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# MATHEMATICAL MODELING THE RUNUP OF NONLINEAR SURFACE GRAVITY WAVES

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**Abstract:** This paper considers two-dimensional numerical simulation of the run-up of nonlinear surface gravity waves on the basis of Navier-Stokes equations. The statement of the problem is formulated and its boundary and initial conditions are described. A discrete model is constructed using the method of splitting with respect to physical processes. A discrete finite-element model of this problem is developed taking into account the cell fill factor. The conservativeness of the discrete model was investigated and the approximation error of the finite-difference scheme is found. The results of a two-dimensional numerical simulation of the run-up of nonlinear surface gravity waves on coastal structures of shallow-water offshore areas are presented.

**AMS Subject Classification:** 76D33
**Key Words:** Navier-Stokes equations, numerical simulation, splitting method, run-up of nonlinear surface gravity waves

## Introduction

The effect of wave processes on coastal and hydraulic structures has been sufficiently relevant. These processes have often been described by using different modifications of shallow-water equations. However, an adequate description of the effect of surface gravity waves on coastal structures requires consideration

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of turbulent effects and bottom friction. The Navier-Stokes equation makes it possible to both describe the nonlinear effects and take into account turbulent processes in a viscous incompressible fluid. Therefore, along with other hydrodynamic models, the Navier-Stokes equation is used to describe the run-up of surface gravity waves on coastal slopes of shallow-water areas.

In Vinje and Brevik, 1981, the authors described a numerical simulation of the evolution of the profile of a breaking surface wave. The wave profiles were constructed numerically on the basis of a solution of a system of Laplace equations.

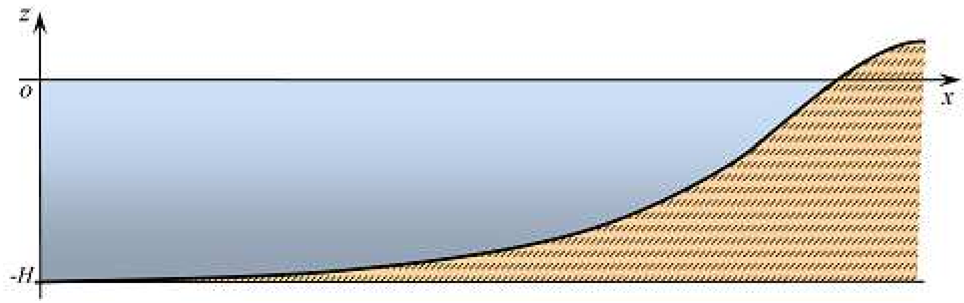
Paper of Elgar et all, 1998, is devoted to numerical modeling and experimental observations of the impact of nonlinear interaction effects, reflection, and attenuation on the propagation of surface gravity waves in a coastal zone. The nonlinear effects are considered in the framework of the Boussinesq approximation.

The problem of sea wave run-up on the shore in the framework of exact solutions of the nonlinear shallow-water theory is considered in Zahibo et all, 2006. Depending on a suitable waveform, different formulas for the height of wave run-up on the shore are obtained. An analytical formula for the height of solitary wave run-up on the shore was proposed as applied to tsunami waves.

Paper of Kawasaki, 1999, describes a two-dimensional numerical model for the effect of a submerged breakwater on wave propagation. The model is based on the Navier-Stokes equation. The wave is simulated both before and after the breaking. The wave profile transformation and the wave steepness relation with its spectral composition are analyzed.

Paper of Zhao et all, 2004, is devoted to a two-dimensional turbulent model of breaking waves. The free surface of the fluid is described by the Marker and Cell (MAC) method based on Reynolds-Averaged Navier-Stokes (RANS) equations. The profiles of breaking waves propagating over a sloping bottom were compared.

This paper describes a two-dimensional numerical model for the run-up of surface gravity waves to shore slopes. The simulation is based on hydrophysical conditions of the Azov Sea. The problems of analytical and numerical modeling of the propagation of nonlinear surface gravity waves for the given shallow-water basin were described in Abbasov, 2011 and Abbasov, 2012.




1. $\lambda = 3$ m; H=0.2 m; T=5.0 s; $a$ =0.128 m [5];
2. $\lambda = 10$ m; H=2 m; t=5.4 s; $a$ =1 m; c=3,6 m/s; kH=1,3; $\varepsilon$ =0,5 (considered model).

Figure 1: Geometry of the problem run-up of surface gravity wave.

## 1. Statement of the Problem

According to the geometry of the problem, the $Ox$-axis of the coordinate system is aligned with the undisturbed fluid surface and is directed toward the coast; the $Oz$-axis is directed vertically upward (see Figure 1). At the initial time, the liquid is at rest. At some distance from the coast (at a point $x = 0$), a disturbance is set as a pressure pulse that varies harmonically. It is required to track the subsequent motion of water mass.

To describe the two-dimensional surface gravity waves on the fluid surface with viscosity, the two-dimensional Navier-Stokes equation, the continuity equation for incompressible fluid, and the equation of hydrodynamic pressure (Fletcher, 1988) are used:

— the Navier-Stokes equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \eta\frac{\partial^2 u}{\partial z^2} \quad , \tag{1}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \mu\frac{\partial^2 w}{\partial x^2} + \eta\frac{\partial^2 w}{\partial z^2} + g, \tag{2}$$

— the continuity equation for incompressible fluid:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{3}$$

— the full hydrodynamic pressure with account of depth is determined by:

$$P(x,z,t) = p(x,z,t) + \rho g z, \tag{4}$$

where $u$and$w$ are the horizontal and vertical components of the water particle motion velocity vector $V(u,w)$, $\rho$ is the fluid density, $g$ is the gravitational constant, and $\mu$ and $\eta$ are the horizontal and vertical components of the turbulent viscosity coefficient.

On the free surface of the fluid, the following conditions are assumed:

$$P(x,z,t) = P_{atm.}, \quad w(x,z,t) = \frac{1}{g\rho}\frac{\partial P}{\partial t}; \tag{5}$$

— at the bottom of area conditions impermeable and slidings are assumed:

$$\frac{\partial}{\partial n}V(x,z,t) = 0, \quad \rho\eta\frac{\partial}{\partial z}u(x,z,t) = -\tau_x(t), \quad \rho\mu\frac{\partial}{\partial x}w(x,z,t) = -\tau_z(t); \tag{6}$$

— on the left lateral border (there is a source):

$$\frac{\partial}{\partial n}P(x,z,t) = \alpha, \qquad u(x,z,t) = u^{(0)}, \qquad w(x,z,t) = w^{(0)} \tag{7}$$

— the right lateral border is continuation of a bottom, conditions are similar; where $\tau_x(t),\ \tau_z(t)$ are the tangential stress components at the fluid bottom, and $\alpha$ is a parameter specified from initial conditions.

The tangential stress components are defined by following expressions:

$$\tau_x(t) = \rho C_p(|V|)u\,|V|\,, \quad \tau_z(t) = \rho C_p(|V|)w\,|V|\,, \tag{8}$$

where

$$C_p(|V|) = \begin{cases} 0,0088; & |V| < 6,6\,m/s, \\ 0,0026; & |V| \geq 6,6\,m/s. \end{cases}$$

The horizontal and vertical components of the turbulent viscosity coefficient accept following values $\mu,\ \eta$ ($0<\mu<1\mathrm{m}^2$/s, $0<\eta<1\mathrm{m}^2$/s).

At the initial time, the liquid is at rest, and following conditions are satisfied: $t=0$: $P(x,z,0) = \rho g z,\ u(x,z,0) = 0, \quad w(x,z,0) = 0.$

## 2. Splitting of Equations

The time approximation of the original equations was performed using the method of splitting with respect to physical processes (Harlow & Welch, 1965). This method implies a three-stage calculation:

— at the first stage, the velocity field is calculated on the basis of equations;

$$\frac{u^{n+\sigma}-u^n}{\tau}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=\mu\frac{\partial^2 u}{\partial x^2}+\eta\frac{\partial^2 u}{\partial z^2}\quad, \tag{9}$$

$$\frac{w^{n+\sigma}-w^n}{\tau}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=\mu\frac{\partial^2 w}{\partial x^2}+\eta\frac{\partial^2 w}{\partial z^2}+g, \tag{10}$$

— at the second stage, the pressure is calculated;

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}=\frac{\rho}{\tau}\left(\frac{\partial u^{n+\sigma}}{\partial x}+\frac{\partial w^{n+\sigma}}{\partial z}\right), \tag{11}$$

— at the third stage, the velocity field is adjusted with respect to pressure:

$$\frac{u^{n+1}-u^{n+\sigma}}{\tau}=-\frac{1}{\rho}\frac{\partial P}{\partial x}\quad,\quad \frac{w^{n+1}-w^{n+\sigma}}{\tau}=-\frac{1}{\rho}\frac{\partial P}{\partial z}, \tag{12}$$

where $u^n$ and $w^n$ are the velocity vector components at the current time step;

$u^{n+\sigma}$ and $w^{n+\sigma}$ are the velocity vector components at the auxiliary time step;

$u^{n+1}$ and $w^{n+1}$ are the velocity vector components at the next time step.

## 3. Construction and Investigation of the Discrete Model

A discrete finite-volume model with the cell fill factor was developed. The computational domain is rectangular in space directions. A uniform grid is used for the numerical implementation of the discrete mathematical model. The original differential equations are approximated in spatial coordinates using an integro-interpolation method.

Discrete analogs of equations for calculating the velocity vector components and pressure field, as well as discrete analogues of boundary conditions were obtained. The discrete model was checked for conservativeness. For the finite-difference analog of the differential equation, the integral law of momentum conservation was found to hold. The approximation error of the finite-difference scheme was estimated to be $O(\tau+h_x^2+h_z^2)$. A stability analysis was performed for the problem using the maximum principle and limitations on time and spatial steps were obtained.

Here, the discrete equations for the velocity vector components and pressure field are calculated with the help of an implicit scheme, while the discrete equations for adjusting the velocity components are calculated using an explicit scheme. The most time consuming is the second stage of calculations, when equation (11) is solved using the upper relaxation method to find pressure. A program code was developed for calculating the two-dimensional fields of velocity and pressure of water medium in the numerical simulation of the run-up and breaking of nonlinear surface gravity waves.

## 4. Results of Numerical Modeling and Their Analysis

It is noteworthy that the shallow-water condition for the Navier-Stokes equation need not be satisfied. Therefore, the initial parameters of surface gravity waves will have a shallow-water parameter within $kH \geq 1$, but with further approaching to the shore, the depth will be reduced and the surface wave will meet the shallow-water conditions.

As a shallow-water model, we use the hydrophysical conditions of the Taganrog Bay of the Azov Sea. In our case, the length of the shallow-water area is determined by the grid size. Also, to ensure that the physical process is correctly described, the nonlinear effects in the propagation of surface waves must be accumulation gradually. The grid size will be 200x600, and the water surface level is 100 vertically. The bay should be as extended along the $Ox$-axis as to involve at least two wavelengths.

In view of the bay depth of $H \quad \leq 5$ m, the segment of the water area will have the following geometric dimensions: a length of 30 m, a vertical length of 10 m (from the bottom), a horizontal step of $h_x \quad = 0.05$ m, and a vertical step of $h_z \quad = 0.05$ m. The length of the surface wave will be $\lambda \quad \leq 15$ m, the initial values of the shallow-water parameter will be in the range $1 \leq kH \leq 10$, the rate of surface wave propagation will depend on frequency based on a dispersion relation.

Then, we modeled the coastal slopes of varying steepness under bay conditions. The bottom lines were constructed on the basis of graphs of power functions. The depth decreases from $H \quad = 5$ m to zero, and the slope steepness does not exceed $0.2^{\circ}$.

Figure 2 shows the dynamics of arrival of nonlinear surface gravity waves on a flat coastal slope. On the left lateral boundary, harmonic disturbance as a

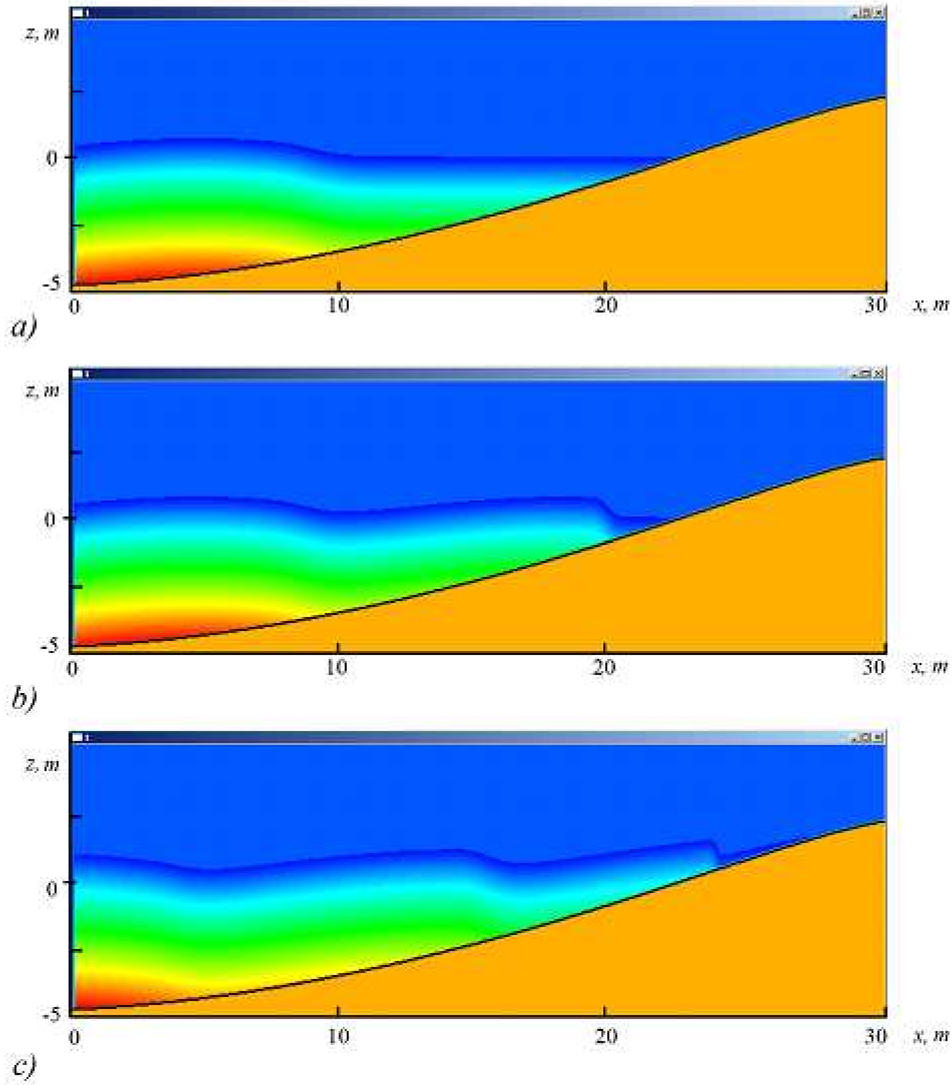



Figure 2: Dynamics of change in the profile of the surface gravity wave arriving on a flat coastal slope; the initial wave parameters are: f=0.39 Hz; $\lambda$ =10 m; c=4 m/s; H=5 m; a=0.5 m; kH=3.14; $\varepsilon$ =0.1; and the time steps are: (a) t=1.6s; (b) t= 3.6s and (c) t= 6.8s.

pressure pulse is specified. The surface gravity wave satisfies the initial shallow-water requirements $H/\lambda < 1/2$, and its length is twice the depth (Whitham, 1974).

As approaching to the coast, the bay depth is reduced, the wave begins to touch the bottom and, thus, the impact of nonlinear effects increases. This leads to a steepened front of the surface wave crest. The trajectories of medium particles on the wave forefront become vertical, the forefront becomes sheer, and the wave is broken. This process can be observed at time step of $t$ =3.6s (see Figure 2b). When breaking, the wave falls apart, which leads to a reduced wave height and an underflooded coastal slope. Later, during the wave run-up, the water flow begins to run back along the inclined slope. The wave that runs back knocks the next wave, which further enhances the steepening and accelerates the breaking of the next wave.

The numerical simulation of the run-up of nonlinear surface gravity waves on coastal structures of shallow-water areas revealed the following:

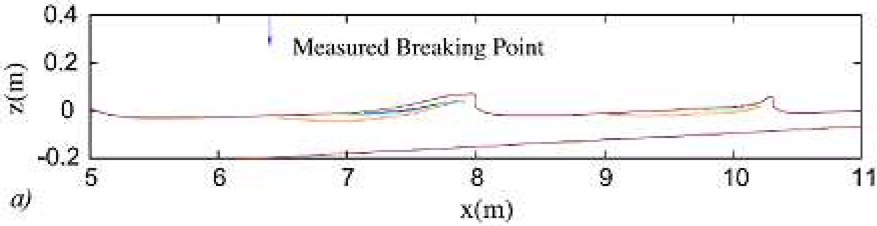
— an increase in the initial steepness of the surface wave leads to wave breaking in the course of run-up to a dry coastal zone; with a decreased initial steepness, the run-up occurs without wave breaking;

— with a decreased wavelength, the shallow-water conditions are fulfilled worse, the nonlinear distortions of the wave profile are reduced due to depth, and the run-up to the coast occurs as a continuous water flow. With an increased length of the surface wave, the profile distortions are amplified due to nonlinearity, and the wave crest becomes broken.

To check the adequacy of the model developed, the results of two-dimensional numerical simulation of arriving nonlinear surface gravity waves on coastal structures were compared with existing numerical (Zhao et all, 2004) and experimental (Kimmoun & Branger, 2007) data. For comparison, Figure 3 shows the simulation results calculated in the framework of the model under consideration and the results obtained in (Zhao et all, 2004). The comparison allows us to note the following:

— the profile of the surface wave propagating on a sloping bottom begins to distort due to the impact of depth and the forefront slope of the wave crest becomes steeper;

— with a further decrease in the depth, the wave breaking is accompanied with a curl on the crest (Figure 3a); our model generates no curl because the two-phase air-water interaction was not considered; therefore, the point of breaking corresponds to a vertical position of the front slope of the wave crest (Figure 3b).



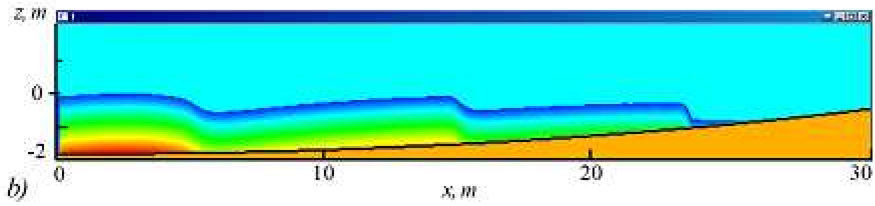




Figure 3: The comparison of run-up surface gravity wave on a flat coastal slope on the basis of different models.

In conclusion, we can note the good agreement of our results with data obtained by other authors, especially in the early stages of distortion in the surface wave profile.

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