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GENERATING INFORMATION MEASURES VIA MATRIX OVER PROBABILITY SPACES

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Abstract: In the present communication, we have generated some new information theoretic measures through the diagonal and non-diagonal elements of the matrix of probability terms obtained by post multiplication of a column vector by a row vector. The elements of columns of such a matrix consists of linear function whereas the row elements are in the form of linear, power, exponential, trigonometric and hyperbolic function. The concept will find application in measurement of diversity in biological systems and in other disciplines using information theory.

AMS Subject Classification: 94A
Key Words: uncertainty, entropy, concavity, symmetry, expansibility

1. Introduction

With the publication of Shannon's [11] path-breaking works in the form of his first paper "A Mathematical Theory of Communication" and consequently providing the discovery of the fundamentals laws of data compression and transmission marks the birth of Information Theory. Zadeh [17] has remarked that uncertainty is an attribute of information and the theory provided by Shannon [11] which has profound intersections with many mathematical disciplines, has

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led to a universal acceptance that information is statistical in nature. A logical consequence of Shannon's [11] theory of uncertainty, in whatever form it is, is that it should be dealt with through the use of probability theory. Shannon [11] introduced the concept of information theoretic entropy by associating uncertainty with every probability distribution $P = (p_1, p_2, ..., p_n)$ and found that there is a unique function that can measure the uncertainty, is given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i, \tag{1}$$

He called it as entropy. The probabilistic measure of entropy (1.1) possesses a number of interesting properties. Immediately, after Shannon [11] gave his measure, research workers in many fields saw the potential of the application of this expression and a large number of other information theoretic measures were derived.

Measures of information are widely used in Biology for the measurement of diversity of communities. Some of the common ecological diversity indices are due to Shannon [11], Simpson [12], McIntosh [4] and Pielou [10]. Jost [3] discussed the concept of number equivalents which can be used more effectively in describing the communities. Gregorius [2] remarked that the total diversity of a subdivided collection should be composed of diversity within sub collections and discussed the effect of differentiation in terms of diversity components. Information measures can be used to measure diversity at α,β and γ levels where the relation between α, β and γ levels has been provided by the multiplicative law due to Whittaker [16].

The concept of diversity has also been extended to measurement of economic biodiversity (Baumgärtner [1]). Some of such measures are due to Weitzman [15], Solow [13], Weikard [14] and Nehring and Puppe [5, 6]. Recently, Parkash and Thukral [8] proved that statistical measures can also be used as information theoretic measures. The same concept was successfully applied to sampling distributions by Parkash, Thukral and Gandhi [9] whereas coefficient of non-determination was used by Parkash, Mukesh and Thukral [7] for measuring randomness of association.

A study of the information measures revealed that in many cases, post multiplication of a column vector $f_1(p_i)$ by a row vector $f_2(p_i)$ produces a matrix, the elements of which may be used to derive a measure of information. Such a matrix is given by

$$A = \begin{matrix} \\ f_1(p_1) \\ f_1(p_2) \\ \vdots \\ f_1(p_n) \end{matrix} \begin{matrix} \begin{matrix} f_2(p_1) & f_2(p_2) & \cdots & f_2(p_n) \end{matrix} \\ \begin{pmatrix} f_1(p_1)f_2(p_1) & f_1(p_1)f_2(p_2) & \cdots & f_1(p_1)f_2(p_n) \\ f_1(p_2)f_2(p_1) & f_1(p_2)f_2(p_2) & \cdots & f_1(p_2)f_2(p_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(p_n)f_2(p_1) & f_1(p_n)f_2(p_2) & \cdots & f_1(p_n)f_2(p_n) \end{pmatrix} \end{matrix} \quad (2)$$

and the information content from its elements may be obtained from the following mathematical expression:

$$I = g\left(\sum_{i=1}^{n}\sum_{j=1}^{n} f_1(p_i)f_2(p_j)\right), \quad (3)$$

In this communication, we derive new information measures from diagonal and non-diagonal elements of some matrices.

2. Diagonal Elements of Matrix as Measures of Information

I If we take $f_1(p_i) = f_2(p_i) = p_i$ in the matrix (1.2), then the diagonal elements can be obtained as ${}_1H_n(P) = \sum_{i=1}^{n} p_ip_i$, that is

$${}_1H_n(P) = \sum_{i=1}^{n} p_i^2, \quad (4)$$

which is a well known Simpson's concentration. Thus, $\sum_{i=1}^{n} p_ip_i$ will also act as an information measure.

II If we take $f_1(p_i) = p_i$ and $f_2(p_i) = p_i^{\frac{1}{2}}$ in the matrix (1.2), then the sum of diagonal elements is given by $\sum_{i=1}^{n} p_ip_i^{\frac{1}{2}}$.

With some additional constant, the sum of diagonal elements gives the following mathematical expression:

$${}_2H_n(P) = 1 - \sum_{i=1}^{n} p_i^{\frac{3}{2}}, \quad (5)$$

Now, we prove (2.2) as a theoretical measure of information. For this purpose, we have studied the following properties:

1. Obviously ${}_2H_n(P) \geq 0$ for $0 \leq p_i \leq 1$.
2. ${}_2H_n(P)$ is a continuous function of p_i.
3. ${}_2H_n(P)$ is permutationally symmetric function of p_i.
4. Concavity: To study its concavity, we proceed as follow:
   We have

$$\frac{\partial({}_2H_n(P))}{\partial p_i} = -\frac{3}{2} p_i^{\frac{1}{2}}.$$

   Also

$$\frac{\partial^2({}_2H_n(P))}{\partial p_i^2} = -\frac{3}{4} p_i^{-\frac{1}{2}} < 0 \,\forall\, p_i > 0.$$

   Thus, ${}_2H_n(P)$ is a concave function of p_i.
5. Expansibility: We have

$${}_2H_{n+1}(p_1, p_2, ..., p_n, 0) = 1 - \sum_{i=1}^{n+1} p_i^{\frac{3}{2}} = 1 - \sum_{i=1}^{n} p_i^{\frac{3}{2}} =_2 H_n(p_1, p_2, ..., p_n).$$

   Thus, ${}_2H_n(P)$ does not change by the inclusion of an impossible event.
6. For degenerate distributions, we have

$${}_2H_n(1, 0, ..., 0) = 1 - ((1)^{\frac{3}{2}} + (0)^{\frac{3}{2}} + \cdots + (0)^{\frac{3}{2}}) = 0.$$

   Thus for certain events, ${}_2H_n(P) = 0$.
7. For obtaining maximum value, we consider the Lagrange's function given by

$$L \equiv (1 - \sum_{i=1}^{n} p_i^{\frac{3}{2}}) - \lambda(\sum_{i=1}^{n} p_i - 1).$$

   Thus, we have

$$\frac{\partial L}{\partial p_1} = -\frac{3}{2} p_1^{\frac{1}{2}} - \lambda, \frac{\partial L}{\partial p_2} = -\frac{3}{2} p_2^{\frac{1}{2}} - \lambda, \ldots, \frac{\partial L}{\partial p_n} = -\frac{3}{2} p_n^{\frac{1}{2}} - \lambda.$$

   For maximum value, we put

$$\frac{\partial L}{\partial p_1} = \frac{\partial L}{\partial p_2} = \cdots = \frac{\partial L}{\partial p_n} = 0,$$

which gives $p_1 = p_2 = \cdots = p_n$. Also $\sum_{i=1}^{n} p_i = 1$ gives $p_i = \frac{1}{n}\ \forall i = 1, 2, \cdots, n$. Thus the maximum value arises when the distribution is uniform.

8. The maximum value $\phi(n)$ of the proposed entropy measure is $\phi(n) = 1 - n^{-\frac{1}{2}}$ which gives $\phi'(n) = \frac{1}{2} n^{-\frac{3}{2}} > 0\ \forall n = 2, 3, 4, \cdots$, which shows that the maximum value $\phi(n)$ is an increasing function of n.

   Next, we have presented ${}_2H_n(P)$ and obtained Figure 1 which clearly shows that ${}_2H_n(P)$ is a concave function.

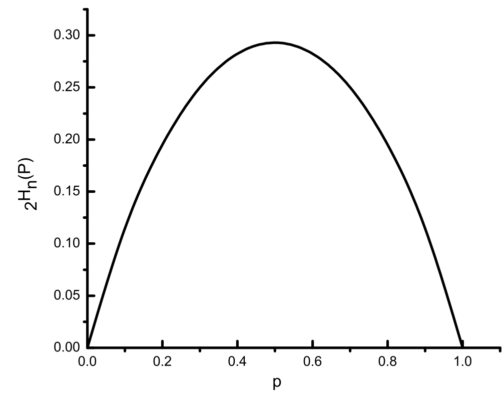

Figure 1: Concavity of ${}_2H_n(P)$ with respect to P.

Under the above properties, we see that ${}_2H_n(P)$ is an information measure and consequently, from equation (2.2), we conclude that $1 - \sum_{i=1}^{n} p_i p_i^{\frac{1}{2}}$ will act as an information measure.

III If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \exp(p_i)$ in the matrix (1.2), then the diagonal elements with some additional constant give the following expression

$$ {}_3H_n(P) = \exp(1) - \sum_{i=1}^{n} p_i \exp(p_i), \tag{6}$$

Now, we prove (2.3) as a theoretical measure of information through the study of following properties:

1. Obviously ${}_3H_n(P) \geq 0$ for $0 \leq p_i \leq 1$.
2. ${}_3H_n(P)$ is a continuous function of p_i.
3. ${}_3H_n(P)$ is permutationally symmetric function of p_i.
4. Concavity: From equation (2.3), we have

$$\frac{\partial^2_3 H_n(P)}{\partial p_i^2} = -(2+p_i)\exp(p_i) < 0 \,\forall\, 0 \leq p_i \leq 1.$$

which shows that ${}_3H_n(P)$ is a concave function of p_i.

5. Expansibility: We have

$$\begin{aligned} {}_3H_{n+1}(p_1, p_2, ..., p_n, 0) &= \exp(1) - \sum_{i=1}^{n+1} p_i \exp(p_i) \\ &= \exp(1) - \sum_{i=1}^{n} p_i \exp(p_i) \\ &= {}_3H_n(p_1, p_2, ..., p_n). \end{aligned}$$

Thus, ${}_3H_n(P)$ does not change by the inclusion of an impossible event.

6. For degenerate distributions, we have

$${}_3H_n(1, 0, ..., 0) = \exp(1) - (1\exp(1) + 0\exp(0) + \cdots + 0\exp(0)) = 0.$$

Thus for certain events, ${}_3H_n(P) = 0$.

7. For obtaining maximum value, we consider the Lagrange's function given by

$$L \equiv (\exp(1) - \sum_{i=1}^{n} p_i \exp(p_i)) - \lambda(\sum_{i=1}^{n} p_i - 1),$$

And observe that the maximum value arises when the distribution is uniform.

8. The maximum value $\phi(n)$ of the proposed entropy measure is $\phi(n) = \exp(1) - \exp(\frac{1}{n})$, which gives $\phi^{'}(n) = \frac{1}{n^2}\exp(\frac{1}{n}) > 0$, numerically $\forall n = 2, 3, 4, \cdots$, which shows that the maximum value $\phi(n)$ is an increasing function of n.

Under the above properties, we see that ${}_3H_n(P)$ is an information measure and consequently, from equation (2.3), we conclude that $\exp(1) - \sum_{i=1}^{n} p_i \exp(p_i)$ will act as an information measure.

Proceeding as above, we can prove that the following functions can be treated as information measures:

IV If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \sin p_i$ in the matrix (1.2), then the diagonal elements with some additional constant give the following expression

$$ {}_4H_n(P) = \sin 1 - \sum_{i=1}^{n} p_i \sin p_i. \tag{7} $$

V If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \cos p_i$ in the matrix (1.2), then the diagonal elements with some additional constant give the following expression

$$ {}_5H_n(P) = \sum_{i=1}^{n} p_i \cos p_i - \cos 1. \tag{8} $$

VI If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \sinh p_i$ in the matrix (1.2), then the diagonal elements with some additional constant give the following expression

$$ {}_6H_n(P) = \sinh 1 - \sum_{i=1}^{n} p_i \sinh p_i. \tag{9} $$

VII If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \tanh p_i$ in the matrix (1.2), then the diagonal elements with some additional constant give the following expression

$$ {}_7H_n(P) = \tanh 1 - \sum_{i=1}^{n} p_i \tanh p_i. \tag{10} $$

3. Non-Diagonal Elements of Matrix as Measures of Information

I If we take $f_1(p_i) = f_2(p_i) = p_i$ in the matrix (1.2), then the non-diagonal elements can be obtained as follows:

$$ \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j, i \neq j. $$

This equation gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j = 1 - \sum_{i=1}^{n} p_i^2. \tag{11}$$

Since R.H.S. $(1-\sum_{i=1}^{n} p_i^2)$ of equation (3.1) is a well known Gini-Simpson's index, so $\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j$ will also act as an information measure.

II If we take $f_1(p_i) = p_i$ and $f_2(p_i) = p_i^{\frac{1}{2}}$ in the matrix (1.2), then the non-diagonal elements can be obtained as follows:

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i^{\frac{1}{2}} = \sum_{i=1}^{n} p_i^{\frac{3}{2}} + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j^{\frac{1}{2}}, i \neq j.$$

This equation gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j^{\frac{1}{2}} = \sum_{i=1}^{n} p_i^{\frac{1}{2}} - \sum_{i=1}^{n} p_i^{\frac{3}{2}}. \tag{12}$$

Now, we show that the R.H.S. of equation (3.2) is a theoretical measure of information. For this purpose, we take

$${}^{1}H_n(P) = \sum_{i=1}^{n} p_i^{\frac{1}{2}} - \sum_{i=1}^{n} p_i^{\frac{3}{2}},$$

and study the following properties:

1. Obviously ${}^{1}H_n(P) \geq 0$ as $1 \geq p_i$ or $p_i^{\frac{1}{2}} \geq p_i^{\frac{3}{2}}$ or $\sum_{i=1}^{n} p_i^{\frac{1}{2}} \geq \sum_{i=1}^{n} p_i^{\frac{3}{2}}$.
2. ${}^{1}H_n(P)$ is a continuous function of p_i.
3. ${}^{1}H_n(P)$ is permutationally symmetric function of p_i.
4. Concavity: We have $\frac{\partial^2({}^{1}H_n(P))}{\partial p_i^2} = -\frac{1}{4}(\frac{1}{p_i^{\frac{3}{2}}} + \frac{3}{p_i^{\frac{1}{2}}}) < 0 \,\forall p_i > 0$, which shows that ${}^{1}H_n(P)$ is a concave function of p_i.

5. Expansibility: We have

$$
\begin{aligned}
{}^1H_{n+1}(p_1, p_2, ..., p_n, 0) &= \sum_{i=1}^{n+1} p_i^{\frac{1}{2}} - \sum_{i=1}^{n+1} p_i^{\frac{3}{2}} \\
&= \sum_{i=1}^{n} p_i^{\frac{1}{2}} - \sum_{i=1}^{n} p_i^{\frac{3}{2}} \\
&= {}^1H_n(p_1, p_2, ..., p_n).
\end{aligned}
$$

Thus, ${}^1H_n(P)$ does not change by the inclusion of an impossible event.

6. For degenerate distributions, we have

$$
\begin{aligned}
{}^1H_n(1, 0, ..., 0) &= ((1)^{\frac{1}{2}} + (0)^{\frac{1}{2}} + \cdots + (0)^{\frac{1}{2}}) \\
&\quad -((1)^{\frac{3}{2}} + (0)^{\frac{3}{2}} + \cdots + (0)^{\frac{3}{2}}) \\
&= 0.
\end{aligned}
$$

Thus for certain events, ${}^1H_n(P) = 0$.

7. For obtaining maximum value, we consider the Lagrange's function given by

$$
L \equiv (\sum_{i=1}^{n} p_i^{\frac{1}{2}} - \sum_{i=1}^{n} p_i^{\frac{3}{2}}) - \lambda(\sum_{i=1}^{n} p_i - 1).
$$

Thus, we have

$$
\frac{\partial L}{\partial p_1} = (\frac{1}{2}p_1^{-\frac{1}{2}} - \frac{3}{2}p_1^{\frac{1}{2}}) - \lambda, \frac{\partial L}{\partial p_2} = (\frac{1}{2}p_2^{-\frac{1}{2}} - \frac{3}{2}p_2^{\frac{1}{2}}) - \lambda, \ldots,
$$
$$
\frac{\partial L}{\partial p_n} = (\frac{1}{2}p_n^{-\frac{1}{2}} - \frac{3}{2}p_n^{\frac{1}{2}}) - \lambda.
$$

Thus, for maximum value, we have

$$
\frac{1}{2}p_1^{-\frac{1}{2}} - \frac{3}{2}p_1^{\frac{1}{2}} = \frac{1}{2}p_2^{-\frac{1}{2}} - \frac{3}{2}p_2^{\frac{1}{2}} = \cdots = \frac{1}{2}p_n^{-\frac{1}{2}} - \frac{3}{2}p_n^{\frac{1}{2}},
$$

which further gives $p_1 = p_2 = \cdots = p_n$. Also using the constraint $\sum_{i=1}^{n} p_i = 1$, we get $p_i = \frac{1}{n}\ \forall i = 1, 2, \cdots, n$. Thus the maximum value arises when the distribution is uniform.

8. The maximum value $\phi(n)$ of the proposed entropy measure is $\phi(n) = n^{\frac{1}{2}} - n^{-\frac{1}{2}}$ which gives $\phi^{'}(n) = \frac{1}{2}(\frac{1}{n^{\frac{1}{2}}} + \frac{1}{n^{\frac{3}{2}}}) > 0 \; \forall n = 2, 3, 4, \cdots$.
   This shows that the maximum value $\phi(n)$ is an increasing function of n.
   Moreover, with the help of numerical data, we have presented ${}^1H_n(P)$ as shown in the figure-2.

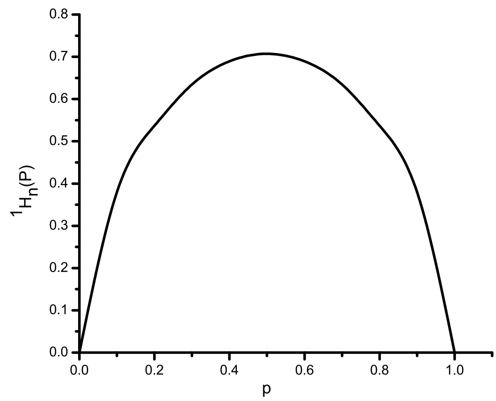

Figure 2: Concavity of ${}^1H_n(P)$ with respect to P.

Under the above properties, we see that $\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j^{\frac{1}{2}}$ is an information measure.

Proceeding as above, we can prove that the following theoretical functions can be treated as information measures:

III If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \exp(p_i)$ in the matrix (1.2), then the non-diagonal elements can be obtained as follows:

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} \exp(p_i) = \sum_{i=1}^{n} p_i \exp(p_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \exp(p_j), i \neq j.$$

This gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i \exp(p_j) - (n-1) = \sum_{i=1}^{n} \exp(p_i) - \sum_{i=1}^{n} p_i \exp(p_i) - (n-1). \quad (13)$$

IV If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \sin p_i$ in the matrix (1.2), then the non-diagonal elements can be obtained as follows:

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} \sin p_i = \sum_{i=1}^{n} p_i \sin p_i + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \sin p_j, i \neq j.$$

This gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i \sin p_j = \sum_{i=1}^{n} \sin p_i - \sum_{i=1}^{n} p_i \sin p_i. \quad (14)$$

V If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \cos p_i$ in the matrix (1.2), then the non-diagonal elements can be obtained as follows:

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} \cos p_i = \sum_{i=1}^{n} p_i \cos p_i + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \cos p_j, i \neq j.$$

This gives

$$(n-1) - \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \cos p_j = (n-1) - \sum_{i=1}^{n} \cos p_i + \sum_{i=1}^{n} p_i \cos p_i. \quad (15)$$

VI If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \sinh p_i$ in the matrix (1.2), then the non-diagonal elements with some additional constant can be obtained as follows:

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} \sinh p_i = \sum_{i=1}^{n} p_i \sinh p_i + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \sinh p_j, i \neq j.$$

This gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i \sinh p_j = \sum_{i=1}^{n} \sinh p_i - \sum_{i=1}^{n} p_i \sinh p_i. \quad (16)$$

VII If we take $f_1(p_i) = p_i$ and $f_2(p_i) = \tanh p_i$ in the matrix (1.2), then the non-diagonal elements with some additional constant can be obtained as follows:

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} \tanh p_i = \sum_{i=1}^{n} p_i \tanh p_i + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \tanh p_j, i \neq j.$$

This gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i \tanh p_j = \sum_{i=1}^{n} \tanh p_i - \sum_{i=1}^{n} p_i \tanh p_i. \qquad (17)$$

Concluding Remarks: We have established that the diagonal and non-diagonal elements of several matrices can be used to derive new information measures. This concept may be extended to other such matrices.

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