_1 x$ for some $m_1 \in M$ and $\gamma_1 \in \Gamma$. If $x\Gamma x \subseteq I$, then $m\gamma x \leq m_1\gamma_1 x\gamma x \in I$, a contradiction. $\square$

We now construct a new graph on the scole of M, denoted by $\Gamma(I)$, is the graph whose vertices are the set of minimal ideals($\neq I$) of I in M with distinct vertices J and K adjacent if and only if $J\Gamma K \subseteq I$. A graph in which each pair of distinct vertices is joined by an edge is called complete graph. We use K_n for the complete graph with n vertices. The following theorem gives a relationship between $\Gamma_I(M)$ and $\Gamma(I)$.

Theorem 2.12. *Suppose $M = \sum(I)$. Then the complete graph on n vertices K_n is a subgraph of $\Gamma(I)$ if and only if K_n is a subgraph of $\Gamma_I(M)$.*

Proof. Suppose that K_n is a subgraph of $\Gamma_I(M)$. Then there exist $n-$ vertices $x_i \in M \backslash I$ such that $x_i\Gamma x_j \subseteq I$ for all $i \neq j$. We now show that these x_i belongs to distinct minimal ideals in M.

If x_i and x_j belongs to $I \cup (M\Gamma x_i]$ and $I \cup (M\Gamma x_j]$, respectively, and if $(M\Gamma x_i] = (M\Gamma x_j]$, then by Lemma 2.10, we have $(I : x_i) = (I : x_j)$. Since $x_i\Gamma x_j \subseteq I$, we have $x_i\Gamma x_i \subseteq I$, a contradiction to Theorem 2.11.

If $(I \cup \{x_i\}]$ or $(I \cup \{x_j\}]$ are minimal ideals with $(M\Gamma x_i] \subseteq I$ or $(M\Gamma x_j] \subseteq I$, then x_i and x_j lies in different minimal ideal of M. Consequently, K_n is a subgraph of $\Gamma(I)$.

Conversely, if K_n is a subgraph of $\Gamma(I)$, then there exist n distinct minimal ideals I_i of I in M with $I_i \Gamma I_j \subseteq I$ for $i \neq j$. These ideals are either of the form $(M\Gamma x_i] \nsubseteq I$ or $(I \cup \{x_i\}]$ with $(M\Gamma x_i] \subseteq I$ for $x_i \in M \backslash I$. Using these n distinct x_i, we can construct K_n as a subgraph of $\Gamma_I(M)$. □

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