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SOME REMARKS ON TOPOLOGICAL LATTICES

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Abstract: In this paper we give a lattice-theoretic approach to compactification of topological spaces. We use Stone-Cech compactification for representing the remainder of the locally compact spaces.

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1. Introduction

Among the other results we investigate a lattice theoretic approach for the remainder $\beta X \setminus X$ of a topological space X. The problem of compactification of certain topological spaces is considered by many authors in recent years [1,3]. We use an algebraic sstematic way using lattice theoretic techniques.

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2. Preliminaries

Knowledge and some of the basic concepts of topology (such as separation axioms) is assumed.

A subset S of a topological space X is called clopen if S is both open and closed with respect to the topology on X.

A compactification αX of a space X is a compact Hausdorff space that contains X as a dense subset. A space has a compactification iff it is Tychonoff. From now on, unless otherwise noted, spaces will be assumed to be Tychonoff. $\alpha X \setminus X$ denotes the remainder (i.e. the compactification αX minus the original space X) of the compactification αX.

The one-point compactification of X, sometimes denoted ωX, is the compactification of X whose remainder consists of a single point. Locally compact, non-compact spaces are precisely those that have one-point compactification.

The maximum compactification of a space X is called the stone-čech compactification of X and is denoted βX. The stone-čech compactification of X can be characterized as the unique compactification of X such that every bounded continuous function on X can be extended to a continuous function on βX. If X is compact and connected, it is called continuum.

2.1. Definitions, Lattice Theoretic

A partial order $\leq$ is a binary relation that is reflexive, transitive, and antisymmetric. A partially ordered set $(P, \leq)$ is a set P paired with a partial order $\leq$. We usually just write P instead of $(P, \leq)$.

Given a partially ordered set P and elements $a, b \in P$, the supremum of a and b (denoted $a \vee b$) is the least element that is greater than or equal to both a and b. The infimum of a and b (denoted $a \wedge b$) is the greatest element that is less than or equal to both a and b. If L is a partially ordered set (Poset) where $a \vee b$ and $a \wedge b$ exist for all $a, b \in L$, then we call L a *lattice*.

A Lattice is called distributive if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for all $a, b \in L$.

A bounded lattice is a lattice with both greatest and a least element (usually denoted 1 and 0, respectively).

A filter p on a lattice L is a subset of L such that

(i). $0 \notin p$

(ii). $a, b \in p$ implies that $a \wedge b \in p$.

(iii). $a \in p$ and $b \in L$ with $b \geq a$ implies that $b \in p$.

An ultrafilter is a maximal filter (with respect to set inclusion).

Let $a \in L$. Then $a' \in L$ is said to be the pseudocomplement of a if $a \wedge a' = 0$ and $b \wedge a = 0 \Rightarrow b \leq a'$, $\forall b \in L$.

A lattice is called pseudocomplemented if every $a \in L$ has a pseudocomplement.

Let $a \in L$. Then $a^c \in L$ is said to be the complement of a if $a \wedge a^c = 0$ and $a \vee a^c = 1$. A lattice B where every element of B has a complement is called a Boolean algebra.

If B is a Boolean algebra, then the stone space of B, denoted $St(B)$, is the space formed by the set of all ultrafilters on B. We also use this notation to describe the collection of ultrafilters of a lattice (i.e. if L is a lattice, $St(L)$ is the space of all ultrafilters on L).

2.2. General Lattices

Suppose that L is a bounded distributive lattice. A filter on L is said to be prime if $a \vee b \in L$ implies that $a \in L$ or $b \in L$. With the help of the following lemma, we show that every ultrafilter is prime.

Lemma 2.2.1. *Let p be an ultrafilter of a lattice L. Suppose $b \in L$ and $\forall a \in p$, $a \wedge b \neq 0$. Then $b \in p$.*

Proof. Let

$$q = \{c \in L : \exists a \in p,\ c \geq a \wedge b\}$$

It is easy to show that q is a fitter. Let $a \in p$. Since $a \geq a \wedge b$, then $a \in q$. So $p \subseteq q$. But p is an ultrafilter, so $p = q$. since for any $a \in p$, $b \geq b \wedge a$, then $b \in q$. But $p = q \Rightarrow b \in p$. □

Proposition2.2.2. *Suppose that p is a filter of a lattic L: then p is maximal $\Rightarrow$ p is prime.*

Proof. Let p be a maximal filter of L. Let $x, y \in L$ such that $x \vee y \in p$. Then $x \vee y \in p \Rightarrow (x \vee y) \wedge a \neq 0, \forall a \in p \Rightarrow (x \wedge a) \vee (y \wedge a) \neq 0, \forall a \in p$ suppose that $\exists a, b \in p$ such that $x \wedge a = 0$ and $y \wedge b = 0$. Then $(x \wedge (a \wedge b)) \vee (y \wedge (a \wedge b)) = 0$, which contradict above since $a \wedge b \in p$. So either $(x \wedge a \neq 0, \forall a \in p)$ or $(y \wedge a \neq 0, \forall a \in p)$ so by lemma 1.3.1., either $x \in p$ or $y \in p$. □

Now let's define the topology on $St(L)$ by considering a base for the closed sets.

Notation. Let $[f] = \{p \in St(L) : f \in p\}$, for each $f \in L$, and let $B(St(L)) = \{[f] : f \in L\}$.

Obviously $B(St(L))$ is a base for the closed sets of some topology on $St(L)$, which denoted τ_B.

The following are fundamental properties of these basic closed sets.

Lemma 2.2.3. *(i)* $[a] \cup [b] = [a \vee b], \forall a, b \in L$

(ii) $[a] \cap [b] = [a \wedge b], \forall a, b \in L$.

Proof. The proof is obvious.

Theorem 2.2.4. *A subset E of a lattice L has the finite intersection property $\Rightarrow$ E is contained in an ultrafilter of L.*

Proof. Assume $E \subseteq L$ has the f.i.p. So, if p_0 is the filter generated by E, then p_0 is proper. Let $P = \{q : q \supseteq p_0,\ q \text{ is a filter of } L\}$ clearly P is non-empty and can be partially ordered by set inclusion. For any non-empty chain C in P, $\cup C$ is clearly a filter containing each member of C. Finally, apply zorn's lemma. □

Corollary 2.2.5. *Any filter p of a lattice L is contained in an ultrafilter.*

With this corollary, we prove that $St(L)$ is compact.

Proposition 2.2.6. *$St(L)$ is compact.*

Proof. Let C be a collection of basic closed sets of $St(L)$ with the finite intersection property, denote $C = \{[f_\alpha] : \alpha \in \Delta\}$. Let S be the collection of all finite infimums of the $f'_\alpha s, \alpha \in \Delta$. (we will show that S is a filter base.)

(i). Let $g \in S$. Then $g = f_{\alpha_1} \wedge \cdots \wedge f_{\alpha_n}$ since C has the $f.i.p., \exists\ p \in [f_{\alpha_1}] \cap \cdots \cap [f_{\alpha_n}] \Rightarrow f_{\alpha_1}, \cdots, f_{\alpha_n} \in p \Rightarrow f_{\alpha_1} \wedge \cdots \wedge f_{\alpha_n} \in p \Rightarrow f_{\alpha_1} \wedge \cdots \wedge f_{\alpha_n} \neq 0$ since $0 \notin p$.

(ii). Let $g_1, g_2 \in S$. Then $g_1 = f_{\alpha_1} \wedge \cdots \wedge f_{\alpha_n}$ and $g_2 = f_{\beta_1} \wedge \cdots \wedge f_{\beta_m}$. So $g_1 \wedge g_2 = f_{\alpha_1} \wedge \cdots f_{\alpha_n} \wedge f_{\beta_1} \wedge \cdots \wedge f_{\beta_m} \in S$. So, by (i) and (ii), S is a base for some filter p, and thus by corollary 1.3.5. $\exists$ an ultrafilter p' such that $p' \supseteq p$. By construction, $f_\alpha \in p', \forall \alpha \in \Delta$. Hence $p' \in [f_\alpha], \forall \alpha \in \Delta$. So $\cap C \neq 0$. Therefore $St(L)$ is compact. □

Next, we'll consider the following lattice property, labeled $(*)$:

$(*)$ If $p, q \in L$ such that $p \wedge q = 0$, then $\exists a, b \in L$ such that $p \wedge a = 0, q \wedge b = 0$, and $a \vee b = 1$.

Definition2.2.7. A lattice L is called normal if L has the (*) property.

Proposition 2.2.8. *If L is normal, then $St(L)$ is T_2.*

Proof. Let $p, q \in St(L)$ such that $p \neq q$. Then $\exists f \in p$ and $g \in q$ s.t. $f \wedge g = 0$. So by $(*)$, $\exists a, b \in L$ s.t. $f \wedge a = 0, g \wedge b = 0$, and $a \vee b = 1$ so $p \notin [a]$ and $q \notin [b]$. Also by Lemma 1.3.3 $[a] \vee [b] = [a \vee b] = [1] = St(L)$, so $St(L)$ is T_2. □

Definition 2.2.9. *A lattice L is called extermal provided that 1 is the only element of L that is in every ultrafilter on L.*

Proposition 2.2.10. *Assume that L is extermal. If $St(L)$ is T_2, then L is normal.*

Proof. Suppose $a, b \in L$ s.t. $a \wedge b = 0$. Then $[a] \cap [b] = \phi$ (i.e. they have no ultrafilters in common). Since $St(L)$ is T_2 and compact, $St(L)$ is normal. Note that the base $B = \{[a] : a \in L\}$ for the closed sets of $St(L)$ is closed under finite intersections. This, combined with the fact that $St(L)$ is normal and compact, implies the following:

$$\exists [c], [d] \in B s.t. [a] \cap [c] = \phi, [b] \cap [d] = \phi$$

and $s.t.$ $[c] \cup [d] = St(L)$.

Suppose $a \wedge c \neq 0$. Then $\exists$ an ultrafilter $p \in St(L)$ s.t. $a \wedge c \in p$. But then $a, c \in p$ (since $a, c \geq a \wedge c$), which means that $p \in [a] \cap [c]$, a contradiction. Hence $a \wedge c = 0$.

Similarly, $h \wedge d = 0$. Since $St(L) = [c] \cup [d] = [c \vee d]$ and L assumed to be extermal, then $c \vee d = 1$. Hence L is normal. □

Corollary 2.2.11. *Let L be extermal. Then L is normal iff $St(L)$ is normal.*

We now consider the disjunction property and how it relates to our extermal property. First let's define the disjuction property as it is given in [3].

Definition. 2.2.12. A lattice L has the disjunction property provided that whenever $a, b \in L$ s.t. $a \neq b$, $\exists c \in L \backslash \{0\}$ s.t. one of $a \wedge c$ and $b \wedge c$ is 0 and the other is not 0.

Theorem 2.2.13. *L has the disjunction property iff there is a 1-1 correspoundence between the elements of L and the elements of $B(St(L)) = \{[f] : f \in L\}$.*

Proof. [3].

The following proposition can easyly proved.

Proposition 2.2.14. *Suppose that $f : L \to L'$ is a lattice embedding and that p is a prime filter on L'. Then $f^{-1}(p)$ is a prime filter on L.*

Definition 2.2.15. Suppose $f : L \to L'$ is a lattice embedding and that L is normal. Define $f^D : St(L') \to St(L)$, the stone-dual map of f, by $f^D(p) = q_p$, where q_p is the unique ultrafilter in L containing $f^{-1}(p)$.

If $p \in St(L')$, then p is also prime. Also $f^{-1}(p)$ is a prime filter on L. Since L is normal, then $f^{-1}(p)$ is contained in a unique ultrafilter on L. Thus f^D is well-defined.

Furthermore,

Proposition 2.2.16. *f^D is onto.*

Proof. Use Corollary 1.3.5.

3. $L(X)$

This section will begin by defining a lattice based on the closed sets of some topological space X. Ultimately the goal is to use this lattice, which we will call $L(X)$, to characterize $\beta X \backslash X$.

Notation 2.1. Suppose X is a topolgical space and F is a closed subset of X. Then we define:
$[F] = \{G \subseteq X : G$ closed and $cl_X(F \Delta G)$ is compact$\}$.

Using the above defined equivalence class, we can construct a lattice:

Definition 2.2. For a topological space X, let $L(X) = \{[F] : F$ is closed in $X\}$, with the operations $\wedge$ and $\vee$ defined by $[F_1] \wedge [F_2] = [F_1 \cap F_2]$ and $[F_1] \vee [F_2] = [F_1 \vee F_2]$.

In the what follows we shall give a partial proof of the fact that $L(X)$ is a lattice.

Proposition 2.3. *The operations $\wedge$ and $\vee$ defined above are well defined.*

Proof. We'll just show this for $\vee$ as the proof that $\wedge$ is well defined in nearly identical. Suppose that $[F_1] \vee [F_2] = [F]$ and $[F_1] \vee [F_2] = [G]$ (i.e. $Cl_X(F\Delta(F_1 \cup F_2))$ and $Cl_X(G\Delta(F_1 \cup F_2))$ are compact). We'd like to show that $Cl_X(F\Delta G)$ is compact. We'll start by observing that $F\Delta G = (F\backslash G) \cup (G \backslash F) \subseteq (F\backslash(F_1 \cup F_2)) \cup (F_1 \cup F_2)\backslash F) \cup (G\backslash(F_1 \cup F_2)) \cup ((F_1 \cup F_2)\backslash G)$. So $Cl_X((F\backslash G) \cup (G\backslash F)) \subseteq Cl_X((F\backslash(F_1 \cup F_2) \cup ((F_1 \cup F_2)\backslash F) \cup (G\backslash(F_1 \cup F_2)) \cup ((F_1 \cup F_2)\backslash G)) \subseteq Cl_X((F\backslash(F_1 \cup F_2)) \cup (F_1 \cup F_2)\backslash F)) \cup Cl_X((G\backslash(F_1 \cup F_2)) \cup ((F_1 \cup F_2)\backslash G) = Cl_X(F\Delta(F_1 \cup F_2)) \cup Cl_X(G\Delta(F_1 \cup F_2))$.

Since the union of two compact sets is compact, then $Cl_X(F\Delta G)$ is a closed subset of a compact set, and therefore compact. So $[F] = [G]$.

Proposition 2.4. *$L(X)$ with the given operations is a bounded distributive lattice.*

Proof. The non-trivial part is stated below:

$[F] < [G]$ is equivalent to $[F] \vee [G] = [G]$ (or $[F] \wedge [G] = [F]$). The lattice bounded, as $[X]$ and $[\phi]$ are the 1 and 0 elements respectively. □

From this point on, unless otherwise stated, we will assume that our topological spaces are locally compact. In what follow, we will show that for any space X, $\beta X\backslash X$ is homeomorphic to $St(L(X))$. We will first introduce some notation and provide proofs of some necessary results to $\beta X\backslash X$. Then we will define the condidate for our homeomorphism, show that it's well defined, and show that it is, in fact, a homeomorphism.

Notation 2.5. Let A be a subset of X. Then the notation A^* means $(Cl_{\beta X}(A))\backslash X$.

Lemma 2.6. *(i) $(F \cup G)^* = F^* \cup G^*$ (ii) $(F \cap G)^* = f^* \cap G^*$.*

Proof. We prove only union case. The intersection will follow similarly.

There are two parts:

($\supseteq$) Let $x \in F^* \cup G^*$. Then $x \in f^*$ or $x \in G^*$. Let U be an open neighborhood (in βX) of x. Then U intersects F or U intersects G. Hence $U \cap (F \cup G) \neq \phi$. So $x \in Cl_{\beta X}(F \cup G)$ and $x \notin X$. So $x \in (F \cup G)^*$.

($\subseteq$) Let $x \in \beta X\backslash X$ s.t. $x \notin (F^* \cup G^*)$ (i.e. $x \notin F^*$ and $x \notin G^*$). So $\exists$ an open (in βX) neighborhood V_1 of x s.t. $V_1 \cap F = \phi$ and $\exists$ an open (in βX) neighborhood V_2 of x s.t. $V_2 \cap G = \phi$. Let $U = V_1 \cap V_2$. Then U is an open neighborhood (in βX) of x s.t. $U \cap (F \cup G) = \phi$.

Therefore $x \notin (F \cup G)^*$. □

Proposition 2.7. *$\mathcal{F} = \{F^* : F$ closed in $X\}$ is a base for the closed sets of $\beta X\backslash X$.*

Proof. Let C be a closed set in $\beta X\backslash X$ and let $x \in (\beta X\backslash X)\backslash C$. Since X is locally compact, C is closed in βX. Let V and W be open sets in βX s.t. $x \in V \cap W$, $Cl_{\beta X}(V) \subseteq W$, and $W \cap C = \phi$. Note that $V' = V \cap X$ is open in X.

Let $F = X\backslash V'$. Clearly F is closed in X. Since $\beta X\backslash X = X^* = (V' \cup F)^* = (V')^* \cup F^*$ and $(V')^* \cap C = \phi$ (because $(V')^* \subseteq (Cl_{\beta X}(V)\backslash B) \subseteq W$), then $C \subseteq F^*$. Additionally, $\beta X\backslash V$ is a closed set in βX containing $F \Rightarrow Cl_{\beta X}(F) \subseteq \beta x\backslash V \Rightarrow F^* \subseteq \beta X\backslash V$.

Since $x \in V$, then $x \notin F^*$. Therefore $\mathcal{F}$ is a base for the closed sets of $\beta X \backslash X$. □

Notation 2.8. *Let* $\mathcal{F}_p = \{F^* : [F] \in p\}$.

Proposition 2.9. *For any* $p \in St(L(X)), |\cap \mathcal{F}_p| = 1$.

Proof. We'll present the proof in two parts:

1. For any $p \in St(L(X)), \cap \mathcal{F}_p \neq \phi$:

Let $p \in St(L(X))$ suppose $F_1^*, F_2^*, \cdots, F_n^* \in \mathcal{F}_p$. Then $[F_1], [F_2] \cap \cdots \cap F_n] \in p$. Hence $F_1^* \cap F_2^* \cap \cdots \cap F_n^* = (F_1 \cap F_2 \cap \cdots \cap F_n)^* \in \mathcal{F}_p$. Therefore $\mathcal{F}_p$ is a family of closed sets with the finite intersection property. Since X is locally compact, $\beta X \backslash X$ is closed ([2]) and thus compact, so $\cap \mathcal{F}_p \neq \phi$.

2. For any $p \in St(L(X)), \cap \mathcal{F}_p$ contains only one point:

Let $p \in St(L(X))$ suppose $\exists r, s \in \cap \mathcal{F}_p$ with $r \neq s$. Then $\exists$ some basic closed set G^* (with G closed in X) with $r \in G^*$ and $s \notin G^*$. Then $[G] \notin p$ because $s \notin G^*$. Since p is maximal, $\exists [F] \in p$ s.t. $[\phi] = [F] \wedge [G] = [F \cap G]$. Hence $F \cap G$ is compact so $\phi = (F \cap G)^* = F^* \cap G^* \supseteq \{r\}$, a contradiction. □

We are now ready to define our homeomorphism candidate.

Definition 2.10. Let $h : St(L(X)) \to \beta X \backslash X$ be defined by $h(p) = r_p$, where r_p is the unique point in $\cap \mathcal{F}_p$.

By proposition 3.9. it is clear that h is well defined. That it is a homeomorphism comes from the result that follow.

Proposition 2.11. *The function* $h : St(L(X)) \to \beta X \backslash X$ *as defined above is 1-1.*

Proof. Suppose $p \neq q$, and that $h(p) = r_p, h(q) = r_q$ (show that $r_p \neq r_q$). Suppose that $r_q \in F^*, \forall [F] \in p$. Let $[G] \in q$. Then $r_q \in F^* \cap G^* = (F \cap G)^*, \forall [F] \in p$.

This implies that $[\phi] \neq [F \cap G] = [F] \wedge [G], \forall [F] \in p$. Since P is maximal, $[G] \in$, Hence $q \subseteq p$. Since q is maximal $p = q$ a contradiction. Therefore $\exists [F] \in p$ s.t. $r_q \notin F^*$. So $r_q \neq h(p) = r_p$. So h is 1-1. □

Proposition 2.12. *h is onto.*

Proof. Let $x \in \beta X \backslash X$. Let $\mathcal{G} = \{G^* : G$ is closed in X and $x \in G^*\}$.

Let $p = \{[F] : F^* \in G\}$.

Show that $p \in St(L(X))$:

1. p is a filter:

(i) Since $x \notin \phi = \phi^*$, then $\phi^* \notin \mathcal{G}$ and $[\phi] \notin p$.

(ii) Suppose $[F], [G] \in p$. So $x \in F^* \cap G^* = (F \cap G)^*$ since $F \cap G$ is closed, $(F \cap G)^* \in \mathcal{G}$. Thus $[F \cap G] \in p$. But $[F \cap G] = [F] \wedge [G]$.

(iii) Suppose $[F] \in p$ and $[G] \in L(X)$ such that $[F] < [G]$. So $[F] = [F] \wedge [G] = [F \cap G]$. So $Cl_X(F \Delta (F \cap G))$ is compact $\Rightarrow Cl_X(F \backslash (F \cap G)) \cup ((F \cap G) \backslash F))$ is compact $\Rightarrow Cl_X((F \backslash G) \cup \phi)$ is compact $\Rightarrow cl_X(F \backslash G)$ is compact $\Rightarrow (F \backslash G)^* = \phi$. Thus $F^* = ((F \backslash G) \cup (F \cap G))^* = (F \backslash G)^* \cup (F \cap G)^* = (F \cap G)^* = F^* \cap G^*$. Therefore $F^* \subseteq G^* \Rightarrow x \in G^* \Rightarrow [G] \in p$.

2. p is maximal:

Suppose $[G] \wedge [F] \neq \phi, \forall [F] \in p$ (show $[G] \in p$). Since $[\phi] \neq [G] \wedge [F] = [G \cap F]$, then $\forall [F] \in p, G \cap F$ is not compact. Hence $\phi = (G \cap F)^* = G^* \cap F^*, \forall [F] \in p$. Suppose that $x \notin G$. Then $\exists$ a basic closed set of the form J^* (with J closed in X) s.t. $x \in J^*$ and $J^* \cap G^* = \phi$ (because $\beta X \backslash X$ is regular). But $x \in J^* \Rightarrow [J] \in p$, which contradict that $G^* \cap F^* \neq \phi, \forall [F] \in p$. Hence $x \in G^*$, i.e. $[G] \in p$. Thus p is maximal.

Therefore by 1. and 2. $p \in St(L(X))$ and $h(p) = x$, so h is onto. □

Notation 2.13. In the following proof $[[F]]$ will be used for the set of ultrafilters on $L(X)$ that contain $[F]$ (i.e. $[[F]] = \{p \in St(L(X)) : [F] \in p\}$). As discussed before, the $[[F]]'s$ form a base for closed sets of $St(L(X))$.

Proposition 2.14. *h is continuous.*

Proof. Let F^* be a basic closed set in $\beta X \backslash X$. The following will show that $h^{-1}(F^*) = [[F]]$: $(\subseteq)$ let $p \in h^{-1}(F^*)$ (so $h(p) \in F^*$) and suppose that $[F] \notin p$. Then $\exists [G] \in p$ s.t. $[\phi] = [F] \wedge [G] = [F \cap G]$ (since p is maximal). So $Cl_X(F \cap G)$ is compact. Hence $\phi = (F \cap G)^* = F^* \cap G^*$, which contradicts that $h(p) \in F^* \cap G^*$. Hence $[F] \in p$, so $p \in [[F]]$.

$(\supseteq)$ let $p \in [[F]]$. Then $[F] \in P \Rightarrow h(p) \in F^*$ (by definition of h)$\Rightarrow p \in h^{-1}(F^*)$. Therefore, $h^{-1}(F^*) = [[F]]$ and since $[[F]]$ is closed in $St(L(X))$, h is continuous. □

Now since any bijective continuous function from a compact space to a Hausdorff space is a homeomorphism we have the following result.

Main Theorem 3.16. *$St(L(X)) \cong \beta X \backslash X$.*

4. Example $L(H)$

Let H be the halfline $[0, \infty)$ with the usual topology. We now present a couple of facts concerning the associated lattice $L(H)$.

Proposition 4.1. *$L(H)$ is normal.*

Proof. This proof is a little long and interested reader may ask this proof from the authors. □

Proposition 4.2. *$L(H)$ is not pseudocomplemented.*

Proof. Look at the closed set $C \subseteq H$ given by $C = [1,2] \cup [3,4] \cup [5,6] \cup \cdots = \cup[2i-1, 2i], \forall i$. Clearly C is not compact, hence $[C] \neq [\phi]$ suppose that $[F]$ is the pseudocomplement of $[C]$. Then $C \cap F$ is compact since $C \cap F \subseteq H$, then $C \cap F$ is bounded, call the upper bound $M (\in H)$. Let $G = F \cup [a, a+1] \cup S$, where a is some odd integer greater than M and S is defined in the following way:

For each $[2i-1, 2i] \subseteq C$ with $i > (a+1)/2$, we know that $2i \notin F$. So look at the sequence $((2i)_n)$ defined by $(2i)_n = 2i + (1/2n)$. Since $(2i)_n \to 2i$, then $\exists$ an $m_{2i} \in \mathbb{N}$ s.t. $(2i)_{m_{2i}} \notin F$. Let $S = \{(2i)_{m_{2i}} : i > (a+1)/2\}$. Notice that for any $n \in \mathbb{N}$, $2i < 2i + (1/2n) < 2i + 1$, so $S \cap C = \phi$. Since S is closed, unbounded and $S \cap F = \phi$, then $Cl_H(G \backslash F) = G \backslash F$ is not compact. This implies that $[G] \neq [F]$ since $G \supseteq F$ then $[G] > [F]$. But by construction $G \cap C = (F \cup [a, a+1] \cup S) \cup [a, a+1]$ which is compact. Thus $[G] \wedge [C] = [\phi]$, which contradicts that $[F]$ is pseudocomplement of $[C]$. Therefore C has no psedudocomplement, and thus $L(H)$ is not pseudocomplemented. □

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