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BALANCED GROWTH PATH IN A CAPITAL-RESOURCE GROWTH MODEL

Paolo Russu

DiSEA Department of Economics and Business
University of Sassari
Via Muroni 25, 07100 Sassari, ITALY

Abstract: We study a growth model with environmental externalities where a single consumption good is obtained using a renewable resource in combination with physical capital. Both inputs are essential for production and technical substitutes. In this context we analyze the issues of sustainability, long-run, associated with the competitive equilibrium solution trajectories. We show that long-run growth and sustainability are both compatible in a natural resource based production economy.

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1. Introduction

Sustainable development is a topic of concern among economists and natural scientists, as well as among development agencies and the general public, even though the concept carries different meanings for these different actors ([8]). The objective of this paper is to study the relationships between physical and natural capital in the evolution of an economic activity which depends on the amount and quality of the natural environment, such as resource-based economy.

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We analyze a growth model with environmental externalities. In particular, we study the equilibrium growth dynamics of an economy constituted by a continuum of identical agents. At each instant of time t, the representative agent produces the output $Y(t)$ by fraction of environmental resource γ at disposal E, by the accumulated physical capital $K(t)$ and by the stock $E(t)$ of a free-access renewable environmental resource ([4], [2]).

Over the years many studies have been conducted on models of growth with environmental constraints (i.e. [1], [3], [5],[6], [7], [10], [11], [12]).

In [1], the economy-wide aggregate production $\overline{Y}(t)$ negatively affects the stock of the environmental resource; however, the value of $\overline{Y}(t)$ is considered as exogenously determined by the representative agent, so that economic dynamics is affected by negative environmental externalities. In our model, instead, the aggregate fraction $\overline{N} = \overline{\gamma}E$ negatively affects the stock of the environmental resource and it is consider as exogenously determinate by representative agent.

2. The Model Economy

The economy we analyze is constituted by a continuum of identical economic agents. At each instant of time $t \in [0\infty)$, the representative agent produces an output $Y(t)$ by the following Cobb-Douglas technology

$$Y(t) = K(t)^{\alpha}(N(t))^{1-\alpha}, \tag{1}$$

where $K(t)$ is the stock of physical capital accumulated by the representative agent and $N(t)$ is the quote of stock of an open-access renewable natural resource required for production of the final good in our economy. The stock of this natural capital at date t is denoted by $E(t)$. Following [2], we assume that such a stock is composed of homogeneous units and that it changes over time because of two different flows that have opposite and offsetting effects on the stock. First, in the absence of any human economically-based intervention, the natural resource evolves according to a biotic law of motion that suggests an exponential growth at a constant rate $\epsilon > 0$. This implies that our natural resource is not subject to the traditional biological laws that apply to animal species, commonly represented by the logistic equation, or that we abstract from the negative feedbacks associated with overcrowding and environmental resistance. Beyond such phenomena, the intrinsic constant growth rate ϵ still may be considered as the net result of different exogenous natural processes: births, deaths and human ecologically-based interventions.

Second, the stock of natural capital is subject to an economically motivated extraction process, or harvesting activity [1], because it is necessarily required for production of the final good in our economy. We define $\gamma(t)$ as the aggregate extraction rate, with $\gamma(t) \in [0; \epsilon]$, and assume that there are many individual firms, each of them extracting a percentage $\gamma_i(t)$ from the aggregate stock, therefore, we get: $\gamma(t) = \sum_i \gamma_i(t)$.

Finally, we assume a linear harvesting function according to which the renewable resource diminishes, each period, by the amount $\gamma(t)E(t)$. In short, we assume that resources used for harvesting are homogeneous, all the harvesters have the same objective function and the marginal product to effort is equal to the average one.

3. The Optimal Control Problem in a Decentralized Economy

There are a constant number of identical individuals indexed by the superscript i, each of which maximizes intertemporal utility

$$\int_0^\infty U(C_i(t), \gamma_i(t))e^{-\rho t}dt, \tag{2}$$

where $\rho > 0$ is the rate of time preference, while the utility function, increasing and concave is

$$U\left[C_i(t), \gamma_i(t)\right] = \frac{\left[C_i(t)\gamma_i(t))^{-\psi}\right]^{1-\sigma} - 1}{1-\sigma},$$

where ψ, $\sigma > 0$ and $\sigma \neq 1$. and $\psi > 0$ gives the (dis)utility of additional harvest of resource. To see the effect of γ_i on the marginal utility of consumption we compute the cross-derivative of the utility function given by:

$$\frac{\partial^2 C_i}{\partial C_i \partial \gamma_i} = -\psi(1-\sigma)C_i^{-\sigma-1}\gamma_i^{-\psi(1-\sigma)} > (<)0 \rightarrow \frac{1}{\sigma} < (>)1. \tag{3}$$

Equation (3) shows that the marginal utility of consumption declines when extraction rate γ_i rises if the inter-temporal elasticity of consumption is larger. If the inter-temporal elasticity of consumption is smaller, the negative effect of an additional unit of γ_i is smaller the higher the consumption. The latter means that a rise in consumption reduces the negative effect of extraction rate at the

[1]In modelling production activity based on open-access natural resources (for example, fishery, forestry and nature-based tourism)

margin. Equation (3) suggests that consumption and a preservation of the natural resource, i.e. a small value of γ_i, are complementary for $\frac{1}{\sigma} > 1$ because, in this case, the marginal utility of consumption rises with a decline in the level of extraction. This means that the higher the marginal utility of consumption, the cleaner the environment. For $\frac{1}{\sigma} < 1$, the consumption and extraction rate can be considered as substitutes because the marginal (dis)utility of additional value of γ_i declines with a rising level of consumption.

The evolution of $K_i(t)$ (assuming, for simplicity, the depreciation of K_i to be zero) is represented by the differential equation

$$\dot{K_i} = K_i^{\alpha}(\gamma E)^{1-\alpha} - C_i, \tag{4}$$

where $\dot{K_i}$ is the time derivative of K_i. This production function exhibits constant returns to scale over private internal factors. In order to model the dynamics of E we obtain the following law of motion

$$\dot{E} = \epsilon E - \overline{N}, \tag{5}$$

where $\overline{N} = \overline{\gamma_i}E$ is the extraction natural resource average. Dynamic (5) says that the representative agent considers $\overline{N}$ as exogenously determined, being economic agents a continuum, the impact on $\overline{N}$ of each one is null. However, since agents are identical, ex post $\overline{N} = \gamma_i E$ holds. This implies that the trajectories resulting from our model are not socially optimal but Nash equilibria, because no agent has an incentive to modify his choices if the others don't modify theirs (see i.e. [1], [13]).

The current value Hamiltonian associated with the previous intertemporal optimization problem (2) subject to (4)-(5) may be written as (from now on we omit the superscript i)

$$\mathcal{H} = \frac{C^{1-\sigma}\gamma^{-\psi(1-\sigma)} - 1}{1-\sigma} + \lambda\dot{K}, \tag{6}$$

where λ is the positive costate variable for K . We solve for a symmetric equilibrium then the first order necessary conditions are:

$$\frac{\dot{\lambda}}{\lambda} = \rho - \alpha K^{\alpha-1}(\gamma E)^{1-\alpha} \tag{7}$$

$$H_C = C^{\sigma}\gamma^{-\psi(1-\sigma)} - \lambda = 0 \tag{8}$$

$$H_\gamma = -\psi\frac{C}{\gamma}\lambda + (1-\alpha)K^{\alpha}(\gamma E)^{-\alpha}E\lambda = 0. \tag{9}$$

From (9) we obtain

$$C = \frac{1-\alpha}{\psi} K^{\alpha} (\gamma E)^{1-\alpha}, \tag{10}$$

which, differentiate once leads to

$$\frac{\dot{C}}{C} = \alpha \frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{\gamma}}{\gamma} + (1-\alpha)\frac{\dot{E}}{E}. \tag{11}$$

Furthermore, differentiating w.r.t , equation (8) and using equation (7) we show that

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(\alpha K^{\alpha-1}(\gamma E)^{1-\alpha} - \rho). \tag{12}$$

Dividing, the equation (4), by K, and inserting the equation (10), we can write

$$\frac{\dot{K}}{K} = \phi K^{\alpha-1}(\gamma E)^{1-\alpha}, \tag{13}$$

with $\phi = 1 - \dfrac{1-\alpha}{\psi}$.

Recalling that $\dfrac{\dot{E}}{E} = \epsilon - \gamma$ and obtaining the dynamic of γ from equations (11) and (12), the system to be analyzed becomes

$$\frac{\dot{K}}{K} = \phi \left(\frac{E}{K}\right)^{1-\alpha} \gamma^{1-\alpha} \tag{14}$$

$$\frac{\dot{E}}{E} = \epsilon - \gamma \tag{15}$$

$$\frac{\dot{\gamma}}{\gamma} = \frac{\alpha}{1-\alpha}\left(\frac{1}{\sigma} - \phi\right)\left(\frac{E}{K}\right)^{1-\alpha} \gamma^{1-\alpha} + \gamma - \frac{\rho}{\sigma(1-\alpha)} - \epsilon, \tag{16}$$

which is a three-equation system in the paths of K,E,γ.

3.1. The Balanced Growth Path (BGP)

In this section, we derive the centralized solution of the model, when the central planner takes γ as control variable. Now, we define a balance growth path as follows (see [9])

Definition 1. A *BGP* is one where the economy is in equilibrium and where the consumption, natural resource, and private capital grow at the same strictly positive constant growth rates, i.e. $\dfrac{\dot{C}}{C} = \dfrac{\dot{E}}{E} = \dfrac{\dot{K}}{K} = g$, $g > 0$, $g =$ *constant*, and $\dfrac{\dot{\gamma}}{\gamma} = 0$.

This definition implies that C, K, E, grow at the same rate on BGP. As concerns γ, it is either constant, (i.e. its growth rate is zero), on the BGP.

Proposition 2. Under assumptions that $1-\alpha < \psi < min\{1,\sigma\}$, the variables $C(t)$, $E(t)$, $K(t)$, grows at constant rate $g = \epsilon - \gamma^\star = \dfrac{\rho\phi}{\alpha - \sigma\phi}$.

Moreover

$$E(t) = E_0 e^{gt} \tag{17}$$

$$K(t) = \frac{E_0}{\gamma^\star}\left(\frac{\alpha-\phi\sigma}{\rho}\right)^{\frac{1}{1-\alpha}} e^{gt} \tag{18}$$

$$C(t) = \frac{1-\alpha}{\psi}\left(\frac{\epsilon-\gamma^\star}{\phi}\right) K(t). \tag{19}$$

Proof. We can reduce the above system (14)-(16) by introducing the auxiliary variable $x = \dfrac{E}{K}$, giving

$$\frac{\dot{x}}{x} = \epsilon - \gamma - \phi(x\gamma)^{1-\alpha} \tag{20}$$

$$\frac{\dot{\gamma}}{\gamma} = \frac{\alpha}{1-\alpha}\left(\frac{1}{\sigma}-\phi\right)(x\gamma)^{1-\alpha} + \gamma - \frac{\rho}{\sigma(1-\alpha)} - \epsilon. \tag{21}$$

It is easily see that $\dot{x} = \dot{\gamma} = 0$, leads to

$$\gamma^\star := \gamma = \epsilon - \frac{\rho\phi}{\alpha-\sigma\phi} \quad \text{and} \quad x^\star = \left(\frac{\rho}{\alpha-\phi\sigma}\right)^{\frac{1}{1-\alpha}} \frac{1}{\gamma^\star}. \tag{22}$$

Solving differential equation (15) and from $K(t) = \frac{E(t)}{x^\star}$, we obtain (17) and (18), while $C(t)$ can be obtain noting that $(\gamma^\star x^\star)^{\frac{1}{1-\alpha}} = \frac{\epsilon-\gamma^\star}{\phi}$ and substituting in (10).

Being

$$\frac{\dot{\lambda}}{\lambda} = \rho - \alpha\frac{\epsilon-\gamma^\star}{\phi} = -\frac{\rho\phi}{\alpha-\phi\sigma},$$

the transversality condition, that is $\lim_{t\to\infty}\lambda(t)K(t)e^{-\rho t} = 0$, implies the following condition $\phi - \alpha < 0$, that together with the conditions $\alpha - \phi\sigma > 0$ and $\phi > 0$ lead to $1-\alpha < \psi < min\{1,\sigma\}$ and this conclude the proof. □

Proposition 2 states that the economy is lying on the balanced growth path for sufficiently low values of control variable ψ. It also provides the initial values of state variables K, E, C and γ, so that the economy is on BGP. Note that given the value of K_0, it is possible to determine E_0 and consequently determine the initial values of the two control variables C_0 and $\gamma_0 = \gamma^\star$.

The social welfare function is given by $W = \int_0^\infty \frac{(C\gamma^{-\psi})^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$; and if the economy initially is on the steady state equilibrium growth path, it can be shown that

$$W = \frac{C_0^{1-\sigma}(\gamma^\star)^{-\psi(1-\sigma)}}{(1-\sigma)(\rho - g(1-\sigma))} - \frac{1}{\rho(1-\sigma)}. \tag{23}$$

4. Conclusion

We analyze a growth model with environmental externalities. In particular, we study the equilibrium growth dynamics of an economy constituted by a continuum of identical agents. At each instant of time t, the representative agent produces the output $Y(t)$ by fraction of environmental resource $\gamma(t)$ at disposal E, by the accumulated physical capital $K(t)$ and by the stock $E(t)$ of a free-access renewable environmental resource.

We determine the dynamics of both the state and control variables on the balanced growth path (BGP).

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