International Journal of Pure and Applied Mathematics
Volume 81 No. 5 2012, 663-669
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF JANOWSKI-SAKAGUCHI TYPE FUNCTIONS

N. Shilpa[1], S. Latha[2 §]

[1,2]Department of Mathematics
Yuvaraja's College
University of Mysore
Mysore, 570 005, INDIA

Abstract: In this paper we derive sufficient condition involving coefficient inequalities for the functions belonging to the classes $\mathcal{K}(A, B, s, t)$, $\mathcal{S}^*(A, B, s, t)$, $\mathcal{K}_\lambda(A, B, s, t)$, $\mathcal{S}^*_\lambda(A, B, s, t)$ defined by using Janowski class and Sakaguchi type functions.

AMS Subject Classification: 30C45
Key Words: coefficient inequalities, Janowski-Sakaguchi type functions

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathcal{P}$ denote the class of analytic functions p defined on $\mathcal{U}$ satisfying $p(0) = 1, \Re\{p(z)\} > 0, z \in \mathcal{U}$. This class $\mathcal{P}$ can be completely characterized in terms of subordination. A function p is in $\mathcal{P}$ if and only if $p(z) \prec \dfrac{1+\omega(z)}{1-\omega(z)}$ where ω is analytic in $\mathcal{H}$ with $\omega(0) = 0, |\omega(z)| < 1$ on $\mathcal{U}$.

Let $\mathcal{H} = \{\omega, \omega \text{ analytic in } \mathcal{U}, \omega(0) = 0, |\omega(z)| < 1, z \in \mathcal{U}\}$.

Received: November 11, 2011

§Correspondence author

Let $\mathcal{P}(A,B)$ denote the Janowski class (see [3]) containing functions p of the form

$$p(z)=\frac{1+A\omega(z)}{1+B\omega(z)}, -1\leq B<A\leq 1, \omega\in\mathcal{H}.$$

Further we have, $\mathcal{P}(1,-1)\equiv\mathcal{P}$.

Now we introduce the following classes of analytic functions:

i. A function $f\in S^{\square}(A,B,s,t)$, if $\dfrac{(s-t)zf^{'}(z)}{f(sz)-f(tz)}\in P(A,B)$.

ii. $f\in S^{\square}_{\lambda}(A,B,s,t)$, if $\dfrac{e^{i\lambda}\dfrac{(s-t)zf^{'}(z)}{f(sz)-f(tz)}-i\sin\lambda}{\cos\lambda}\in P(A,B)$.

The classes $K(A,B,s,t)$ and $K_{\lambda}(A,B,s,t)$ are defined by $f\in K(A,B,s,t)$, if $zf^{'}\in S^{\square}(A,B,s,t)$ and $f\in K_{\lambda}(A,B,s,t)$, if $zf^{'}\in S^{\square}_{\lambda}(A,B,s,t)$ where $s,t\in\mathcal{R}$ with $s\neq t$ for all $z\in\mathcal{U}$ and λ is real and satisfies $|\lambda|<\frac{\pi}{2}$.

For $A=1-2\alpha, B=-1$, this class reduces to the class $S(\alpha,s,t)$ defined by Frasin, see [1]. When $A=1-2\alpha, B=-1, s=1$ we have the class $S(\alpha,1,t)$ introduced by Owa (see [2]), and for $A=1-2\alpha, B=-1, s=1$ and $t=-1$ we have the class $S(\alpha,1,-1)$ introduced and studied by Sakaguchi, see [4]. In the next section we obtain the sufficient conditions involving coefficient inequalities for f to be in the classes $\mathcal{S}^{\square}(A,B,s,t)$ and $\mathcal{K}(A,B,s,t)$.

2. Coefficient Inequalities

Theorem 1. *A function $f\in\mathcal{A}$ is in the class $\mathcal{S}^{\square}(A,B,s,t)$ if and only if*

$$1+\sum_{n=2}^{\infty}A_n z^{n-1}\neq 0 \tag{2}$$

where

$$A_n=\frac{[n-u_n(s,t)]+(nB-Au_n(s,t))\rho}{(B-A)\rho} \quad \text{and} \quad u_n(s,t)=\sum_{j=1}^{n}s^{n-j}t^{j-1}.$$

Proof. A function f belongs to $\mathcal{S}^{\square}(A,B,s,t)$ if and only if

$$\frac{(s-t)zf^{\square}(z)}{f(sz)-f(tz)}\in\mathcal{P}(A,B), \quad \text{for} \quad s\neq t,\ -1\leq B<A\leq 1.$$

But the function $p(z) = \dfrac{1 + A\omega(z)}{1 + B\omega(z)}, -1 \leq B < A \leq 1, \omega \in \mathcal{H}$ are subordinate to $\dfrac{1 + Az}{1 + Bz}$. They map the unit circle $|z| = 1$ onto the boundary of the circle on the line joining $\dfrac{1 - A}{1 - B}$ and $\dfrac{1 + A}{1 + B}$ as diameter. When $B = -1$, the image of the unit circle is the line $\Re\{p(z)\} = \dfrac{1 - A}{2}, -1 < A \leq 1$. The functions $\dfrac{1 + A\omega(z)}{1 + B\omega(z)}$ are analytic and hence map regions onto regions. Therefore every point in the interior of the unit disc goes over to an interior point of the image disc. Thus $f \in \mathcal{S}^{*}(A, B, s, t)$ is equivalent to

$$\frac{(s - t)zf'(z)}{f(sz) - f(tz)} \neq \frac{1 + A\rho}{1 + B\rho}, \quad |\rho| = 1, \quad B\rho \neq -1.$$

This reduces to

$$(1 + B\rho)(s - t)zf'(z) - (1 + A\rho)(f(sz) - f(tz)) \neq 0.$$

Thus, we have

$$(B - A)\rho z + \sum_{n=2}^{\infty}[n(1 + B\rho) - u_n(s, t)(1 + A\rho)]a_n z^n \neq 0$$

where, $u_n(s, t) = \sum_{j=1}^{n} s^{n-j}t^{j-1}$.

Equivalently,

$$(B - A)\rho z \left(1 + \sum_{n=2}^{\infty} \frac{[n(1 + B\rho) - u_n(s, t)(1 + A\rho)]}{(B - A)\rho} a_n z^{n-1}\right) \neq 0. \tag{3}$$

Now dividing both sides of (3) by $(B - A)\rho z, (z \neq 0)$ we obtain

$$1 + \sum_{n=2}^{\infty} \frac{[(n - u_n(s, t)) + (nB - Au_n(s, t))\rho]}{(B - A)\rho} a_n z^{n-1} \neq 0,$$

which completes the proof. □

Theorem 2. *If $f \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^{n} \left[\sum_{j=1}^{k} (-1)^{k-j}(j - u_j(s, t)) \binom{\beta}{k - j} a_j \right] \binom{\gamma}{n - k} \right.\right.$$

$$+ \sum_{k=1}^{n} \left[\sum_{j=1}^{k} (-1)^{k-j} (jB - Au_j(s,t)) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \Bigg| \Bigg) \leq B - A,$$

then $f \in \mathcal{S}^{\square}(A, B, s, t)$.

Proof. For $z \in \mathcal{U}, \beta, \gamma \in \mathcal{R}$ since, $(1-z)^{\beta} \neq 0$ and $(1+z)^{\gamma} \neq 0$ if the inequality

$$\left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right) (1-z)^{\beta} (1+z)^{\gamma} \neq 0 \tag{4}$$

holds true then, we have

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0$$

which is the relation (2) of Theorem (1).

Relation (4) is equivalent to

$$\left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right) \left(\sum_{n=0}^{\infty} (-1)^n \binom{\beta}{n} z^n\right) \left(\sum_{n=0}^{\infty} \binom{\gamma}{n} z^n\right) \neq 0.$$

Equivalently, we have

$$\left(1 + \sum_{n=2}^{\infty} B_n z^{n-1}\right) \left(\sum_{n=0}^{\infty} \binom{\gamma}{n} z^n\right) \neq 0$$

where,

$$B_n = \sum_{j=1}^{n} (-1)^{n-j} A_j \binom{\beta}{n-j}.$$

Further simplification yields,

$$1 + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n} B_k \binom{\gamma}{n-k}\right) z^{n-1} \neq 0, \quad z \in \mathcal{U}.$$

Equivalently, we have

$$1 + \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n} \left(\sum_{j=1}^{k} (-1)^{k-j} A_j \binom{\beta}{k-j}\right) \binom{\gamma}{n-k}\right] z^{n-1} \neq 0.$$

That is, if $f \in \mathcal{A}$ satisfies the succeeding inequality

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}A_j\binom{\beta}{k-j}\right)\binom{\gamma}{n-k}\right| \leq 1.$$

That is, if

$$\frac{1}{\rho(B-A)}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}\left[(j-u_j(s,t))\right.\right.\right.$$
$$\left.\left.\left.+(jB-Au_j(s,t))\rho\right]a_j\binom{\beta}{k-j}\right)\right)\binom{\gamma}{n-k}\right|$$
$$\leq \frac{1}{(B-A)}\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}(j-u_j(s,t))\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}(jB-Au_j(s,t))\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right) \leq 1$$

for, $-1 \leq B < A \leq 1, \rho \in \mathcal{C}, |\rho| = 1$, then $f \in \mathcal{S}^{*}(A,B,s,t)$. □

Since $zf' \in \mathcal{S}^{*}(A,B,s,t)$ if and only if $f \in \mathcal{K}(A,B,s,t)$, from Theorem 2, we have the following result.

Theorem 3. *If $f \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}j(j-u_j(s,t))\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right.\right.$$
$$\left.\left.+\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}j(jB-Au_j(s,t))\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right) \leq B-A,$$

then $f \in \mathcal{K}(A,B,s,t)$.

3. Coefficient Conditions for Functions in the Classes $\mathcal{S}^*_\lambda(A,B,s,t)$ and $\mathcal{K}_\lambda(A,B,s,t)$

For $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$ we define the classes $\mathcal{S}^{*}_\lambda(A,B,s,t)$ and $\mathcal{K}_\lambda(A,B,s,t)$ as follows:

A function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^{*}_{\lambda}(A, B, s, t)$ if and only if

$$\frac{e^{i\lambda}\dfrac{(s-t)zf'(z)}{f(sz)-f(tz)} - i\sin\lambda}{cos\lambda} \in \mathcal{P}(A, B).$$

The class $\mathcal{K}_{\lambda}(A, B, s, t)$ is the subclass of $\mathcal{A}$ consisting of functions f such that $zf' \in \mathcal{S}^{*}_{\lambda}(A, B, s, t)$.

Theorem 4. *A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^{*}_{\lambda}(A, B, s, t)$ if and only if*

$$1 + \sum_{n=2}^{\infty} c_n z^{n-1} \neq 0, \tag{5}$$

where

$$c_n = \frac{(n - u_n(s,t)) + (nB - \gamma u_n(s,t))\rho}{(B - |\gamma|)\rho}, \quad \gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda},$$

$$u_n(s,t) = \sum_{j=1}^{n} s^{n-j} t^{j-1}.$$

Proof. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^{*}_{\lambda}(A, B, s, t)$ if and only if

$$\frac{e^{i\lambda}\dfrac{zf'(s-t)}{f(sz)-f(tz)} - i\sin\lambda}{\cos\lambda} \neq \frac{1+A\rho}{1+B\rho} \qquad (\rho \in \mathcal{C},\ \rho \in \mathcal{C},\ |\rho| = 1).$$

This simplifies into

$$(1 + B\rho)(s-t)zf'(z) - (1+\gamma\rho)(f(sz) - f(tz)) \neq 0,$$
$$\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}.$$

The rest of the proof follows as in Theorem 1. □

Theorem 5. *If $f \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(j - u_j(s,t))\binom{\beta}{k-j}a_j\right)\binom{\gamma}{n-k} + \sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(jB - \gamma u_j(s,t))\binom{\beta}{k-j}a_j\right)\binom{\gamma}{n-k}\right|\right] \leq |B - \gamma|,$$

then $f \in \mathcal{S}^{*}_{\lambda}(A, B, s, t)$ where $\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}$, $-1 \leq B < A \leq 1$.

Proof. Applying the same method as in Theorem 2 we get the result. □

Since $zf' \in \mathcal{S}^*_\lambda(A,B,s,t)$ if and only if $f \in \mathcal{K}_\lambda(A,B,s,t)$, from Theorem 5, we have the following result.

Theorem 6. *If $f \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}j(j-u_j(s,t))\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right.\right.$$
$$\left.\left. + \sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}j(jB-\gamma u_j(s,t))\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right) \leq |B-\gamma|,$$

then $f \in \mathcal{K}_\lambda(A,B,s,t)$, where $\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}$, $-1 \leq B < A \leq 1$.

References

[1] B.A. Frasin, Coefficient inequalities for certain classes of Sakaguchi type functions, *IJNS 1749-3889*, **10**. No. 2 (2010), 206-211.

[2] S. Owa, T. Sekine, R. Yamakawa, On Sakaguchi type functions, *Appl. Math. and Comp.*, **187**, No. 1 (2007), 356-361.

[3] K.S. Padmanabhan, M.S. Ganeshan, Convolution conditions for certain classes of analytic functions, *Indian J. Pure Appl. Math.*, **15**, No. 7 (1984), 777-780.

[4] K. Sakaguchi, On a certain univalent mapping, *J. Math. Soc. Japan.*, **11** (1959), 72-75.