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A NEW APPROACH FOR NONCONVEX SIP

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Abstract: We propose a new method for solving nonconvex semi-infinite problems by using a concave overestimation function of the semi-infinite constraints. At each iteration we solve a nonlinear programming problem locally which gives a feasible point, for certain problems the feasibility is so important than the optimality (e.g. in control systems design). If we decide to stop our algorithm after a finite number of iterations, we have an optimal solution or an approximate solution which is feasible.

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1. Introduction

We consider the following problem:

$$(SIP) \left\{ \begin{array}{c} \min f(x) \\ g(x,s) \leq 0, \forall s \in S = [s_0, s_1] \subset R \\ x \in X \in R^n \end{array} \right.$$

with$X = [x^l, x^u] \subset R^n, x_i^l < x_i^u, i = 1, ..., n$, S a compact of R, $f \in C^2(R^n, R)$ and $g \in C^2(R^n \times R, R)$, f and/or g are nonconvex. Problems of this type, in which a finite-dimensional decision variables is subject to infinitely many inequality constraints, are called semi-infinite. The existing methods to solve

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(SIP) problem like the exchange method, the discretization method, the reduction method are based on the reduction to a sequence of constrained nonlinear programming, see the paper of [4] which has a fairly complete bibliography on the subject. In fact, discretization and exchange methods approximate the feasible set of (SIP) by finitely many inequalities corresponding to finitely many indices in S, yielding an outer approximation of this set, and reduction-based methods solve the Karush-Kuhn-Tucker system of (SIP) by a Newton-SQP approach. As a consequence, the iterates of these methods are not necessarily feasible for (SIP), but only their limit might be. In [1] the interval method is used for the upper bound and a relaxation method for the discretized problem is used to compute the lower bound. In [3] an adaptive convexification algorithm for a feasible point is presented which consists in the use of the αBB method for the lower problem, the goal of the method is to compute a stationnary method of semi-infinite problem, the index set is equal to $[0, 1]$ while in [11] they consider the problem with an arbitrary index sets. In [8] for the lower bound they use the discretization method with the first and the second necessary conditions for the upper bound, the Mc Cormick relaxation is used for the lower level problem. An hybrid algorithm is used in [2] which combines the deterministic and stochastic global optimization algorithms, a penalty method is applied for the semi-infinite problem, the genetic algorithm solves the penalty problem while the penalty term is computed by the interval analysis method. In [7] a homotopy method is considered which consists in the reformulation of the SIP as two systems of KKT, he ensures the feasibility of the stationnary point of SIP problem by using a homotopy function. Techniques of global optimization are used in [5] for semi-infinite and generalized semi-infinite programs, for the later the index set depends on x.In our method, by using our upper bound function of the semi-infinite constraints, we do not need to use the NCP functions but we obtain directly a mathematical problem to which an interior point method is applied, the iterates generated by our algorithm are always feasible. The paper is organized as follows: In the Section 2, we present a method how to construct an upper bound function. Section 3 contains the different steps of our method, the algorithm and its convergence are presented in Section 4. An application is given in the Section 5.

2. Upper Bound Function

We now explain how to construct an upper bound function of a function of class C^2 on an interval $[a, b]$.

In [6] a lower bound function of f is proposed

$$Lf(x) := f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a} - \frac{1}{2}\underline{K}(x-a)(b-x) \leq f(x),\ \forall x \in [a,b]\,.$$

$\underline{K}$ satisfies the inequality $\underline{K} \geq \max\{0, f''(x)\}, \forall x \in [a,b]$

In the same way we introduce a quadratic upper bound function of f:

Theorem 2.1. $\forall x \in [a,b]$,

$$Uf(x) := f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a} + \frac{1}{2}\overline{K}(x-a)(b-x) \geq f(x)$$

with $\overline{K} \geq \max\{0, -f''(x)\}$.

Proof. Consider the function ϕ defined on $[a,b]$ by

$$\phi(x) := Uf(x) - f(x) = f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a} + \frac{1}{2}\overline{K}(x-a)(b-x) - f(x).$$

It is clair that $\phi''(x) = -\overline{K} - f''(x) \leq 0, \forall x \in [a,b]$, so ϕ is concave function $\forall x \in [a,b]$ one has

$$\phi(x) \geq \min\{\phi(x) : x \in [a,b]\} = \phi(a) = \phi(b) = 0.$$

Then the inequality is proved. □

3. The Method

The problem

$$(SIP)\begin{cases} \min f(x) \\ g(x,s) \leq 0, \forall s \in S \in R \\ x \in R^n \end{cases}$$

is equivalent to Stackelberg Game problem see [3]

$$(SG)\begin{cases} \min f(x) \\ g(x,s) \leq 0, s \ solves\ Q(x) \\ x \in R^n \end{cases}$$

with $Q(x)$ the lower level problem

$$(Q(x))\begin{cases} \max g(x,s) \\ s \in S = [s_0, s_1] \in R \end{cases}$$

If g is concave with respect to s then the lower problem is equivalent to the KKT system, but in general this condition isn't satisfied. We use an overestimation function of g,

$$Ug(x,s) = g(x,s_0)\frac{s_1 - s}{s_1 - s_0} + g(x,s_1)\frac{s - s_0}{s_1 - s_0} + \frac{1}{2}\overline{K}_g(s - s_0)(s_1 - s),$$

$$\overline{K}_g \geq \max\{0, -g''_s(x,s)\}\forall(x,s) \in X \times S,$$

$\overline{K}_g$ is computed by the interval method or by the method developed in [5].

$Ug(x,s)$ is quadratic concave wrt s which is different that of [3].

We have the following problem which gives a feasible solution of (SIP)

$$(P^U)\begin{cases} \min f(x) \\ Ug(x,s) \leq 0, \forall s \in S \in R \\ x \in R^n \end{cases}$$

It is equivalent to Stackelberg Game problem

$$(SG^U)\begin{cases} \min f(x) \\ Ug(x,s) \leq 0, s \ solves \ Q^U(x) \\ x \in R^n \end{cases}$$

with $Q^U(x)$ the lower level problem

$$(Q^U(x))\begin{cases} \max Ug(x,s) \\ s \in S = [s_0, s_1] \in R \end{cases}$$

which is equivalent to the KKT system

$$(KKTS)\begin{cases} -(Ug(x,s))' + \lambda_1 - \lambda_2 = 0 \\ \lambda_1(s_1 - s) = 0 \\ \lambda_0(s - s_0) = 0 \\ \lambda_1 \geq 0, \lambda_0 \geq 0 \end{cases}$$

Instead of using the regularized NCP functions as in [3], we use the equation $-(Ug(x,s))' + \lambda_1 - \lambda_0 = 0$, we find

$$s = \frac{s_1 + s_0}{2} + \frac{g(x,s_1) - g(x,s_0)}{\overline{K}_g(s_1 - s_0)} + \frac{\lambda_0 - \lambda_1}{\overline{K}_g}.$$

We replace in the complementarity constraints one obtains

$$\frac{s_1 - s_0}{2}(\lambda_0 - \lambda_1) + \frac{g(x,s_2) - g(x,s_1)}{\overline{K}_g(s_1 - s_0)} + \frac{\lambda_0^2 - \lambda_1^2}{\overline{K}_g} = 0.$$

Then the problem (P^U) is equivalent to

$$\begin{cases} \min f(x) \\ Ug(x, \frac{s_1+s_0}{2} + \frac{g(x,s_1)-g(x,s_0)}{\overline{K}_g s_1 - s_0)} + \frac{\lambda_0 - \lambda_1}{\overline{K}_g}) \leq 0, \\ \frac{s_1-s_0}{2}(\lambda_0 - \lambda_1) + \frac{g(x,s_2)-g(x,s_1)}{\overline{K}_g(s_1-s_0)}(\lambda_0 + \lambda_1) + \frac{\lambda_0^2 - \lambda_1^2}{\overline{K}_g} = 0 \\ x \in R^n, \lambda_0 \geq 0, \lambda_1 \geq 0 \end{cases}$$

At the iteration k: Let $U_i g(x,s)$ is a upper bound function of $g(x,s)$ on the interval $[s_0^i, s_1^i]$, λ_0^i, λ_1^i the corresponding multipliers for the lower level problem for $i = 1, ..., k$. We obtain the following problem

$$\left(PM^{kU}\right) \begin{cases} \min f(x) \\ U_i g(x, \frac{s_1^i+s_0^i}{2} + \frac{g(x,s_1^i)-g(x,s_0^i)}{\overline{K}_g(s_1^i-s_0^i)} + \frac{\lambda_0^i - \lambda_1^i}{\overline{K}_g}) \leq 0, i = 1, ..., k \\ \frac{s_1^i-s_0^i}{2}(\lambda_0^i - \lambda_1^i) + \frac{g(x,s_1^i)-g(x,s_0^i)}{\overline{K}_g s_1^i - s_0^i)}(\lambda_0^i + \lambda_1^i) + \frac{(\lambda_0^i)^2 - (\lambda_1^i)^2}{\overline{K}_g} = 0, \\ \qquad i = 1, ..., k \\ x \in R^n, \lambda_0^i \geq 0, \lambda_1^i \geq 0, i = 1, ..., k \end{cases}$$

which we solve by interior point algorithm see [9].

Remark 3.1. x^{kU} is the solution of problem (PM^{kU}) and $s^i(x^{kU})$ the solution of lower level problem $Q^{iU}(x)$ for $i = 1, ..., k$.

For all $i = 1, ..., k$ such that $U_i g(x^{kU}, s^i(x^{kU}) = 0$, S_i is subdivided via $s^i(x^{kU})$, we also subdivide the interval S_i with the largest length (i.e. we us the exhaustive $w - subdivision$).

4. Algorithm and its Convergence

4.1. Algorithm

Step 1: Initialization: Compute $\overline{K}_g$ by the interval analysis.

Step 2: For k=1,2,.... Solve the problem$(PM^{kU})$by interior point algorithm, let x^{kU} its solution

Step 3: If x^{kU} is a stationary point for (SIP) (in sense of John see [3]) stop x^{kU} is optimal else subdivide the intervals which correspond to the active constraints and the interval with the largest length and replace them with the new sub-intervals in the problem (PM^{kU}), let k:=k+1 and go to step 2.

Remark 4.1. We can compute $\overline{K}_{gi}$ for each sub-interval S_i to accelerate the convergence.

Lemma 4.1. *$U_s g(x,s) - g(x,s) \leq \frac{(\underline{K}_g+\overline{K}_g)h^2}{8}, \forall s \in S$ with h the length of S and $\underline{K}_g \geq \max\{0, g''_s(x,s)\} \forall (x,s) \in X \times S$.*

Proof. One has

$$
\begin{aligned}
Lg(x,s) &= g(x,s_0)\frac{s_1-s}{s_1-s_0} + g(x,s_1)\frac{s-s_0}{s_1-s_0} - \frac{1}{2}\underline{K}_g(s-s_0)(s_1-s) \\
&\leq g(x,s) \leq Ug(x,s) \\
&= g(x,s_0)\frac{s_1-s}{s_1-s_0} + g(x,s_1)\frac{s-s_0}{s_1-s_0} + \frac{1}{2}\overline{K}_g(s-s_0)(s_1-s),
\end{aligned}
$$

$$
Ug(x,s) - g(x,s) \leq Ug(x,s) - Lg(x,s) = \frac{1}{2}\overline{K}_g(s-s_0)(s_1-s)
$$

$$
+\frac{1}{2}\underline{K}_g(s-s_0)(s_1-s) = \frac{1}{2}((\overline{K}_g+\underline{K}_g)(s-s_0)(s_1-s) \leq \frac{(\underline{K}_g+\overline{K}_g)h^2}{8}. \quad \square
$$

Remark 4.2. The consequence of this lemma is that $g(x,s) \leq U_s g(x,s) \leq g(x,s) + \frac{(\underline{K}_g+\overline{K}_g)h_i^2}{8}, \forall s \in S_i$ with h_i the length of S_i.

Let $D^{h^k} = \{x \in R^n / g(x,s) + \frac{(\underline{K}_g+\overline{K}_g)h^2}{8} \leq 0, \forall s \in S\}$ with $h = max_{i=1,\dots,k} h_i$,

$$D = \{x \in R^n / g(x,s) \leq 0, \forall s \in S\}$$

the domain of (SIP), and

$$D^{kU} = \{x \in R^n / U_s g(x,s) \leq 0, \forall s \in S_i, \cup S_i = S\}.$$

Lemma 4.2. *For $h^1 \leq h^2$ then $D^{h^1} \supset D^{h^2}$.*

Proof. The proof follows from the inequalities $g(x,s) + \frac{(\underline{K}_g+\overline{K}_g)(h^1)^2}{8} \leq g(x,s) + \frac{(\underline{K}_g+\overline{K}_g)(h^2)^2}{8} \leq 0$ $\square$

Remark 4.3. D^{h^k} is an increasing sequence of domains when the subdivision is refined(i.e. when k is increasing).

Lemma 2.3. *Let D the domain of (SIP), D^{kU} the domain of (MP^{kU}) then we have the inclusions $D \supset D^{kU} \supset D^{h^k}$.*

Proof. One has $g(x,s) \leq 0, \forall s \in S$ is equivalent to $g(x,s) \leq 0, \forall s \in S_i$ with $\cup S_i = S$ see [3].

$\forall s \in S_i, g(x,s) \leq U_s g(x,s) \leq g(x,s) + \frac{(\underline{K}_g+\overline{K}_g)h_i^2}{8} \leq g(x,s) + \frac{(\underline{K}_g+\overline{K}_g)h^2}{8} \leq 0, h = \max h_i, i = 1, \dots., k, \cup S_i = S$ which implies $D \supset D^{kU} \supset D^{h^k}$. $\square$

Theorem 4.2. *The sequence* $\{x^{kU}\}_k$ *generated by our algorithm converges to the solution of* (SIP).

Proof. As the problems (MP^{kU}) and (SIP) have the same objective function then it suffices to show that the domain of (MP^{kU}) tends to the domain of (SIP) when $k \to \infty$.

By the above lemma one has $D \supset D^{kU} \supset D^{h^k}$ and $k \to \infty \Leftrightarrow h \to 0$ because we have an exhaustive $w-subdivision$ then $D^{h^k} \to D$ when $k \to \infty$(i.e. by the inequality $g(x,s) \leq g(x,s) + \frac{(\underline{K}_g + \overline{K}_g)h^2}{8}$).

Hence by "sandwitch" effect, the domain D^{kU} of (MP^{kU}) tends to D the domain of (SIP) when $k \to \infty$ and the solution x^{kU} of the problem (MP^{kU}) tends to the solution of (SIP) when $k \to \infty$ □

5. Conclusion

We have developed in this work a new method for solving nonconvex semi-infinite problems by using a convexification of the lower level problem(i.e. maximization of concave function). By our convexification method, instead of the using the regularized NCP functions, we transform the KKT system for the lower level problem and we obtain a mathematical problem which can be solved by the interior point algorithm. An w-subdivision for the index set is used and the convergence of our algorithm is shown. This work is theoretical, we don't claim its superiority vis-a-vis other works but it is an alternative to solve nonconvex semi-infinite problems.

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