MORE ON THE DIOPHANTINE EQUATION

$8^x + 19^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution. The solution $(x, y, z)$ is $(1, 0, 3)$.

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1. Introduction

In 2012, Peker and Cenberci [4] suggested that the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. In fact, $8^1 + 19^0 = 9 = 3^2$. In this paper, we will show that $(1, 0, 3)$ is a unique solution $(x, y, z)$ for the Diophantine equation $8^x + 19^y = z^2$ where $x, y$ and $z$ are non-negative integers. For related papers, we list them as follows.

In 2007, Acu [1] showed that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions $(x, y, z)$ for the Diophantine equation $2^x + 5^y = z^2$ where $x, y$ and $z$ are non-negative integers. In 2011, Suvarnamani, Singta and Chotchaisthit [6] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. In the same year, Suvarnamani [5] found some non-negative integer solutions for the Diophantine equation of type $2^x + p^y = z^2$ where $p$ is a positive prime number. In 2012, Chotchaisthit [2] gave all non-negative integer solutions for the Diophantine equation of type $4^x + p^y = z^2$.
where \( p \) is a positive prime number.

2. Preliminaries

The Catalan’s conjecture is a well known conjecture. This conjecture states that \((3, 2, 2, 3)\) is a unique solution \((a, b, x, y)\) for the Diophantine equation \(a^x - b^y = 1\) where \(a, b, x\) and \(y\) are integers with \(\min\{a, b, x, y\} > 1\). In 2004, this conjecture was proven in 2004 by Mihailescu [3].

**Proposition 2.1.** [3] \((3, 2, 2, 3)\) is a unique solution \((a, b, x, y)\) for the Diophantine equation \(a^x - b^y = 1\) where \(a, b, x\) and \(y\) are integers with \(\min\{a, b, x, y\} > 1\).

Next, we will prove two Lemmas by Proposition 2.1.

**Lemma 2.2.** \((1, 3)\) is a unique solution \((x, z)\) for the Diophantine equation \(8^x + 1 = z^2\) where \(x\) and \(z\) are non-negative integers.

*Proof.* Let \(x, y\) and \(z\) be non-negative integers such that \(8^x + 1 = z^2\). If \(x = 0\), then \(z^2 = 2\) which is impossible. Then \(x \geq 1\). Thus, \(z^2 = 8^x + 1 \geq 8^1 + 1 = 9\). Then \(z \geq 3\). Now, we consider on the equation \(z^2 - 8^x = 1\). By Proposition 2.1, we have \(x = 1\). Then \(z = 3\). Hence, \((1, 3)\) is a unique solution \((x, z)\) for the equation \(8^x + 1 = z^2\) where \(x\) and \(z\) are non-negative integers. \(\square\)

**Lemma 2.3.** The Diophantine equation \(1 + 19^y = z^2\) has no non-negative integer solution.

*Proof.* Suppose that there are non-negative integers \(y\) and \(z\) such that \(1 + 19^y = z^2\). If \(y = 0\), then \(z^2 = 2\) which is impossible. Then \(y \geq 1\). Thus, \(z^2 = 1 + 19^y \geq 1 + 19^1 = 20\). Then \(z \geq 5\). Now, we consider on the equation \(z^2 - 19^y = 1\). By Proposition 2.1, we have \(y = 1\). Then \(z^2 = 20\). This is a contradiction. Hence, the equation \(1 + 19^y = z^2\) has no non-negative integer solution. \(\square\)

3. Results

In [4], the Diophantine equation \(8^x + 19^y = z^2\) has no non-negative integer solution. But we will show in this section that \((1, 0, 3)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(8^x + 19^y = z^2\) where \(x, y\) and \(z\) are non-negative integers.
**Theorem 3.1.** \((1,0,3)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(8^x + 19^y = z^2\) where \(x, y\) and \(z\) are non-negative integers.

**Proof.** Let \(x, y\) and \(z\) be non-negative integers such that \(8^x + 19^y = z^2\). By Lemma 2.3, we have \(x \geq 1\). Thus, \(z\) is odd. Then there is a non-negative integer \(t\) such that \(z = 2t + 1\). Thus, \(8^x + 19^y = 4(t^2 + t) + 1\). This implies that \(19^y \equiv 1 \pmod{4}\). Then \(y\) is even. Now, we will divide the number \(y\) into two cases.

Case \(y = 0\). By Lemma 2.2, we have \(x = 1\) and \(z = 3\).

Case \(y \geq 2\). Let \(y = 2k\) where \(k\) is a positive integer. Then \(z^2 - 19^{2k} = 2^{3x}\). Then \((z - 19^k)(z + 19^k) = 2^{3x}\). Thus, \(z - 19^k = 2^u\) where \(u\) is a non-negative integer. Then \(z + 19^k = 2^{3x-u}\). Thus, \(2(19^k) = 2^{3x-u} - 2^u = 2^u(2^{3x-2u} - 1)\). We have two subcases.

Subcase \(u = 0\). Then \(z - 19^k = 1\). Thus, \(z\) is even. This is a contradiction.

Subcase \(u = 1\). Then \(2^{3x-2} - 1 = 19^k\). Thus, \(2^{3x-2} - 19^k = 1\). If \(x = 1\), then \(k = 0\) so \(y = 0\). Thus, \(x \geq 2\). By Proposition 2.1, we have \(k = 1\). Then \(2^{3x-2} = 20\). This is impossible.

Therefore, \((1,0,3)\) is a unique solution \((x, y, z)\) for the equation \(8^x+19^y = z^2\) where \(x, y\) and \(z\) are non-negative integers. \(\square\)

**Corollary 3.2.** The Diophantine equation \(8^x + 19^y = w^4\) has no non-negative integer solution.

**Proof.** Suppose that there are non-negative integers \(x, y\) and \(w\) such that \(8^x + 19^y = w^4\). Let \(z = w^2\). Then \(8^x + 19^y = z^2\). By Theorem 3.1, we have \((x, y, z) = (1,0,3)\). Then \(w^2 = z = 3\). This is a contradiction. Hence, the equation \(8^x + 19^y = w^4\) has no non-negative integer solution. \(\square\)

**4. Open Problem**

In [4], \((2,1,9)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(8^x + 17^y = z^2\) where \(x, y\) and \(z\) are non-negative integers. But \(8^1 + 17^0 = 9 = 3^2\). Thus, we may pose a question that "What’s the set of all solutions \((x, y, z)\) for the Diophantine equation \(8^x + 17^y = z^2\) where \(x, y\) and \(z\) are non-negative integers?".
References


