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A DISCRETE APPROXIMATE SOLUTION FOR THE ASYMPTOTIC TRACKING PROBLEM IN AFFINE NONLINEAR SYSTEMS

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Abstract: This paper deals with a general tracking problem for affine nonlinear systems. The approach is quite general. In fact, a Lyapunov function is chosen and a stabilizing structure of the solution to solve a general tracking problem is proposed. Nevertheless, the general final solution is a discrete one and consists of an approximation of the continuous solution. Moreover, there is an assumption which should be guaranteed in order to obtain the explicit expression of the digital final solution. Despite this assumption the solution remains valid for a wide range of applications according to practical tests. The importance of the solution consists of its generality. In fact, any kind of nonlinearity could be taken into account. Also technical nonlinearities such as hysteresis, saturations, and creep could be considered.

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1. Introduction

One of the most important requirements in control system application is to guarantee a tracking of some trajectories. In practical applications it is often

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required to obtain an asymptotic tracking with an arbitrarily time convergence. Moreover, practical applications need discrete solutions for such a kind of problem. To obtain that this paper proposes an approximate discrete solution starting from a continuous one which is calculated using Lyapunov approach. The result is a quite general solution which depends on some parameters which can determinate the velocity of the tracking convergence. It is shown that the tracking convergence can be obtained arbitrarily fast choosing these coefficients. Through the paper the meaning and the limits of this approximate solution which depends on an assumption are discussed. The topic of tracking trajectories is known in the literature. In fact, works as [1] and [2] try to define a perfect tracking problem and to solve it using the assumption that the system is flat. In [3] an algorithm for tracking trajectory is developed basically using a flatness approach. In the presented work, no structural assumptions on the system are used except the affinity to obtain an asymptotic tracking. Moreover, the solution allows us to obtain an arbitrarily fast convergence just setting a parameter. In practical application the affinity hypothesis is a no conservative one. In fact, most of the applications present an affine structure. In [4] a practical application shows the importance of the asymptotic tracking problem. Moreover, [5] an application is pointed out and an asymptotic tracking is proposed using a Lyapunov approach. The paper is organized in the following way. Section 1.1 presents the problem formulation together with a possible solution using Lyapunov approach. A final remark is devoted to the discussion of the assumptions and of the limits of the presented solution. The section of the conclusion closes the paper.

1.1. Problem Formulation

Problem 1. *Let define the following affine nonlinear system:*

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}), \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, *and* $\mathbf{y}(t) \in \mathbb{R}^p$, *with* $n, m, p \in \mathbb{N}$. *If a state trajectory* $\mathbf{x}_d(t)$ *is given to be tracked, find a possible general* $\mathbf{u}(t)$ *which realizes the tracking for the system described by (1) and (2).*

In order to propose a possible solution of **Problem 1**, the following field is defined:

$$\mathbf{K}(t) = \mathbf{G}\big(\mathbf{x}_d(t) - \mathbf{x}(t)\big), \tag{3}$$

where

$$\mathbf{G} = \begin{bmatrix} \lambda_{11} & 0 & ...0_{1n} \\ 0 & 0 & \lambda_{2n} \\ ... & ... & ... \\ ... & ... & ... \\ ... & ... & ... \\ 0 & 0 & \lambda_{mn} \end{bmatrix}$$

with $\lambda_{i,j} \in \mathbb{R}$ and $i = 1, 2, ..., p,\ j, = 1, 2, ..., n$; variable $\mathbf{x}_d(t)$ represents the vector of the desired trajectories. If the following Lyapunov function is defined:

$$\mathbf{V}(\mathbf{K}_i) = \frac{\mathbf{K}_i^2(t)}{2}, \tag{4}$$

then it follows that:

$$\dot{\mathbf{V}}(\mathbf{K}_i) = \mathbf{K}_i(t)\dot{\mathbf{K}}_i(t). \tag{5}$$

In order to find the stability of the solution, it is possible to choose the following functions:

$$\dot{\mathbf{V}}(\mathbf{K}_i) = -\mathbf{M}_d{\mathbf{K}_i}^2(t), \tag{6}$$

where $\mathbf{M}_d$ indicates a diagonal positive definite matrix which is responsible for the velocity of the tracking convergence. $\mathbf{K}_i$ represents each component of the vector field $\mathbf{K}$, with $i = 1, 2, ..., p$. As above mentioned, p is the number of outputs of the system defined in (1). Comparing (5) with (6), the following relation is obtained:

$$\mathbf{K}_i(t)\dot{\mathbf{K}}_i(t) = -\mathbf{M}_d\mathbf{K}_i^2(t), \tag{7}$$

and finally

$$\mathbf{K}_i(t)\big(\dot{\mathbf{K}}_i(t) + \mathbf{M}_d\mathbf{K}_i(t)\big) = 0. \tag{8}$$

The no trivial solution follows from the condition

$$\dot{\mathbf{K}}_i(t) + \mathbf{M}_d\mathbf{K}_i(t) = 0. \tag{9}$$

From (3) it follows:

$$\dot{\mathbf{K}}_i(t) = \mathbf{G}\big(\dot{\mathbf{x}}_d(t) - \dot{\mathbf{x}}(t)\big) = \mathbf{G}\dot{\mathbf{x}}_d(t) - \mathbf{G}\dot{\mathbf{x}}(t). \tag{10}$$

The main idea is to find a $\mathbf{u}_{eq}(t)$, an equivalent input, and after that a $\mathbf{u}(t)$, such that $\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_d(t)$. For that, from (1) it follows that:

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_d(t) = \mathbf{f}(\mathbf{x}_d(t)) + \mathbf{g}(\mathbf{x}_d(t))\mathbf{u}(t), \tag{11}$$

and from (10) the following relation is obtained:

$$\dot{\mathbf{K}}_i(t) = \mathbf{G}\dot{\mathbf{x}}_d(t) - \mathbf{G}\mathbf{f}(x_d(t)) - \mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\mathbf{u}(t) = \mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big(\mathbf{u}_{eq}(t) - \mathbf{u}(t)\big), \quad (12)$$

where $\mathbf{u}_{eq}(t)$ is the equivalent input which, in our case, assumes the following expression:

$$\mathbf{u}_{eq}(t) = \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1}\mathbf{G}\Big(\dot{\mathbf{x}}_d(t) - \mathbf{f}(x_d(t))\Big). \quad (13)$$

After inserting (12) into (9), the following relation is obtained:

$$\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big(\mathbf{u}_{eq}(t) - \mathbf{u}(t)\big) + \mathbf{M}_d\mathbf{K}_i(t) = 0, \quad (14)$$

and in particular:

$$\mathbf{u}(t) = \mathbf{u}_{eq}(t) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1}\mathbf{M}_d\mathbf{K}_i(t). \quad (15)$$

Normally, it is a difficult job to calculate $\mathbf{u}_{eq}(t)$. If equation (12) is rewritten in a discrete form using backward Euler approximation, then it follows:

$$\frac{\mathbf{K}_i((k+1)T_s) - \mathbf{K}_i(kT_s)}{T_s} = \mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big(\mathbf{u}_{eq}(kT_s) - \mathbf{u}(kT_s)\big). \quad (16)$$

Parameter $T_s \in \mathbb{N}$ and represents the sampling time of the discretisation. If equation (15) is also rewritten in a discrete form, then:

$$\mathbf{u}(kT_s) = \mathbf{u}_{eq}(kT_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1}\mathbf{M}_d\mathbf{K}_i(kT_s). \quad (17)$$

Equation (16) can be also rewritten as:

$$\mathbf{u}_{eq}(kT_s) = \mathbf{u}(kT_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1}\frac{\mathbf{K}_i((k+1)T_s) - \mathbf{K}_i(kT_s)}{T_s}. \quad (18)$$

Equation (18) can be estimated to one-step backward in the following way:

$$\mathbf{u}_{eq}((k-1)T_s) = \mathbf{u}((k-1)T_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1}\frac{\mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)}{T_s}. \quad (19)$$

Because of function $\mathbf{u}_{eq}(t)$ being a continuous one, we can write:

$$\mathbf{u}_{eq}(kT_s) \approx \mathbf{u}_{eq}((k-1)T_s). \quad (20)$$

Considering equation (20), then equation (19) becomes:

$$\mathbf{u}_{eq}(kT_s) = \mathbf{u}((k-1)T_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1}\frac{\mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)}{T_s}. \quad (21)$$

Inserting (21) into (17) it follows:

$$\mathbf{u}(kT_s) = \mathbf{u}((k-1)T_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1} \Big(\mathbf{M}_d\mathbf{K}_i(kT_s) + \frac{\mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)}{T_s}\Big), \quad (22)$$

and finally:

$$\mathbf{u}(kT_s) = \mathbf{u}((k-1)T_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))T_s\big)^{-1} \Big(\mathbf{M}_dT_s\mathbf{K}_i(kT_s) + \mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)\Big). \quad (23)$$

Remark 1. Equation (23) states the solution of the proposed problem. This solution depends on different aspects: parameters T_s, λ_{ij} and matrix $\mathbf{M}_d$. The unique critical parameter is parameter T_s which states the sampling time for the discrete approximation in (16). Parameter T_s is responsible for the validity of assumption stated in (20) which the proposed solution is based on. In fact, the convergence of the result is shown just for the continuous case mathematically and through the assumption of equation (20) the result is extended to the discrete case. To conclude, if the dynamics of the considered system is very high (very fast dynamics), then parameter T_s must be chosen very small in order to guarantee assumption (20) and thus the convergence of the proposed solution. The coefficients of diagonal positive definite matrix $\mathbf{M}_d$ are responsible for the velocity of tracking convergence according to equation (6).

2. Conclusions

This paper deals with a general tracking problem for affine nonlinear systems. The approach and its solution are quite general. The importance of the solution consists of any kind of nonlinearity which could be taken into account. Also technical nonlinearities such as hysteresis, saturations, and creep could be considered.

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