International Journal of Pure and Applied Mathematics
Volume 82 No. 1 2013, 53-63
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

THE INVERSE SURFACES OF TANGENT DEVELOPABLE OF A TIMELIKE CURVE IN MINKOWSKI SPACE $\mathbb{E}_1^3$

M. Evren Aydin[1 §], Mahmut Ergüt[2]

Department of Mathematics
Firat University
Elazig, 23119, TURKEY

Abstract: In this paper, we study inverse surfaces in Minkowski space $\mathbb{E}_1^3$. We obtain various relations between these surfaces. Also we give some necessary and sufficient conditions so that the the inverse surface of tangent developable of a timelike curve is flat or minimal in $\mathbb{E}_1^3$.

AMS Subject Classification: 11A25, 53B30
Key Words: inverse surface, timelike curve, tangent developable surface

1. Introduction

The use of developable surfaces has a long history and studied by many mathematicians. See [4, 7, 9, 10]. Real developable surfaces have natural applications in many areas of engineering and manufacturing. For instance, an aircraft designer uses them to design the airplane wings, and a tinsmith uses them to connect two tubes of different shapes with planar segments of metal sheets. In computer graphics, in modelling and animating objects seen in everyday life can be approximated by piecewise continuous developable surfaces.

On the other hand, a conformal map is a function which preserves the angles. It is an important technique used in complex analysis and has many applications

Received: September 11, 2012

[§]Correspondence author

in different physical situations. An inversion is a conformal mapping and also is differentiable. Further, it is a self-inverse transformation defining between open subsets of $\mathbb{E}_1^3$. The image of a surface under an inversion is called its inverse surface.

This study is organised as follows: Firstly, we obtain relations between the geometric properties of inverse surfaces in $\mathbb{E}_1^3$. Secondly, we calculate the fundamental forms and curvatures of inverse surface of timelike tangent developable by the help of these of tangent developable surface. Finally, we give some necessary and sufficient conditions for the curvatures of inverse surface of the timelike tangent developable to be identically zero.

2. Preliminiaries

The Lorentz-Minkowski space is the metric space $\mathbb{E}_1^3 = \left(\mathbb{R}^3, \langle , \rangle\right)$, where the metric $\langle , \rangle$ is given by

$$\langle u, v\rangle_L = -u_1v_1 + u_2v_2 + u_3v_3, \quad u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3). \tag{2.1}$$

The Lorentz vector product of the vectors u and v is defined by

$$u \wedge_L v = (u_2v_3 - u_3v_2, u_1v_3 - u_3v_1, u_2v_1 - u_1v_2).$$

A vector $v \in \mathbb{E}_1^3$ is called a spacelike vector, a lightlike (null) vector or timelike vector if $\langle v, v\rangle_L > 0$ (or $v = 0$), $\langle v, v\rangle_L = 0$ and $v \neq 0$, or $\langle v, v\rangle_L < 0$, respectively. The light-cone of $\mathbb{E}_1^3$ is defined as the set of all lightlike vectors of $\mathbb{E}_1^3$ that is,

$$\Lambda = \left\{(v_1, v_2, v_3) \in \mathbb{E}_1^3 : -v_1^2 + v_2^2 + v_3^2 = 0\right\} - \{(0, 0, 0)\}.$$

The norm of a vector is defined by $\|v\|_L = \sqrt{|\langle v, v\rangle_L|}$ and v is called a unit vector provided $\|v\|_L = 1$.

Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ be a smooth regular curve in $\mathbb{E}_1^3$ (i.e. $\alpha'(t) \neq 0$ for any $t \in I$). α is spacelike (resp. timelike, lightlike) at t if $\alpha'(t)$ is a spacelike (resp. timelike, lightlike) vector. The curve α is called spacelike (resp. timelike, lightlike) if it is for any $t \in I$.

Let consider the timelike curve α parametrized by the arclength s. $T(s) = \alpha'(s)$ is called the tangent vector at s. Then $T'(s) \neq 0$ is a spacelike vector independent with $T(s)$. The curvature of α at s is defined as $\kappa(s) = \|T'(s)\|_L$. The normal vector $N(s)$ is given as follows

$$N(s) = \frac{T'(s)}{\kappa(s)}.$$

The binormal vector $B(s)$ is defined by

$$B(s) = T(s) \wedge_L N(s).$$

The vector $B(s)$ is spacelike. For each s, $\{T, N, B\}$ is an orthonormal base of $\mathbb{E}^3_1$ which is called the Frenet trihedron of α. The torsion of α at s is given by

$$\tau(s) = \langle N'(s), B(s)\rangle_L.$$

Thus we have Frenet equaitons as [8]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.2}$$

Let $\mathfrak{M}$ be a surface in $\mathbb{E}^3_1$ and

$$X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)),$$

is a patch on the surface $\mathfrak{M}$. Denote by $X_u = \frac{\partial X(u,v)}{\partial u}$ and $X_v = \frac{\partial X(u,v)}{\partial v}$. Then the first fundamental form of $\mathfrak{M}$ is defined by

$$g = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2, \tag{2.3}$$

where

$$g_{11} = \langle X_u, X_u\rangle_L, \; g_{12} = \langle X_u, X_v\rangle_L, \; g_{22} = \langle X_v, X_v\rangle_L.$$

The surface $\mathfrak{M}$ is called spacelike, timelike or lightlike if

$$\det(g) = g_{11}g_{22} - g_{12}^2 > 0, \quad < 0 \text{ or } = 0.$$

A unit normal spacelike vector field on a timelike surface $\mathfrak{M}$ defined by

$$U = \frac{X_u \wedge_L X_v}{\|X_u \wedge_L X_v\|_L}$$

Thus the second fundamental form of the timelike surface $\mathfrak{M}$ given by

$$L = L_{11}du^2 + 2L_{12}dudv + L_{22}dv^2, \tag{2.4}$$

where

$$L_{11} = \langle X_{uu}, U\rangle_L, \; L_{12} = \langle X_{uv}, U\rangle_L, \; L_{22} = \langle X_{vv}, U\rangle_L.$$

On the other hand, the Gaussian curvature K and the mean curvature H are defined by, respecetively,

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \tag{2.5}$$

$$H = \frac{L_{11}g_{22} - 2L_{12}g_{12} + g_{11}L_{22}}{2\left(g_{11}g_{22} - g_{12}^2\right)}. \tag{2.6}$$

3. Inversions of Curves and Surfaces in $\mathbb{E}_1^3$

Let $c \in \mathbb{E}_1^3$, $r \in \mathbb{R}^+$ and Λ_c be light-cone at the point c. We denote that

$$\left(\mathbb{E}_1^3\right)^* = \mathbb{E}_1^3 - (\{c\} \cup \Lambda_c).$$

An inversion of $\mathbb{E}_1^3$ with the center $c \in \mathbb{E}_1^3$ and the radius r is the map

$$\Phi[c,r] : \left(\mathbb{E}_1^3\right)^* \longrightarrow \left(\mathbb{E}_1^3\right)^*$$

given by

$$\Phi[c,r](p) = c + \lambda(p)(p-c), \tag{3.1}$$

where $\lambda(p) = \frac{r^2}{\|p-c\|_L^2}$ is real valued differentiable function.

Let $\alpha : I \subset \mathbb{R} \longrightarrow \left(\mathbb{E}_1^3\right)^*$ be a curve with arbitrary speed. Inverse curve of the curve α with respect to $\Phi[c,r]$ defined by

$$\Phi[c,r] \circ \alpha = c + \lambda(\alpha)(\alpha - c), \tag{3.2}$$

where $\lambda(\alpha) = \frac{r^2}{\|\alpha-c\|_L^2}$.

Denote by β the inverse curve of the curve α with respect to $\Phi[c,r]$. Differentiating (3.2)

$$\beta' = \lambda\left(\alpha' - \frac{2\langle\alpha', (\alpha - c)\rangle_L}{\|\alpha - c\|_L^2}(\alpha - c)\right),$$

also taking $\alpha(t) = p$, $\alpha'(t) = v_p$, and $\beta'(t) = \Phi_{*p}(v_p)$, then we have the tangent map of the $\Phi[c,r]$ as follows

$$\Phi_{*p}(v_p) = \lambda(p)\left(v_p - \frac{2\langle v_p, (p-c)\rangle_L}{\|p-c\|_L^2}(p-c)\right). \tag{3.3}$$

Considering (3.3),

$$\langle\Phi_{*p}(v_p), \Phi_{*p}(v_p)\rangle_L = \lambda^2(p)\langle v_p, v_p\rangle_L. \tag{3.4}$$

Thus, from (3.4), we have

Proposition 3.1. *In Minkowski space $\mathbb{E}_1^3$, the causal character of a tangent vector are invariant under the tangent map of an inversion, so causal character of a curve is.*

Now, let us assume that $X : U \subset \mathbb{E}^2 \longrightarrow \left(\mathbb{E}_1^3\right)^*$ is the patch on a surface $\mathfrak{M}$. The inverse patch of $X(u,v)$ with respect to $\Phi[c,r]$ is the patch defined by

$$Y(u,v) = (\Phi[c,r] \circ X)(u,v). \tag{3.5}$$

Throughout this paper, we assume that X is a patch in $\left(\mathbb{E}_1^3\right)^*$and Y is inverse patch of X with respect to $\Phi[c,r]$. Differentiating (3.5) with respect to u and v

$$\begin{cases} Y_u = \lambda(X)\left(X_u - \frac{2\langle X_u, X-c\rangle}{\|X-c\|_L^2}(X-c)\right) \\ Y_v = \lambda(X)\left(X_v - \frac{2\langle X_v, X-c\rangle}{\|X-c\|_L^2}(X-c)\right) \end{cases}, \tag{3.6}$$

where $\lambda(X) = \frac{r^2}{\|X-c\|_L^2}$. From this, we can easily obtain

$$(g_{ij})_Y \circ \Phi = \lambda^2(X)(g_{ij})_X \tag{3.7}$$

where $(g_{ij})_X$ and $(g_{ij})_Y$ are the coefficients the first fundamental form (metric) of X and Y, respectively. Denote by g_X and g_Y metrics of X and Y. Then

$$g_Y \circ \Phi_* = \lambda^2(X)\, g_X. \tag{3.8}$$

Proposition 3.2. *Let X be a patch in $\left(\mathbb{E}_1^3\right)^*$and Y be inverse patch of X with respect to $\Phi[c,r]$. Then,*

$$(g_{11})_Y (g_{22})_Y - (g_{12})_Y^2 = \lambda^4(X)\left((g_{11})_X (g_{22})_X - (g_{12})_X^2\right),$$

and causal character of a surface are invariant under an inversion.

Producting the vectors Y_u and Y_v and to norm these, we have

$$U_y \circ \Phi = \frac{Y_u \times Y_v}{\|Y_u \times Y_v\|_L} = -U_X + \frac{2\langle U_X, X-c\rangle_L}{\|X-c\|_L^2}(X-c), \tag{3.9}$$

where U_x and U_y are unit normal vector fields of X and Y, respectively. Differentiating (3.6) with respect to u, v and to product U_y, we obtain

$$(L_{ij})_Y \circ \Phi = -\lambda(X)\left((L_{ij})_X + \frac{2\lambda(X)\eta(X)}{r^2}(g_{ij})_X\right), \tag{3.10}$$

where $(L_{ij})_X$ and $(L_{ij})_Y$ respectively are the coefficients the second fundamental forms of X and Y,

$$\lambda(X) = \frac{r^2}{\|X-c\|_L^2} \text{ and } \eta(X) = \langle U_X, X-c\rangle_L.$$

Denote by L_X and L_Y the second fundamental forms of X and Y, respectively. Hence

$$L_Y \circ \Phi_* = -\lambda(X) L_X - \frac{2\lambda^2(X)\eta(X)}{r^2} g_X. \quad (3.11)$$

Lemma 3.3. *Let X be a patch in $(\mathbb{E}_1^3)^*$ and Y be inverse patch of X with respect to $\Phi[c,r]$. Denote by $\mathbf{S}_X$ and $\mathbf{S}_Y$ the shape operators of X and Y, respectively. Then*

$$\mathbf{S}_Y \circ \Phi_* = -\frac{1}{\lambda(X)} \mathbf{S}_X - \frac{2\eta(X)}{r^2} \mathbf{I}_2, \quad (3.12)$$

where $\mathbf{I}_2$ is unit matrix, $\lambda(X) = \frac{r^2}{\|X-c\|_L^2}$ and $\eta(X) = \langle U_X, X-c\rangle_L$.

Proof. From the equalities (3.7) and (3.10), the proof is easily done.

From Lemma 3.3, we have the following

Corollary 3.4. *Because of the equality* (3.12), *we have:*

(i) Umbilical points are invariant under an inversion;

(ii) Flat points are mapped to umbilical points under an inversion;

(iii) Flat points are invariant under an inversion if and only if $\eta(X) = 0$.

Theorem 3.5. *Let X be a patch in $(\mathbb{E}_1^3)^*$ and Y be inverse patch of X with respect to $\Phi[c,r]$. Given W_X and W_Y the third fundamental forms of X and Y, respectively. Then,*

$$W_Y \circ \Phi_* = W_X + 2\xi(X) L_X + \xi^2(X) g_X, \quad (3.13)$$

where g_X and L_X are resopectively the first and second fundamental forms of X and $\xi(X) = \frac{2\langle U_X,(X-c)\rangle_L}{\|X-c\|_L^2}$.

Proof. Assume that p be any point in $X(U)$ and $\Phi(p) = q \in y(U)$. Let the tangent map of Φ at p be $\Phi_{*p} = T_p(X) \longrightarrow T_{\Phi(p)}(Y)$. Also, suppose $u_p, v_p \in T_p(X)$ and $\hat{u}_q, \hat{v}_q \in T_q(Y)$ such that

$$\Phi_{*p}(u_p) = \hat{u}_q \text{ and } \Phi_{*p}(v_p) = \hat{v}_q.$$

Then

$$(W_Y)_q(\hat{u}_q, \hat{v}_q) = g_Y\left(\mathbf{S}_Y^2(\hat{u}_q), \hat{v}_q\right)$$

or

$$(W_Y)_q(\hat{u}_q, \hat{v}_q) = g_Y(\mathbf{S}_Y(\hat{u}_q), \mathbf{S}_Y(\hat{v}_q)). \tag{3.14}$$

Considering (3.8) and (3.12) in (3.14), we write

$$\begin{aligned}(W_Y)_q(\hat{u}_q, \hat{v}_q) &= \lambda^2(X)\, g_X\left(\left(-\lambda^{-1}(X)\,\mathbf{S}_X - \frac{2}{r^2}\eta(X)\,\mathbf{I}_2\right)(u_p),\right.\\ &\qquad \left.\left(-\lambda^{-1}(X)\,\mathbf{S}_X - \frac{2}{r^2}\eta(X)\,\mathbf{I}_2\right)(v_p)\right)\end{aligned}$$

From above equation, we obtain the (3.13).

Lemma 3.6. *Let X be a patch in $(\mathbb{E}^3_1)^*$ and Y be inverse patch of X with respect to $\Phi[c, r]$. Then*

$$k_Y \circ \Phi_* = -\lambda^{-1}(X)\, k_X - \frac{2\eta(X)}{r^2}, \tag{3.13}$$

where k_X and k_Y are the normal curvatures of X and Y,

$$\lambda(X) = \frac{r^2}{\|X - c\|_L^2} \text{ and } \eta(X) = \langle U_X, X - c\rangle_L.$$

Proof. Given $v_p \in T_p(X)$ for any point $p \in X(U)$ and $\Phi_*(v_p) = \hat{v}_q$ such that $\Phi(p) = q$. The normal curvature of Y in direction the vector $\hat{v}_q$ is

$$k_Y(\hat{v}_q) = \frac{L_Y(\hat{v}_q, \hat{v}_q)}{g_Y(\hat{v}_q, \hat{v}_q)}. \tag{3.14}$$

Substituting (3.8) and (3.11) in (3.14), we obtain the desired.

Corollary 3.7. *Since principal curvatures of a surface is extreme values of the normal curvature of the surface, there is a relation between principal curvatures of inverse surfaces as follows*

$$(k_i)_Y \circ \Phi_* = -\lambda^{-1}(X)\,(k_i)_X - \frac{2\eta(X)}{r^2}, \quad i = 1, 2, \tag{3.15}$$

where $(k_i)_X$ and $(k_i)_Y$ are the principal curvatures of X and Y, respectively.

Because of (3.15), we have

$$K_Y = \frac{1}{\lambda^2(X)} K_X + \frac{4\eta(X)}{\lambda(X)\, r^2} H_X + \frac{4\eta^2(X)}{r^4}, \tag{3.16}$$

$$H_Y = -\frac{1}{\lambda(X)} H_X - \frac{2\eta(X)}{r^2}, \tag{3.17}$$

where H_X, K_X are the mean and Gaussian curvature of X and H_Y, K_Y are these of Y.

4. Conclusions of Section 3

In Minkowski space $\mathbb{E}_1^3$, for inverse surfaces, the third fundamental forms, the minimality and flatness are invariant properties under the inversion when $\eta = 0$. We comment the function η to be identically zero as follows: If

$$\eta(X) = \langle U_X, X - c \rangle_L = 0,$$

then the tangent planes at all the points of the surface $X(u, v)$ pass through the center of inversion. In such a case, the properties mentioned above are invariant under the inversion.

5. The Inverse Surface of Tangent Developable of a Timelike Curve

Let $\gamma : I \subset \mathbb{R} \longrightarrow \left(\mathbb{E}_1^3\right)^*$ be a timelike curve with arclength s and $\{T, N, B\}$ be Frenet frame along γ. Denote by κ and τ the curvature and the torsion of the curve γ, respectively. Then the tangent developable of γ is a timelike ruled surface parametrized by

$$M(s, u) = \gamma(s) + uT(s), \tag{4.1}$$

where T is unit tangent vector field of γ. As it is known, the coefficients of the first and second fundamental forms of the surface $M(s, u)$ have following

$$(g_{11})_M = -1 + (u\kappa)^2, \ (g_{12})_M = (g_{22})_M = -1, \tag{4.2}$$

and

$$(L_{11})_M = -sgn(u\kappa)(u\kappa\tau), \ (L_{12})_M = (L_{22})_M = 0. \tag{4.3}$$

The unit normal spacelike vector field on the surface $M(s, u)$ is given by

$$U_{\mathfrak{M}}(s, u) = -sgn(u\kappa) B(s). \tag{4.4}$$

Next the curvatures (mean and Gaussian) and the matrix of shape operator of this surface are respectively as follows

$$H_M = \frac{-sgn(u\kappa)\tau}{2u\kappa}, \quad \text{and} \quad K_M = 0 \tag{4.5}$$

and

$$\mathbf{S}_M = \left(\frac{sgn\,(u\kappa)\,\tau}{u\kappa}\right)\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}. \tag{4.6}$$

We show that N is the inverse surface of the tangent developable surface M with respect to the inversion $\Phi\,[c,r]$. Thus the inverse surface N has following parametrization

$$N = c + \frac{r^2}{\|M-c\|^2}\,(M-c)\,. \tag{4.7}$$

Hence, if we take into account the equalities (3.7) and (3.10), then, the coefficients of the first and second fundamental forms of the inverse surface N are given by

$$(g_{11})_N = \lambda^2\,(M)\left(-1+(u\kappa)^2\right),\;\; (g_{12})_N = (g_{22})_N = -\lambda^2\,(M)\,, \tag{4.8}$$

and

$$\begin{aligned}(L_{11})_N =& sgn\,(u\kappa)\,\lambda u\kappa\tau - \frac{2\lambda^2\,(M)\,\eta\,(M)}{r^2}\left(-1+(u\kappa)^2\right),\\ (L_{12})_N =& (L_{22})_N = \frac{2\lambda^2\,(M)\,\eta\,(M)}{r^2},\end{aligned} \tag{4.9}$$

where $(g_{ij})_N$ and $(L_{ij})_N$ are the coefficients of the first and second fundamental forms of the inverse surface N, respectively, and

$$\lambda\,(M) = \frac{r^2}{\|M-c\|_L^2} \text{ and } \eta\,(M) = \langle U_M, M-c\rangle_L\,.$$

Denote by $\mathbf{S}_N$ the matrix of the shape operator of the inverse surface N, then using by (3.12) $\mathbf{S}_N$ is calculated as follows

$$\mathbf{S}_N = \begin{bmatrix} sgn\,(u\kappa)\,\frac{\tau}{\lambda u\kappa} - \frac{2\eta}{r^2} & sgn\,(v\kappa)\,\frac{\tau}{\lambda v\kappa}\\ 0 & -\frac{2\eta}{r^2}\end{bmatrix}. \tag{4.10}$$

Morever, the Gauss and mean curvatures of the inverse surface N by the help of these of the surface M are respectively, using by (3.16) and (3.17),

$$K_N = \frac{4}{r^2}\eta\left(-sgn\,(u\kappa)\,\frac{\tau}{2\lambda\,(M)\,u\kappa} + \frac{\eta\,(M)}{r^2}\right), \tag{4.11}$$

$$H_N = sgn\,(u\kappa)\,\frac{\tau}{2\lambda\,(M)\,u\kappa} - \frac{2\eta\,(M)}{r^2}, \tag{4.12}$$

where K_N and H_N are the Gauss and the mean curvatures of the inverse surface N, respectively.

Theorem 4.1. *Let N be the inverse surface of the tangent developable surface M with respect to the inversion Φ. Then the inverse surface N is a flat surface if and only if either the normal lines to the surface M or the tangent planes of the surface M pass through the center of inversion.*

Proof. Assume that the inverse surface N is flat, then from (4.11), we can write

$$\frac{4}{r^2}\eta\left(M\right)\left(-sgn\left(u\kappa\right)\frac{\tau}{2\lambda\left(M\right)u\kappa}+\frac{\eta\left(M\right)}{r^2}\right)=0, \tag{4.13}$$

where either

$$\eta\left(M\right)=\langle U_M,\left(M-c\right)\rangle_L=0, \tag{4.14}$$

or

$$sgn\left(u\kappa\right)\frac{\tau}{2\lambda\left(M\right)u\kappa}=\frac{\eta\left(M\right)}{r^2}. \tag{4.15}$$

If the equality (4.14) is satisfied, then the tangent planes of the surface M pass through the center of inversion. If the equality (4.15) holds, then it follows

$$U_M=sgn\left(u\kappa\right)\frac{\tau}{2u\kappa}\left(M-c\right).$$

Namely, the normal lines to the surface M pass through the center of inversion.

The proof of sufficient condition is obvious.

Theorem 4.2. *Let N be the inverse surface of the tangent developable surface M with respect to $S_c\left(r\right)$. The inverse surface N is minimal if and only if the normal lines to the surface M pass through the center of inversion.*

Proof. The proof is similar to that of Theorem 4.1.

References

[1] A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, CRC Press LLC (1998).

[2] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[3] D.E. Blair, *Inversion Theory and Conformal Mapping*, American Mathematical Society (2000).

[4] E. Turhan, T. Korpınar, Characterization tangent developable of null biharmonic curves in the Lorentzian Heisenberg group Heis^3, *World Applied Sciences Journal*, **11**, No. 8 (2010), 893-897.

[5] F. Dillen, W. Kühnel, Ruled Weingarten surfaces in Minkowski 3-space, *Manuscripta Math.*, **98** (1999), 307-320.

[6] H.S.M. Coexeter, Inversive geometry, *Educational Studies in Mathematics*, **3** (1971), 310-321.

[7] P. Alegre, K. Arslan, A. Carriazo, C. Murathan, G. Öztürk, Some special types of developable ruled surface, *Hacettepe Journal of Mathematics and Statistics*, **39**, No. 3 (2010), 319-325.

[8] R. López, Differential geometry of curves and surfaces in Lorentz-Minkowski space, a*ArXiv:* 0810.3351v1 [math.DG] (2008).

[9] S. Izumiya, H. Katsumi, T. Yamasaki, The rectifying developable and the spherical Darboux image of a space curve, *Geometry and Topology of Caustics-Caustics'98*, Banach Center Publications, **50** (1999), 137-149.

[10] S. Izumiya, N. Takeuchi, Special curves and ruled surfaces, In: *Applicable Mathematics in the Golden Age* (Ed. J.C. Misra), Narosa Publishing House, New Delhi (2003), 305-338.