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# ON THE CONNECTED RANK FOR PROJECTIVE VARIETIES

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**Abstract:** We start the discussion of the following query. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. For each $P \in \mathbb{P}^n$ let $r_{X,s}(P)$ (resp. $r_{X,c}(P)$, resp. $r_{X,cc}(P)$) be the minimal degree of a zero-dimensional (resp. zero-dimensional and connected, resp. zero-dimensional connected and curvilinear) scheme $Z$ such that $P \in \langle Z \rangle$, where $\langle \ \rangle$ denote the linear span. Compute the maximal integer $r_{X,s}(P)$ and $r_{X,c}(P)$ when $P$ is arbitrary (resp. general) in $\mathbb{P}^n$.

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**Key Words:** symmetric tensor rank, connected rank, curvilinear scheme, cactus rank

## 1. Introduction

For any subscheme $A$ of a projective space $\mathbb{P}^n$ let $\langle A \rangle$ denote the linear span of $A$, i.e. the intersection of all hyperplanes of $\mathbb{P}^n$ containing $A$, with the convention $\langle A \rangle = \mathbb{P}^n$ if there is no such a hyperplane.

Let $X \subset \mathbb{P}^n$ be a non-degenerate irreducible variety. Fix $P \in \mathbb{P}^n$. The *connected rank* $r_{X,c}(P)$ of $P$ is the minimal integer $t$ such that there is a degree $t$ connected scheme $Z \subset X$ such that $P \in \langle Z \rangle$. If we drop the word " connected ", then we get the usual definition of s*cheme rank* or *scheme length* or ĉactus rank

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$r_{X,s}(P)$ of $P$ (see [5], Definition 5.1, [3], [8], [2]). Notice that $r_{X,c}(P) = 1$ if and only if $P \in X$.

We use this definition for an arbitrary variety $X$, but (in this form) it better behaves when $X$ is a smooth curve. If $X$ is a smooth curve, then $r_{X,c}(P)$ is the minimal integer $t$ such that there is $Q \in X$ with $P \in \langle \{tQ, X\} \rangle$, where $\{tQ, X\}$ denote the degree $t$ effective divisor of $X$ with $Q$ as its support. For interpolation problems for arbitrary $X$ (even for singular curves) it is better to use the following notion of connected curvilinear rank. A zero-dimensional scheme $Z$ is said to be curvilinear if for each $Q \in Z_{red}$ the Zariski tangent space of $Z$ at $Q$ has dimension $\le 1$. If $Z \subset \mathbb{P}^n$, then $Z$ is curvilinear $\Leftrightarrow$ $Z$ is contained in a smooth curve $\Leftrightarrow$ $Z$ is contained in a smooth and connected projective curve $\Leftrightarrow$ $Z$ is contained in the smooth locus of a curve. Every curvilinear scheme $Z \subset \mathbb{P}^n$ is smoothable inside $\mathbb{P}^n$. The *connected curvilinear rank* $r_{X,cc}(P)$ of $P$ is the minimal integer $t$ such that there is a degree $t$ connected and curvilinear scheme $Z \subset X$ such that $P \in \langle Z \rangle$. If $X$ is a smooth curve, then $r_{X,c}(P)) = r_{X,cc}(P)$ for all $P$.

Let $\rho(X, s)$ (resp. $\rho(X, c)$, resp. $\rho(X, cc)$) be the maximal of the integers $r_{X,s}(P)$ (resp. $r_{X,c}(P)$, resp. $r_{X,cc}(P)$) for some $P \in \mathbb{P}^n$. Set $\lambda(X, s) := r_{X,s}(P)$ (resp. $\lambda(X, c) := r_{X,c}(P)$, resp. $\lambda(X, cc) := r_{X,cc}(P)$), where $P$ is a general element of $\mathbb{P}^n$.

**Proposition 1.** *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety which is smooth in codimension* 1*. Set $m := \dim(X)$. Then $r_{X,cc}(P) \le n-m+1$ for all $P \in \mathbb{P}^n$.*

**Proposition 2.** *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety Set $m := \dim(X)$. Then $r_{X,c}(P) \le n - m + 1$ for all $P \in \mathbb{P}^n$.*

**Query 1.** *Assume that $X$ is smooth and $m := \dim(X) \ge 3$. Under which assumptions on $X$ and the embedding $X \hookrightarrow \mathbb{P}^n$ is $\rho(X, s) = \rho(X, c)$ and/or $\lambda(X, s) = \lambda(X, c)$ ?*

We are mainly interested in the case in which $X$ is a Veronese embedding of $\mathbb{P}^m$ (see [3], [8], [2]). In this case sometimes Query 1 has a negative answer, but the meaning of the query is that, quite often, we should have $\rho(X, s) = \rho(X, c)$ and $\lambda(X, s) = \lambda(X, c)$.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. In Remark 2 we point out the differences in positive characteristic.

## 2. Curves

**Remark 1.** Fix any non-degenerate $X \subsetneq \mathbb{P}^n$. If there is $P \in \mathrm{Sing}(X)$ whose Zariski tangent space has dimension $n$, then $r_{X,cc}(P) = r_{X,c}(P)$ for every $P \in \mathbb{P}^n \setminus X$. The converse holds if $\dim(X) < n/2$ and $X$ has only finitely many singular points.

The dimension $\dim(Y)$ of a constructible set $Y \subset \mathbb{P}^n$ is the maximal dimension of an irreducible component of $\overline{Y}$. For any integral variety $X \subset \mathbb{P}^n$ let $\sigma_x(X)$ denote the closure in $\mathbb{P}^n$ of the linear spaces $\langle S\rangle$ with $S \subset X$ a finite set and $\sharp(S) = x$ (we get the same closed irreducible set $\sigma_x(X)$ if we impose that $\sharp(S) \le x$ or (if $x \le \dim(\langle X\rangle) + 1$ we impose that $S$ is linearly independent). For any integer $x > 0$ set $\Phi_x(X, c) := \{P \in \mathbb{P}^n : r_{X,c} = x\}$, $\Psi_x(X, cc) := \{P \in \mathbb{P}^n : r_{X,c} = x\}$, $\Phi'_x(X, c) := \{P \in \mathbb{P}^n : r_{C,c} \le x\}$, $\Psi'_x(C, cc) := \{P \in \mathbb{P}^n : r_{C,c} \le x\}$.

Let $C$ be a smooth curve and $f : C \to \mathbb{P}^n$ be a non-degenerate morphism. For each $Q \in C$ we get a flag $V_i(f, Q)$, $0 \le i \le n$, of linear subspaces of $\mathbb{P}^n$ (the osculating flag of $f$ at $Q$) with the following properties. We have $V_0(f, Q) = \{Q\}$, $V_n(Q) = \mathbb{P}^n$, $\dim(V_i(f, Q)) = i$ for all $i$, $V_i(f, Q) \subset V_{i+1}(f, Q)$ for all $i \in \{0, \dots, n-1\}$ and the order of contacts $o(f, Q, i)$ of $f$ with $V_i(f, Q)$ at $Q$ is a strictly increasing function. For each $P \in \mathbb{P}^n \setminus f(C)$ let $r_f(P)$ be the minimal integer $t$ such that there is $Q \in C$ such that $P \in V_i(f, Q)$ and $o(f, Q, i-1) = t$. If $P \in f(C)$, then set $r_f(P) = 1$.

**Proposition 3.** *Let $C \subset \mathbb{P}^n$ be a smooth and non-degenerate curve. Then $r_{C,c}(P) \le n$ for every $P \in \mathbb{P}^n$ and $r_{C,c}(P) = n$ for a general $P \in \mathbb{P}^n$.*

*Proof.* The second assertion is true for an arbitrary integral and non-degenerate curve (Proposition 5). Fix $P \in \mathbb{P}^n$. If $P \in C$, then $r_{C,c}(P) = 1$. Now assume $P \notin C$. Let $\ell_P : \mathbb{P}^n \setminus \{P\} \to \mathbb{P}^{n-1}$ denote the linear projection from $P$. Set $f := \ell_P|C$. First assume $n \ge 3$. We have $r_{C,c}(P) \le n$ if and only if there is $Q \in C$ such that $f$ has not ramification order $1, \dots, n-1$ at $O$ and its ramification order is strictly (with the usual ordering) larger of the first $n-1$ ramification orders of $Q$ for the inclusion $C \hookrightarrow \mathbb{P}^n$. First assume that $f$ is birational onto its image, i.e. assume $\deg(f(C)) = \deg(C)$. We claim that it it is sufficient to use the Brill-Segre formula (see [6], Theorem 9) for the morphisms $f : C \to \mathbb{P}^{n-1}$ and the inclusion $C \hookrightarrow \mathbb{P}^n$. Since in this case $C$ and the normalization of $C$ have the same genus ($C$ is the normalization of $f(C)$) and $\deg(C) = \deg(f(C))$, the two formulas say that the total weight of the higher ramification point for $f$ is $n/(n+1)$ the one for the inclusion $C \hookrightarrow \mathbb{P}^n$. Hence it is sufficient to check that each ramification point of order $x < n$ of the inclusion $C \hookrightarrow \mathbb{P}^n$ is counted more than $(n+1)/n$ times the one for $f$ with the same ramification sequence. Fix $Q \in C$ and choose a formal coordinate $z$ at $Q$ and a osculating flag of $\hookrightarrow \mathbb{P}^n$ at $P$. Hence there are non-negative integers $\alpha_i$, $1 \le i \le n$, so that the ramification indices $a_i(Q)$, $1 \le i \le n$, of $\hookrightarrow$ at $Q$ are $a_1(Q) = 1$, $a_i(Q) = i + \sum_{j=1}^{i-1} \alpha_i$ for all $i \in \{2, \dots, n\}$. Assume that, up to $n-1$, these invariants are the same for $f$. The weight $w_f(Q)$ of $Q$ for $f$ is $w_f(Q) = \sum_{i=1}^{n-1}(n-1-i)\alpha_i$, while $w_{\hookrightarrow}(Q) = \sum_{i=1}^{n}(n-i)\alpha_i$ (see [4], page 274). Hence $w_f(Q) < n/(n+1)w_{\hookrightarrow}(Q)$ if $\alpha_i > 0$ for some $i$.

Now assume $\deg(f) \ge 2$ and call $Y$ the normalization of the curve $f(C)$. The morphism $f$ induces a morphism $f' : C \to Y$ such that $\deg(f') = \deg(f) \ge 2$. If $f'$ is ramified, then $r_{C,c}(P) = 2$. If $f'$ is unramified, then both $C$ and $Y$ have genus 1 (Riemann-Hurwitz formula and the fact that $\mathbb{P}^1$ is algebraically simply connected). In this case we may apply again the Brill-Segre formula, because in $\mathbb{P}^r$ it gives $(n+1)\deg(C)$, while in $\mathbb{P}^n$ it gives $n\deg(f(C))$, if no point for $f$ has higher ramification, then each ramification point for $f$ gives $\deg(f)$ ramification points for $\hookrightarrow$ with the same ramification sequence.

Now assume $n = 2$. In this case we use again that $f : C \to \mathbb{P}^1$ has at least one ramification point. □

The same proof gives the following result.

**Proposition 4.** *Let $C$ be a smooth curve and $f : C \to \mathbb{P}^n$ a morphism birational onto its image and with $f(C)$ spanning $\mathbb{P}^n$. Then $r_f(P) \le n$ for all $P \in \mathbb{P}^n$.*

**Proposition 5.** *Let $C \subset \mathbb{P}^n$, $n \ge 2$, be a smooth, connected and non-degenerate curve. Fix an integer $x \in \{1, \dots, n\}$. Then $\Phi_x(C, c) \neq \emptyset$ and $\dim(\Phi_x(C, c)) = x$ .*

*Proof.* Since $\mathrm{char}(\mathbb{K}) = 0$ for a general $Q \in C_{reg}$ we have $\dim(\langle\{xQ, C\}\rangle) = x - 1$ (see [6]). Since $C$ is non-degenerate, $\langle\{xQ, C\}\rangle \cap C$ is finite. Hence we have infinitely many distinct $(x-1)$-dimensional spaces $\langle\{xQ, C\}\rangle$. Now take an arbitrary connected and curvilinear scheme $Z \subset C$. Set $Q := Z_{red}$ and write $\{Q_1, \dots, Q_s\} := u^{-1}(Q)$ (as sets). □

**Remark 2.** Assume $p := \mathrm{char}(\mathbb{K}) > 0$. Let $C$ be a smooth and connected and projective curve. Let $f : C \to \mathbb{P}^n$ be a morphism birational onto its image and with $f(C)$ spanning $\mathbb{P}^n$. If $p > \deg(C)$, then the discussion made in characteristic $p$ works. If $p < \deg(C)$, then the generic hermitian invariants may not be the classical ones (see [6], §7) and hence (when $f$ is an embedding) we may have $r_{C,c}(P) > n$ for a general $P \in \mathbb{P}^n$ (see [6], §7). Moreover, two classical definitions of osculating space are different (see [6], Remark at the end of §5).

*Proof of Proposition 1.* We use the statement of Proposition 3 and the proof of [7], Proposition 4.1. We use induction on $m$, the case $m = 1$, being true by Proposition 3. Fix $P \in \mathbb{P}^n$ and take a general hyperplane $H \subset \mathbb{P}^n$. If $P \in X$, then $r_{X,cc}(P) = 1$. Hence we may assume $P \notin X$. Bertini's theorem says that $H \cap X$ is an integral projective variety whose singular locus has dimension $\max\{-1, \dim(\mathrm{Sing}(X)) - 1\}$, with the convention $\dim(\emptyset) = -1$. Hence $X \cap H$ is integral, non-singular in codimension 1. Since $X$ is connected and reduced, the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X\cap H,H}(1) \to 0$$

shows that $X \cap H$ spans $P$. Since $r_{X\cap H,cc}(P) \geq r_{X,cc}(P)$, it is sufficient to apply the inductive assumption. □

*Proof of Proposition 2.* As in the proof of Proposition 1 we reduce to the case $m = 1$. Let $u : C \to X$ be the normalization map of the integral curve $X$ and $f : C \to \mathbb{P}^n$ the composition of $u$ with the inclusion $X \hookrightarrow \mathbb{P}^n$. By Proposition 4 there is $Q \in C$ such that $P$ is contained in the osculating hyperplane of $f$ at $P$. Let $Z := f(\{nQ, C\})$, i.e. let $Z$ the intersection of all closed subschemes $W$ of $X$ such that $\{nQ, C\} \subseteq u^{-1}(W)$. We have $\deg(Z) \leq n$, because the natural map $\mathcal{O}_Z \to \mathcal{O}_{\{nQ,C\}}$ is injective by the definition of $Z$ as a minimal subscheme. We have $P \in \langle Z \rangle$. □

**Remark 3.** For arbitrary $n, m$ with $n \gg m$ it is easy to construct singular $m$-dimensional varieties $X$ with $\rho(X) < \lambda(X, c)$. In the following example we have $\rho(X) = 4$. Fix integers $n, m$ such that $n \gg m > 0$. Let $X \subset \mathbb{P}^n$ be an integral curve with exactly one singular $Q$. Assume $\dim(T_Q X)) = n - 1$, where $T_Q(X)$ denote the Zariski tangent space to $X$ at $Q$. For any $Q' \in X \setminus X \cap T_Q X$ we have $\langle T_Q X \cup \{Q'\}\rangle = \mathbb{P}^n$. Hence $\rho(X, s) \leq 3$. Now assume that the scheme $T_Q X \cap X$ contains the fourth infinitesimal neighborhood of $Q$ in $X$ and that $m > 3m + 1$. In this case we have $\rho(X, c) \geq 4$.

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