International Journal of Pure and Applied Mathematics
Volume 81 No. 5 2012, 693-699
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

COMULTIPLICATION MODULES OVER STRONGLY GRADED RINGS

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Abstract: Let G be a group and $g \in G$. Let R be a commutative G-graded ring and M be a graded R-module. The study of graded R-modules has been investigated for along time, and several results have been found. In this work, we give further results about this topic. Moreover, when R is strongly G-graded ring and M is graded comultiplication R-module, we introduce some theorems on the components M_g of M.

AMS Subject Classification: 13A02
Key Words: graded comuliplication modules, graded modules

1. Introduction

Let (R, G) denote a commutative G-graded ring R over a group G. Let the additive subgroups R_g of R, indexed by the elements $g \in G$, be the homogeneous components. The elements of R_g are called homogeneous elements of degree g. Consider $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. For $x \in R$, x is written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g.

Assume M is a left R-module. If additive subgroups M_g of M, indexed

Received: June 19, 2012

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by the elements $g \in G$, exist such that $M = \bigoplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, then M is called G-graded R-module, and denoted by (M, G). Consider $supp(M, G) = \{g \in G : M_g \neq 0\}$. The study of graded R-modules has received much attention from many authors. For more details, we refer the readers to [1], [2], [3], as well as [4], and the references therein.

2. Definitions

In this section, we review some definitions that will be used in the proofs of our results. Let us start by the following.

Definition 2.1. (see [2]) *A G-graded ring R is said to be strongly graded if and only if for all $g \in G$, $1 \in R_g R_{g^{-1}}$.*

It is clear that, R is a strong G-graded ring if and only if $R_g R_h = R_{gh}$ for all $g, h \in G$.

Definition 2.2. (see [2]) *A G-graded R-module M is said to be strongly graded if $R_g M_h = M_{gh}$ for all $g, h \in G$.*

By this definition, it is easy to show that (R, G) is strong if and only if every graded R-module is strongly graded.

Definition 2.3. (see [3]) Let R be a G-graded ring. An ideal I of R is G-graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$.

Let us give an example showing that, an ideal of a G-graded ring is not G-graded.

Example 2.4. (see [3]) Consider $R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded ring with $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Let $I = \langle 1 + i\rangle$ is ideal of R with $1 + i \in I$. If I is G-graded, then $1 \in I$. Thus, there are $x, y \in \mathbf{Z}$ such that $x + iy \in R$ and $1 = (x + iy)(1 + i)$, and so $x = \frac{1}{2}$, which is a contradiction. Therefore, I is not G-graded.

Definition 2.5. (see [3]) *Let M be a G-graded R-module, and N be an R-submodule of M. We say that N G-graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$.*

An R-submodule of a G-graded R-module need not be G-graded.

Definition 2.6. (see [5]) *A graded R-module M is said to be graded comultiplication R- module if for every graded R-submodule N of M there exists a graded ideal I of R such that $N = (0 :_M I) = \{m \in M : Im = 0\}$.*

Definition 2.7. (see [5]) *A nonzero R-submodule N of M is second R-*

submodule of M if for each $a \in R$, the R-homomorphism mapping $\phi_a : N \longrightarrow N$ defined by $\phi_a(n) = an$ is either surjective or zero.

3. Results

In this section, we introduce and prove our main results. In fact, we establish the following.

Lemma 3.1. *Let M be a graded R-module, and let N be R-submodule of M. Then $Ann(N) = \{r \in R : rN = 0\}$ is a graded ideal of R.*

Proof. Assume that N is an R-submodule of M. On one hand, it is obvious that $Ann(N)$ is an ideal of R. On the other hand, assume that $r \in Ann(N)$, then $r = \sum_{g\in G} r_g$, where $r_g \in R_g$. For any $m \in N$, $\sum_{g\in G} r_g m = rm = 0 \in \{0\}$ which is graded R-submodule of M. So, $r_g m = 0$ for all $g \in G$, which gives that $r_g \in Ann(N)$. Therefore, $Ann(N)$ is graded ideal of R. □

Lemma 3.2. *Let M be a graded R-module. Then M is a graded comultiplication R-module if and only if every graded R-submodule N of M, $N = (0 :_M Ann(N))$.*

Proof. Suppose that M is a graded comultiplication R-module, and that N is a graded R-submodule of M. Then a graded ideal I of R exists such that $N = (0 :_M I)$. In particular, $IN = \{0\}$. So, $I \subseteq Ann(N)$ and hence $(0 :_M Ann(N)) \subseteq (0 :_M I) = N$. This and since $N \subseteq (0 :_M Ann(N))$, we conclude that $N = (0 :_M Ann(N))$. Conversely, by Lemma 3.1, $Ann(N)$ is a graded ideal of R. Thus, M is a graded comultiplication R-module. □

Lemma 3.3. *Let R be a strongly G-graded ring and M be a graded R-module. Assume that I is a graded ideal of R, then $(0 :_M I)_e = (0 :_{M_e} I_e)$.*

Proof. In one side, we assume that $m \in (0 :_M I)_e$, then $m \in M_e$ such that $Im = \{0\}$. So, $I_e m \subseteq Im = \{0\}$, and then $I_e m = \{0\}$, which implies that $m \in (0 :_{M_e} I_e)$. In the other side, we assume that $k \in (0 :_{M_e} I_e)$, then $k \in M_e$ such that $I_e k = \{0\}$. Since R is a strongly graded ring, we get that $I_g k = I_g I_e k = \{0\}$ for all $g \in G$, and then $Ik = \{0\}$. Thus, $k \in (0 :_M I) \bigcap M_e = (0 :_M I)_e$, which yields that $(0 :_M I)_e = (0 :_{M_e} I_e)$. □

As an immediate consequence of Lemmas 3.1 and 3.3, we obtain the following lemma.

Lemma 3.4. *Let R be a strongly G-graded ring and M be a graded R-module. If N is a graded R-submodule of M, then $(0:_M Ann(N))_e = (0:_{M_e} (Ann(N))_e)$.*

Following the same arguments used in the prove of Lemma 3.3, we achieve the following lemma.

Lemma 3.5. *Let R be a strongly G-graded ring. Suppose that M is a graded R-module and that N is a graded R-submodule of M. Then $Ann(N_e) = (Ann(N))_e$.*

Using Lemma 3.4 and Lemma 3.5, we prove the following three theorems.

Theorem 3.6. *Let R be a strongly G-graded ring and M be a graded R-module. Then M is graded comultiplication R-module if and only if M_e is comultiplication R_e-module.*

Proof. Suppose that M is a graded comultiplication R-module. Let N be an R_e-submodule of M_e. Then RN is a graded R-submodule of M, and so $RN = (0:_M Ann(RN))$. Lemmas 3.4-3.5, and since $N = R_eN = (RN)_e = (0:_M Ann(RN))_e = (0:_{M_e} (Ann(RN))_e) = (0:_{M_e} Ann((RN)_e)) = (0:_{M_e} Ann(N))$, we conclude that M is comultiplication R_e-module. Conversely, let N be a graded R-submodule of M. Then N_e is an R_e-submodule of M_e, and thus, by Lemmas 3.4-3.5, $N_e = (0:_{M_e} Ann(N_e) = (0:_{M_e} (Ann(N))_e) = (0:_M Ann(N))_e$. Since R is strongly graded, we obtain that for any $g \in G$, $N_g = R_gN_e = R_g(0:_M Ann(N))_e = (0:_M Ann(N))_g$. Therefore, M is graded comultiplication R-module. □

Theorem 3.7. *Let R be a strongly G-graded ring and M be a graded R-module. If M is comultiplication R_e-module, then $supp(M, G) = \{e\}$.*

Proof. To prove our theorem, let us firstly prove that $Ann(M_g) = Ann(M_e)$ for any $g \in \text{supp}(M, G)$. Given $g \in \text{supp}(M, G)$. Let $r \in Ann(M_g)$. Then $r \in R_e$ such that $rM_g = \{0\}$. As R is strongly graded, $rM_e = R_{g^{-1}}rM_g = \{0\}$ and thus $r \in Ann(M_e)$. Similarly, we show that $Ann(M_e) \subseteq Ann(M_g)$. From this, since M_e and M_g are R_e-submodules of M, and M is comultiplication R_e-module, we conclude that $\{0\} \neq M_g = (0:_{M_e} Ann(M_g)) = (0:_{M_e} Ann(M_e)) = M_e$. Consequently, $g = e$. □

Theorem 3.8. *Let R be a strongly G-graded ring, and let M be a graded R-module. Then M is graded comultiplication R-module if and only if for any R_e-submodule N of M_e and for any ideal I of R_e such that $N \subsetneq (0:_{M_e} I)$, there exists an ideal J of R_e such that $I \subsetneq J$ and $N = (0:_{M_e} J)$.*

Proof. Suppose that M is a graded comultiplication R-module. Let N be an R_e-submodule of M_e and I be an ideal of R_e such that $N \subsetneqq (0 :_{M_e} I)$. It is obvious that $RN = (0 :_M Ann(RN))$, so use Lemmas 3.4-3.5 to obtain that $N = (0 :_{M_e} Ann(N))$. It is clear that $Ann(N) \nsubseteq I$, since if not, then we get that $(0 :_{M_e} I) \subseteq (0 :_{M_e} Ann(N)) = N \subseteq (0 :_{M_e} I)$, which contradicts with the assumption. Choose $J = I + Ann(N)$. Then, since $Ann(N) \nsubseteq I$, we directly derive that $I \subsetneqq J$. Also, we get that $(0 :_{M_e} J) = (0 :_{M_e} I) \bigcap (0 :_{M_e} Ann(N)) = (0 :_{M_e} I) \bigcap N = N$. Conversely, let N be an R_e-submodule of M_e, and let $X = \{A : A \text{ is an ideal of } R_e \text{ with } N \subseteq (0 :_{M_e} A)\}$. Then by Zorn's Lemma, X has a maximal element, say I. If $N \neq (0 :_{M_e} I)$, then by the assumption, an ideal J of R_e exists such that $I \subsetneqq J$ and $N = (0 :_{M_e} J)$, which contradicts with the choice of I. So, $N = (0 :_{M_e} I)$ and hence M_e is comultiplication R_e-module. Therefore, since R is strongly graded, we conclude, by Theorem 3.6, that M is graded comultiplication R-module . □

Theorems 3.6 plus 3.8 and a direct approach, prove the following theorem.

Theorem 3.9. *Let M be a graded comultiplication R-module, and let N be an R_e-submodule of M_e. Suppose that $Ann(N)$ is prime ideal of R_e. Then N is second R_e-submodule of M_e.*

Proof. For $a \in R_e$, let $\phi_a : N \longrightarrow N$ be a R_e-homomorphism mapping defined by $\phi_a(n) = an$. Theorem 3.6 gives that aN is R_e-submodule of M_e such that $aN \subseteq N = (0 :_{M_e} Ann(N))$. If ϕ_a is not surjective, then $aN \subsetneqq N = (0 :_{M_e} Ann(N))$. Thus, by Theorem 3.8, there exists an ideal J of R_e such that $Ann(N) \subsetneqq J$ and $aN = (0 :_{M_e} J)$. Hence, $aJ \subseteq Ann(N)$. Since $Ann(N) \subset J$ is prime ideal of R_e, then $a \in Ann(N)$. Consequently, $aN = \{0\}$ and so ϕ_a is zero map. This completes the proof of the theorem. □

Theorem 3.10. *Let R be a strongly G-graded ring. Assume that M is a graded R-module, and that M_e is a comultiplication R_e-module. If P is a minimal ideal of R_e such that $(0 :_{M_e} P) = \{0\}$, then for any $g \in G$, M_g is cyclic R_e-module.*

Proof. Given $g \in G$. Without loss of generality, we may assume that $M_g \neq \{0\}$. Then there exist a non-zero element $m \in M_g$ and an ideal I of R_e so that $R_{g^{-1}}m = (0 :_{M_e} I)$. By this and the assumption, we obtain that $R_{g^{-1}}m = ((0 :_{M_{ee}} P) :_{M_e} I) = (0 :_{M_e} IP)$. As P is minimal ideal and $\{0\} \subseteq IP \subseteq P$, then either $IP = \{0\}$ or $IP = P$. If $IP = P$, we get that $R_{g^{-1}}m = (0 :_{M_e} P) = \{0\}$, and then $R_em = R_g.\{0\} = \{0\}$, which is a contradiction since $0 \neq m \in R_em$.

Thus, $IP = \{0\}$, and so $R_{g^{-1}}m = M_e$. Therefore, $R_e m = R_g M_e = M_g$, which proves that M_g is cyclic R_e-module. □

Let us denoted by $W(M_g)$, the set of all $a \in R_e$ such that the R_e-homomorphism mapping $\phi_a : M_g \longrightarrow M_g$, defined by $\phi_a(x) = ax$ is not surjective.

Theorem 3.11. *Let R be a strongly G-graded ring and M be a graded R-module. Suppose that M_e is a faithful comultiplication R_e-module. Then, $W(M_g) = Z(R_e)$ for all $g \in G$, where $Z(R_e)$ is the set of zero divisors of R_e.*

Proof. Given $g \in G$. On one hand, let $a \in W(M_g)$, then $aM_g \neq M_g$. Since R is strongly graded, we obtain that $aM_e \neq M_e$, and then an ideal I of R_e exists such that $aM_e = (0 :_{M_e} I)$. Thus, $Ia \subseteq Ann(M_e) = \{0\}$ and hence $a \in Z(R_e)$. On the other hand, let $b \in Z(R_e)$. Then there exists a non-zero element c in R_e such that $cb = 0$. So, $bM_e \subseteq (0 :_{M_e} R_e c)$. Because M_e is faithful comultiplication, we obtain that $(0 :_{M_e} R_e c) \neq M_e$ and then $bM_e \neq M_e$. Consequently, since R is strongly graded, $bM_g \neq M_g$, which means that $b \in W(M_g)$. □

Theorem 3.12. *Let M be a graded comultiplication R-module, and let N be a graded R-submodule of M. Then $N = \bigcap_{g\in G}(N + M_g)$.*

Proof. Let N be a graded R-submodule of M. Then $N \subseteq \bigcap_{g\in G}(N + M_g)$
$\subseteq \bigcap_{g\in G}(M_g :_M Ann(N)) = (\bigcap_{g\in G} M_g :_M Ann(N)) = (0 :_M Ann(N)) = N$.
Therefore, $N = \bigcap_{g\in G}(N + M_g)$. □

Acknowledgments

The paper is partially suported by Yormouk University.

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