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HAMILTONIAN DYNAMICS FOR AN ALTERNATIVE ACTION DESCRIBING MAXWELL'S EQUATIONS

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Abstract: We develop a complete Dirac's canonical analysis for an alternative action that yields Maxwell's four-dimensional equations of motion. We study in detail the full symmetries of the action by following all steps of Dirac's method in order to obtain a detailed description of symmetries. Our results indicate that such an action does not have the same symmetries than Maxwell theory, namely, the model is not a gauge theory and the number of physical degrees of freedom are different.

1. Introduction

A dynamical system is characterized by means of its symmetries which constitute an important information in both the classical and quantum context. The physics of the fundamental interactions based on the standard model [1], is a relevant example where the symmetries of a dynamical system just like gauge covariance, CPT invariance, the identification of the physical degrees of freedom and conserved quantities are useful for understanding the classical and quantum formulation of the theory. However, the Lagrangians studied in the standard model are singular systems and the conventional Hamiltonian anal-

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ysis, is not the correct way for studying them [2]. On the other hand, it is well-known that the analysis of a dynamical system by means of its equations of motion implies that the phase space is not endowed with a natural or preferred symplectic structure as it has been claimed in [3, 4], and the freedom in the choice of the symplectic structure is an important issue because could yield different quantum formulations. Hence, in spite of we have an infinity ways to choose a symplectic structure for any system, the next question arises: are the symmetries of the classical theory preserved in all different choices of the symplectic structure?, the answer in general is not [5, 6]. In fact, we will show along this paper that the alternative action to conventional Maxwell's equations proposed in [7, 8] is not a gauge theory, is not invariant under parity symmetry and the physical degrees of freedom are not those knew for the electromagnetic field. Thus, in the study of the symmetries of a dynamical system must be taken into account an action principle plus their equations of motion, because the action gives the equations of motion and additionally fixes the symplectic structure of the theory, see [4].

The alternative action proposed in [7, 8] is a singular system, and this fact was ignored in those works. Nevertheless, we shall perform our study in a different way, this is, we will develop a complete Dirac's canonical approach. This formalism is an elegant approach for obtaining the relevant physical information of a theory under study, namely, the counting of physical degrees of freedom, the gauge transformations, the study of the constraints, the extended Hamiltonian and the extended action [2], being a relevant information because is the guideline to make the best progress in the analysis of quantum aspects of the theory. All those facts will be explained along the paper.

2. Hamiltonian Dynamics for an Alternative Action Describing Maxwell's Equations

The system that we shall study in this paper is given by the following action principle [7, 8]

$$
\begin{aligned}
S[E_i, B_i] &\\
&= \frac{1}{8\pi}\int\left[B^i\left(\frac{1}{c}\partial_t E_i - \epsilon_i{}^{jk}\partial_j B_k\right) - E^i\left(\frac{1}{c}\partial_t B_i + \epsilon_i{}^{jk}\partial_j E_k\right)\right]dx^4, \quad (1)
\end{aligned}
$$

where E and B represents the electric and magnetic fields and form a set of six dynamical variables.

By considering to E_i and B_j as our set of dynamical variables, the equations of motion obtained from the action (1) are given by

$$\begin{cases} \partial_t E_i = c\epsilon_i{}^{jk}\partial_j B_k, \\ \partial_t B_i = -c\epsilon_i{}^{jk}\partial_j E_k, \end{cases} \tag{2}$$

which correspond to the half of Maxwell's equations. In order to obtain the complete Maxwell equations without sources, we observe that the fields should satisfy

$$\begin{aligned} \nabla \cdot E &= 0, \\ \nabla \cdot B &= 0, \end{aligned} \tag{3}$$

in our analysis we shall take into account the above equations. In [7, 8] the action (1) was proposed as an alternative Lagrangian for describing the dynamics of the electromagnetic field, however, it is easy to see that the action is neither Lorentz invariant nor invariant under parity symmetry; this fact will be reflected in the hamiltonian analysis, in particular in the number of physical degrees of freedom of the theory. Furthermore, we will show below that the action does not have the principal symmetry knew for Maxwell theory namely, gauge invariance.

For our aims, we identify from the action principle that the Hessian given by

$$H^{ij} = \frac{\partial^2 \mathcal{L}}{\partial(\partial_t E_i)\partial(\partial_t B_j)}$$

has entries zero, so the system under study is a singular theory. The Hessian has a rank=0, and six null vectors. Therefore we expect six primary constraints. A pure Dirac's analysis calls for the definition of the momenta (Π^i_E, P^i_M) canonically conjugate to $(E_i,\, B_i)$

$$\Pi^i_E = \frac{\partial \mathcal{L}}{\partial(\partial_t E_i)} = \frac{1}{8\pi c}B^i, \tag{4}$$

$$P^i_M = \frac{\partial \mathcal{L}}{\partial(\partial_t B_i)} = -\frac{1}{8\pi c}E^i. \tag{5}$$

where the following six primary constraints arise

$$\begin{aligned} \Phi^i_E &: \ \Pi^i_E - \frac{1}{8\pi c}B^i \approx 0, \\ \Psi^j_M &: \ P^j_M + \frac{1}{8\pi c}E^j \approx 0. \end{aligned} \tag{6}$$

In order to verify that the above constraints are the correct, we calculate the Jacobian among them

$$
\frac{\partial(\Phi^i_E, \Psi^j_M)}{\partial(E_i, B_j, \Pi^i_E, P^j_M)} = \begin{pmatrix}
0 & 0 & 0 & -\frac{1}{8\pi c} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{8\pi c} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{8\pi c} & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{8\pi c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{1}{8\pi c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{8\pi c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

we observe that the Jacobian has rank=6, and it is constant on the constraints surface. Therefore, the primary constraints given in (6) are the set of correct primary constraints.

Now, the canonical Hamiltonian takes the form

$$
\begin{aligned}
H_c &\equiv \int \mathcal{H}_c d^3x = \int [\dot{E}_i \Pi^i_E + \dot{B}_i P^i_M - \mathcal{L}] d^3x \qquad (7) \\
&= \int [\dot{E}_i(\frac{1}{8\pi c} B^i) + \dot{B}_i(-\frac{1}{8\pi c} E^i) - \frac{1}{8\pi c} B_i \partial_t E_i + \frac{1}{8\pi c} B_i \epsilon_{ijk} \partial_j B_k \\
&+ \frac{1}{8\pi c} E_i \partial_t B_i + \frac{1}{8\pi c} E_i \epsilon_{ijk} \partial_j E_k] d^3x \\
&= \frac{1}{8\pi} \int [B_i \epsilon_{ijk} \partial_j B_k + E_i \epsilon_{ijk} \partial_j E_k] d^3x.
\end{aligned}
$$

Hence, the primary Hamiltonian is given by

$$
H^1 = H_c + \int u_i(x) \Phi^i(x) d^3x,
$$

where u_i are Lagrange multipliers enforcing the primary constraints.

In order to know if there are secondary constraints, we calculate the following matrix whose entries are the Poisson brackets among the primary constraints, this is

$$
\begin{cases}
\left\{\Phi^i_E(x), \Phi^j_E(y)\right\} = 0, \\
\left\{\Psi^i_M(x), \Psi^j_M(y)\right\} = 0, \\
\left\{\Phi^i_E(x), \Psi^j_M(y)\right\} = -\frac{1}{4\pi c} \delta^{ij} \delta(x-y),
\end{cases}
$$

So, the matrix acquires the form

$$W^{\mu\nu} = \left(\begin{array}{ccc} \{\Phi_E^1(x),\Phi_E^1(y)\} & \{\Phi_E^1(x),\Phi_E^2(y)\} & \{\Phi_E^1(x),\Phi_E^3(y)\} \\ \{\Phi_E^2(x),\Phi_E^1(y)\} & \{\Phi_E^2(x),\Phi_E^2(y)\} & \{\Phi_E^2(x),\Phi_E^3(y)\} \\ \{\Phi_E^3(x),\Phi_E^1(y)\} & \{\Phi_E^3(x),\Phi_E^2(y)\} & \{\Phi_E^3(x),\Phi_E^3(y)\} \\ \{\Psi_M^1(x),\Phi_E^1(y)\} & \{\Psi_M^1(x),\Phi_E^2(y)\} & \{\Psi_M^1(x),\Phi_E^3(y)\} \\ \{\Psi_M^2(x),\Phi_E^1(y)\} & \{\Psi_M^2(x),\Phi_E^2(y)\} & \{\Psi_M^2(x),\Phi_E^3(y)\} \\ \{\Psi_M^3(x),\Phi_E^1(y)\} & \{\Psi_M^3(x),\Phi_E^2(y)\} & \{\Psi_M^3(x),\Phi_E^3(y)\} \end{array} \right.,$$

$$\left. \begin{array}{ccc} \{\Phi_E^1(x),\Psi_M^1(y)\} & \{\Phi_E^1(x),\Psi_M^2(y)\} & \{\Phi_E^1(x),\Psi_M^3(y)\} \\ \{\Phi_E^2(x),\Psi_M^1(y)\} & \{\Phi_E^2(x),\Psi_M^2(y)\} & \{\Phi_E^2(x),\Psi_M^3(y)\} \\ \{\Phi_E^3(x),\Psi_M^1(y)\} & \{\Phi_E^3(x),\Psi_M^2(y)\} & \{\Phi_E^3(x),\Psi_M^3(y)\} \\ \{\Psi_M^1(x),\Psi_M^1(y)\} & \{\Psi_M^1(x),\Psi_M^2(y)\} & \{\Psi_M^1(x),\Psi_M^3(y)\} \\ \{\Psi_M^2(x),\Psi_M^1(y)\} & \{\Psi_M^2(x),\Psi_M^2(y)\} & \{\Psi_M^2(x),\Psi_M^3(y)\} \\ \{\Psi_M^3(x),\Psi_M^1(y)\} & \{\Psi_M^3(x),\Psi_M^2(y)\} & \{\Psi_M^3(x),\Psi_M^3(y)\} \end{array} \right),$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \frac{\delta(x-y)}{4\pi c},$$

this matrix has rank=6 and zero null vectors, therefore this theory has not secondary constraints. Because of we do not expect secondary constraints for this theory, the evolution in time of the primary constraints will allow us to know the six Lagrange multiplies introduced above, this is

$$\begin{aligned} \dot{\Phi}_E^n(x) &= \int d^3y \{\Phi_E^n(x), \mathcal{H}_c(y)\} + \int d^3y u_m(y) \{\Phi_E^n(x), \Phi_M^m(y)\}, \\ &= \frac{1}{8\pi}\epsilon_{ijn}\partial_j E_i(x) + \frac{1}{8\pi}\epsilon_{ijn}\partial_j E_i(x) - \frac{1}{4\pi c}u_n(x) \approx 0, \end{aligned} \tag{8}$$

hence,

$$u_i(x) = c\epsilon_{kji}\partial_j E_k(x) = -c\epsilon_i{}^{jk}\partial_j E_k(x), \tag{9}$$

and

$$\begin{aligned} \dot{\Psi}_M^m(x) &= \int d^3y \{\Psi_M^m(x), \mathcal{H}_c(y)\} + \int d^3y v_n(y) \{\Psi_M^m(x), \Phi_E^n(y)\}, \\ &= \frac{1}{8\pi}\epsilon_{ijm}\partial_j B_i(x) + \frac{1}{8\pi}\epsilon_{ijm}\partial_j B_i(x) + \frac{1}{4\pi c}v_m(x) \approx 0, \end{aligned} \tag{10}$$

then

$$v_i(x) = -c\epsilon_{kji}\partial_j B_k(x) = c\epsilon_i{}^{jk}\partial_j B_k(x), \tag{11}$$

thus, the six Lagrange multipliers have been identified.

We have observed that the complete set of constraints are given by (6), which are of second class. In fact, we see that

$$\begin{aligned}
\left\{\Phi^i_E(x), \Phi^j_E(y)\right\} &= 0, \\
\left\{\Phi^i_M(x), \Psi^j_M(y)\right\} &= 0, \\
\left\{\Phi^i_E(x), \Psi^j_M(y)\right\} &= -\frac{1}{4\pi c}\delta^{ij}\delta(x-y).
\end{aligned} \tag{12}$$

With all those results at hand, we are able to calculate the physical degrees of freedom as follows; there are 6 dynamical variables and 6 second class constraints, thus, there are three physical degrees of freedom. However, we need to take into account the equations (3). For this aim, we observe that the constraints (6) satisfy the following two reducibility conditions

$$\begin{aligned}
&\partial_i \Phi^i_E = 0, \\
&\partial_i \Psi^i_M = 0.
\end{aligned} \tag{13}$$

Therefore, reducibility conditions imply that there are $[6-2] = 4$ second class constraints, hence the physical degrees of freedom are four. This result is expected because the action (1) is not invariant under parity, therefore the degrees of freedom are distinguishable under parity, however, we know that in the case of Maxwell theory, the action is invariant under parity and the degrees of freedom are not distinguishable under parity. It is important to remark, that in spite of action (1) yields Maxwell equations of motion, our results indicate that the action (1) does not describe the dynamics of the electromagnetic field at all. In fact, it is well-know that Maxwell theory is a gauge theory and has two physical degrees of freedom; on the other side, Eq. (1) is not a gauge theory and has four physical degrees of freedom. In this manner, we confirm that a dynamical system should be defined by means an action principle plus their equations of motion, and not only by means the equations of motion. Our results obtained in this letter, show relevant differences among the action (1) and conventional Maxwell action at classical level and of course, will be interesting research the quantum differences among them as well, for this aim we will develop all the necessary tools in follow sections.

Because of there are second class constraints in the theory, we shall calculate the Dirac's brackets. Dirac's brackets will be useful in order to study the observables, as well as, for performing the quantization of the theory. Hence, Dirac's brackets are defined by

$$\{F(x), G(y)\}_D \equiv \{F(x), G(y)\} + \int d^3z d^3w \{F(x), \chi^\alpha(z)\} W^{-1}_{\alpha\beta} \{\chi^\beta(w), G(y)\},$$

where $W^{-1}_{\alpha\beta}$ is the inverse of the matrix W defined above, and $\chi^\alpha = (\Phi^j_E, \Psi^i_M)$ are the second class constraints. Hence, we obtain the following Dirac's brackets

$$\begin{aligned} \{E_i(x), \Pi^j_E(y)\}_D &= \{E_i(x), \Pi^j_E(y)\} \\ &\quad + \int d^3z d^3w \{E_i(x), \chi^\alpha(z)\} W^{-1}_{\alpha\beta} \{\chi^\beta(w), \Pi^j_E(y)\} \\ &= \{E_i(x), \Pi^j_E(y)\} \\ &= \frac{1}{3}\delta^j_i \delta(x-y). \end{aligned} \tag{14}$$

$$\begin{aligned} \{E_i(x), B^j(y)\}_D &= \{E_i(x), B^j(y)\} \\ &\quad + \int d^3z d^3w \{E_i(x), \chi^\alpha(z)\} W^{-1}_{\alpha\beta} \{\chi^\beta(w), B^j(y)\} \\ &= 4\pi c \delta^j_i \delta(x-z). \end{aligned} \tag{15}$$

$$\begin{aligned} \{\partial_i E^i(x), B^j(y)\}_D &= \{\partial_{xi} E_i(x), B^j(y)\} \\ &\quad + \int d^3z d^3w \{\partial_{xi} E_i(x), \chi^\alpha(z)\} W^{-1}_{\alpha\beta} \{\chi^\beta(w), B^j(y)\} \\ &= 4\pi c \partial_j \delta(x-y). \end{aligned} \tag{16}$$

$$\begin{aligned} \{B_i(x), P^j_M(y)\}_D &= \{B_i(x), P^j_M(y)\} \\ &\quad + \int d^3z d^3w \{B_i(x), \chi^\alpha(z)\} W^{-1}_{\alpha\beta} \{\chi^\beta(w), P^j_M(y)\} \\ &= \{B_i(x), P^j_M(y)\} \\ &= \frac{1}{3}\delta^j_i \delta(x-y). \end{aligned} \tag{17}$$

$$
\begin{aligned}
\{\partial_i \Pi^i_E(x), B_j(y)\}_D &= \{\partial_i \Pi^i_E(x), B_j(y)\} \\
&\quad + \{\partial_i \Pi^i_E(x), \chi^\alpha(z)\} W^{-1}_{\alpha\beta} \{\chi^\beta(w), B_j(y)_j\} \\
&= \partial_i \{\Pi^i_E(x), B_j(y)\} \\
&\quad + \{\partial_i \Pi^i_E(x), \Psi^i_M(z)\} W^{-1}_{ij} \{\Psi^j_M(w), B_j(y)_j\} \\
&= 0. \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\{\partial_i P^i_M(x), E_j(y)\}_D &= \{\partial_i P^i_M(x), E_j(y)\} \\
&\quad + \{\partial_i P^i_M(x), \chi^\alpha(z)\} W^{-1}_{\alpha\beta} \{\chi^\beta(w), E_j(y)_j\} \\
&= \partial_i \{P^i_M(x), E_j(y)\} \\
&\quad + \{\partial_i \Pi^i_E(x), \Phi^i_E(z)\} W^{-1}_{ij} \{\Phi^j_E(w), E_j(y)_j\} \\
&= 0. \qquad (19)
\end{aligned}
$$

We finish our analysis by calculating the extended action and the extended Hamiltonian. For this aim, we use the Lagrange multipliers found in (9) and (11), and we find

$$
\begin{aligned}
S_E[A_\mu, \pi^\mu, \lambda_j, u_i] &= \int d^4x [\Pi^i_E \dot{E}_i + P^i_M \dot{B}_i - \frac{1}{8\pi}[B_i \epsilon_{ijk} \partial_j B_k + E_i \epsilon_{ijk} \partial_j E_k] \\
&\quad - u_i \Psi^i - v_i \Phi^i - \overline{u}_i \Psi^i - \overline{v}_i \Phi^i] \\
&= \int d^4x [\Pi^i_E \dot{E}_i + P^i_M \dot{B}_i - [c\Pi^i_E \epsilon_{ijk} \partial_j B_k - c P^i_M \epsilon_{ijk} \partial_j E_k] \\
&\quad - \overline{u}_i \Psi^i - \overline{v}_i \Phi^i] \qquad (20)
\end{aligned}
$$

where we can identify the extended Hamiltonian given by

$$
H_E = \int d^3x \left(c\Pi^i_E \epsilon_{ijk} \partial_j B_k - c P^i_M \epsilon_{ijk} \partial_j E_k \right). \qquad (21)
$$

It is easy to see that H_E is of first class. In fact, we have

$$
\begin{aligned}
\{\Phi^i_E, H_E\} &= c\epsilon^{ij}{}_l \partial_j \Psi^l \approx 0, \\
\{\Psi^i_M, H_E\} &= -c\epsilon^{ij}{}_l \partial_j \Phi^l \approx 0. \qquad (22)
\end{aligned}
$$

On the other hand, by using the equations (15)-(20) we observe that the Dirac's bracket among H_E and the second class constraints vanish, therefore H_E is an observable.

From the extended action we calculate the following variations

$$
\begin{aligned}
\delta S_E = & \int d^4x[\delta\Pi^i_E \dot{E}_i + \Pi^i_E \delta\dot{E}_i + P^i_M \delta\dot{B}_i + \delta P^i_M \delta\dot{B}_i - c\delta\Pi^i_E \epsilon_{ijk}\partial_j B_k \\
& - c\Pi^i_E \epsilon_{ijk}\partial_j \delta B_k] \\
& + c\delta P^i_M \epsilon_{ijk}\partial_j E_k + cP^i_M \epsilon_{ijk}\partial_j \delta E_k - \delta\overline{u}_i \Psi^i - \delta\overline{v}_i \Phi^i], \\
= & \int d^4x[\delta\Pi^i_E \partial_t E_i + \partial_t(\Pi^i_E \delta E_i) - \delta E_i \partial_t \Pi^i_E + \partial_t(P^i_M \delta B_i) - \delta B_i \partial_t P^i_M \\
& + \delta P^i_M \partial_t B_i - c\delta\Pi^i_E \epsilon_{ijk}\partial_j B_k] - c\epsilon_{ijk}\partial_j(\Pi^i_E \delta B_k) + c\epsilon_{ijk}\delta B_k \partial_j \Pi^i_E \\
& + c\delta P^i_M \epsilon_{ijk}\partial_j E_k + c\epsilon_{ijk}\partial_j(\delta P^i_M E_k) \\
& - c\epsilon_{ijk}\delta E_k \partial_j P^i_M - \delta\overline{u}_i \Psi^i - \delta\overline{v}_i \Phi^i] \\
= & \int d^4x[(\partial_t E_i - c\epsilon_{ijk}\partial_j B_k)\delta\Pi^i_E - (\partial_t \Pi^i_E + c\epsilon_{kji}\partial_j P^k_M)\delta E_i \\
& - (\partial_t P^i_M - c\epsilon_{kji}\partial_j \Pi^k_E)\delta B_i] \\
& + (\partial_t B_i + c\epsilon_{ijk}\partial_j E_k)\delta P^i_M - \delta\overline{u}_i \Psi^i - \delta\overline{v}_i \Phi^i],
\end{aligned} \tag{23}
$$

where the following equations of motion rise

$$
\begin{aligned}
\delta E_i \ : \ \partial_t \Pi^i_E &= c\epsilon_{ijk}\partial_j P^k_M, \\
\delta B_i \ : \ \partial_t P^i_M &= -c\epsilon_{ijk}\partial_j \Pi^k_E, \\
\delta\Pi^i_E \ : \ \partial_t E_i &= c\epsilon_{ijk}\partial_j B_k, \\
\delta P^i_M \ : \ \partial_t B_i &= -c\epsilon_{ijk}\partial_j E_k, \\
\delta\overline{u}_i \ : \ \Psi^i &\approx 0, \\
\delta\overline{v}_i \ : \ \Phi^i &\approx 0.
\end{aligned} \tag{24}
$$

It is important to remark, that the above equations of motion were not obtained in [7, 8]. In fact, in those works it was ignored that Eq. (1) is a singular system and it was defined a dynamical system by using the equations of motion, then, the system was treated as a non-singular system, however, we have showed that Eq. (1) does not describes Maxwell theory. On the other hand, the Hamiltonian found in [7, 8] is not equivalent to Eq. (21), the Hamiltonian found in this work by following Dirac's method is of first class, and the dynamics of the system is carry out on the constraints surface defined on the full phase space. Nevertheless, in [7, 8] is not possible to talk about first class or second class constraints and the dynamics of the system is carry out on the full phase space, but we have showed that this is not possible because Eq. (1) is singular.

3. A Quantum State of Zero Energy

In order to observe differences among the Eq.(1) and conventional Maxwell theory at quantum level, we will calculate a quantum state of zero energy for the Hamiltonian (21). For this aim, we identify the following classical-quantum correspondence for the canonical momentum $\Pi^i_E \to -i\frac{\delta}{\delta E_i}$ and $P^i_M \to -i\frac{\delta}{\delta B_i}$ for the magnetic field. Hence

$$\begin{gathered}\left(\widehat{E}_i(x)\psi\,(E,B)\right) = \widehat{E}_i\psi(E,B), \left(\widehat{B}_i(x)\psi\,(E,B)\right) = \widehat{B}_i\psi(E,B),\\ \left(\widehat{\Pi}^i_E(x)\psi\,(E,B)\right) = -i\frac{\delta\psi(E,B)}{\delta E_i}, \left(\widehat{P}^i_M(x)\psi\,(E,B)\right) = -i\frac{\delta\psi(E,B)}{\delta B_i},\end{gathered} \quad (25)$$

where $\psi(E,B)$ is an arbitrary function of the fields E, B and represents a quantum state. By using the above correspondence, the classical-quantum representation of the Hamiltonian (21) is given by

$$\widehat{H} = \int \left(-ic\epsilon_i{}^{jk}\partial_j B_k \frac{\delta}{\delta E_i} + ic\epsilon_i{}^{jk}\partial_j E_k \frac{\delta}{\delta B_i} \right), \quad (26)$$

the representation of the vacuum for the theory will correspond to an eigenfunction of zero energy for the Hamiltonian $\widehat{H}$ determined by

$$\widehat{H}\psi(E,B) = 0. \quad (27)$$

Hence, the function that solves exactly Eq. (27) is given by

$$\psi(E,B) = e^{\alpha I(E,B)}, \quad (28)$$

where α is a constant and

$$I(E,B) = \frac{1}{2}\int \left(\epsilon^{ijk}E_i\partial_j E_k + \epsilon^{ijk}B_i\partial_j B_k \right) dx^3. \quad (29)$$

Some remarks are important to comment; the wave function given in (28) does not correspond the Chern-Simons state known for Maxwell theory, because the dynamical variable is the connexion. In fact, it is well-known that in Maxwell theory, the wave function that solves the Hamiltonian for the vacuum is given for the Chern-Simos state for the connexion A, therefore, with this result we confirm that at quantum level, the action given in Eq.(1) does not describes the electromagnetic field. Furthermore, the expression (29) can be written as

$$I(E,B) = \frac{1}{4}\int \left(\epsilon^{ijk}E_i\mathbf{R_{jk}} + \epsilon^{ijk}B_i\mathbf{\Upsilon_{jk}} \right), \quad (30)$$

where $\mathbf{R_{jk}} = \partial_i E_j - \partial_j E_i$ and $\mathbf{\Upsilon_{jk}} = \partial_i B_j - \partial_j B_i$ are some like curvatures of the field E and B respectively. Hence, it is straightforward to show that $I(E, B)$ is invariant under changing

$$\begin{aligned} E_i \to E_i &\quad + \quad \partial_i \theta, \\ B_i \to B_i &\quad + \quad \partial_i \theta, \end{aligned} \tag{31}$$

therefore, $I(E, B)$ is a composition of Chern-Simons terms associated for the fields E and B. Finally, it is easy to show that Eq.(29) viewed as a field theory is topological one and diffeomorphism covariant, thus, while in the action (1) the fields E and B are not gauge fields, for $I(E, B)$ they are. In this manner, the state (28) is not physically accepted for the theory. Of course, will be interesting to perform the path integral quantization of the action (28) in order to obtain a better understanding of the quantization of the system. All these ideas are in progress and will be reported in forthcoming works.

4. Concluding Remarks

In this paper, we have performed a complete Dirac's analysis for an action yielding Maxwell's equations of motion. In our analysis we found that although the non conventional action yields the same equations of motion than Maxwell theory, its symmetries are not those associated to the electromagnetic field. The theory studied is not a gauge theory and does not have the same number of degrees of freedom. In this manner, we need to be careful for defining a dynamical system. If we define a system by using only the equations of motion, we are in the situation that an infinity number of Hamiltonian structures can bee defined for the same system, however, this fact presents a problem because by changing the hamiltonian structure, we could obtain several actions yielding the same equations of motion, but the symmetries of the theory can be lost, such as it was presented in the analysis developed along this paper. Therefore, in order to know the symmetries, a dynamical system must be defined by means of an action principle, because the action contains the relevant information and symmetries of the theory.

It is important to remark that similar results will be found if the canonical analysis is performed for the case of an action close to linearized gravity. In fact, in [7, 8] is proposed an action with similar structure than (1)

$S[E_{ij}, B_{ij}]$

$$= \int \left[B^{ij} \left(\frac{1}{c} \partial_t E_{ij} - \epsilon_i{}^{kl} \partial_k B_{lj} \right) - E^{ij} \left(\frac{1}{c} \partial_t B_{ij} + \epsilon_i{}^{kl} \partial_k E_{lj} \right) \right] d^4x, \quad (32)$$

where E_{ij} and B_{ij} are related with the components of the curvature tensor corresponding to the perturbed metric (see [8, 7] for full details). However, from the analysis performed in this paper, we will found that the action (32) does not describe the dynamics of the linearized gravity theory; the principal symmetry as gauge invariance is lost and the number of physical degrees of freedom do not correspond to those found in linearized gravity [9]. Finally, in [10] can be found an alternative hamiltonian description of gravity, however, in that work the analysis was performed in the same way than [7, 8], this is, defining a dynamical system by means its equations of motion, hence the symmetries well-knew in gravity are lost.

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References

[1] S. Weinberg, *The Quantum Theory of Fields*, Volumes I and II, Cambridge University Press, Cambridge, England (1996).

[2] D.M. Gitman, I.V. Tyutin, *Quantization of Fields with Constraints*, Berlin, Germany, Springer, Springer Series in Nuclear and Particle Physics (1990); A. Hanson, T. Regge, C. Teitelboim, *Constrained Hamiltonian Systems*, Accademia Nazionale dei Lincei, Roma (1978).

[3] Merced Montesinos, G.F. Torres del Castillo, *Phys. Rev. A*, **70** (2004), 032104.

[4] M. Mondragon, M. Montesinos, *J. Math. Phys.*, **47** (2006), 022301.

[5] Alberto Escalante, *Phys. Lett. B*, **676** (2009), 105-111.

[6] A. Escalante, J. Angel López-Osio, Hamiltonian analysis for topological and Yang-Mills theories expressed as a constrained BF-like theory, *IJPAM*, **75** (2012), 339-352.

[7] G.F. Torres del Castillo, D. Acosta Avalos, *Rev. Mex. Fis.*, **40** (1994), 405.

[8] R. Rosas-Rodriguez, *J. Phys. Conf. Ser.*, **24** (2005), 231-235.

[9] J. Barcelos-Neto, T.G. Dargam, *Z. Phys. C*, **67** (1995), 701-706.

[10] R. Rosas-Rodriguez, *Int. J. Mod. Phys. A*, **23** (2008), 895-908.