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BURNSIDE'S LEMMA

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Abstract: There is a famous problem which involves discriminating the faces of a die using 3 colors: how many different patterns can be produced? This article introduces Burnside's lemma which is a powerful method for handling such problems. It requires a knowledge of group theory, but is not so difficult and is likely to be understood by elementary school pupils.

AMS Subject Classification: 19A22, 00A09, 97A20
Key Words: Burnside's counting theorem, Pólya's formula, permutation, orbit, invariant, equivalence, equivalence class

1. Discriminating Dice Using 3 Colors

When I set the following problem in a certain magazine, one reader put forward a wonderful solution using Burnside's lemma (see Nishiyama, 2006).[3] I did not know of Burnside's lemma. It requires a knowledge of group theory and a familiarity with the appropriate symbology, but it's not so difficult and high-school students can probably understand it.

The problem I set was as follows. There are a number of squares divided with diagonal borders and colored differently, as shown in Figure 1. How many different possible patterns are there when 4 such squares forming a 2×2 grid are filled in? Cases of symmetrical colors, rotational symmetries and mirror

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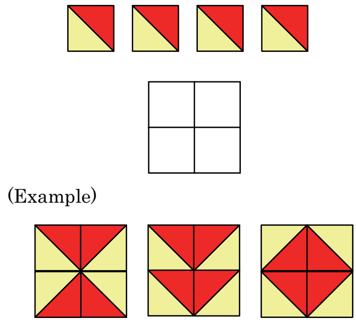

Figure 1: How many ways in total are there to make different patterns?

symmetries are regarded as equivalent.

2. Case-by-Case Solution

The objective in this chapter is to introduce Burnside's lemma, so let's begin the explanation with a well-used example. Suppose there is a die like that shown in Figure 2. When the six faces of this die are divided by painting them each with one of three colors, how many different patterns can be produced?

To begin with, allow me to explain a general case-based solution. Dice are cubes so they have six faces. Suppose they are each painted with one of three colors (say blue, yellow or red). There are three different possible cases for the color of each face. Thus, since each of the faces are independent, in total there are

$$3^6 = 729$$

ways of coloring the faces.

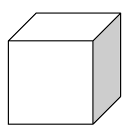

Figure 2: Discriminating the faces of a die using 3 colors

Checking all of these is a laborious task. Let's therefore try thinking about the cases organized in the following way.

Classifying according to how many colors are used yields three cases: 1 color, 2 colors and 3 colors. Let's attempt a top-level classification on this basis. When only 1 color is used, there are 3 cases, i.e., when all 6 faces are simply either blue, yellow, or red.

Next, in the case that 2 colors are used, the ratio of the colors can take 3 different values, 5 : 1 faces, 4 : 2, or 3 : 3. Let's use this as a mid-level classification. There is a further relationship according to which 2 of the 3 colors are chosen. When all 3 colors are used, there are 3 possible ratios of the colors, 4 : 1 : 1 faces, 3 : 2 : 1, or 2 : 2 : 2. Counting up the patterns in this way, there are 57 different cases.

This counting operation is probably impossible with pencil and paper. I actually drew a net of the cube on a computer, and checked the arrangements of the colors over and over again. It turned out that many times, the patterns that I had imagined to be different inside my head were actually the same. In the end I bought a wooden block (with 2 cm edges) from the carpentry section of a DIY store, and sticking colored paper on the faces, confirmed the 57 patterns. Figure 4 shows each of the 57 patterns. Blue is represented by B, yellow by Y and red by R, and the numbers 1 to 6 in the header row correspond to the numbers of the faces in the net (Figure 3).

3. Solution using Burnside's Lemma

No matter how cautiously the equations enumerating the cases are counted up, counting errors and oversights are sometimes bound to happen. For situations like this, there is a powerful method which applies knowledge from group theory known as Burnside's lemma. I'll explain below.

Burnside's lemma is described as follows in the free encyclopedia, Wikipedia.[1] Burnside's lemma is also known as Burnside's counting theorem, Pólya's formula, the Cauchy-Frobenius lemma, and the orbit-counting theorem. These all refer to the same thing. Burnside wrote down this lemma in 1900. According to the history of mathematics, Cauchy wrote it in 1845, and Frobenius in 1887, so Burnside was not the first person to discover it, and some people refer to it formally as 'not-Burnside's lemma'.

When a permutation group G is applied to a set X, if the number of elements which are invariant under an element g of the group G is denoted X^g, then the number of orbits, $|X/G|$, is given by the following formula.

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$

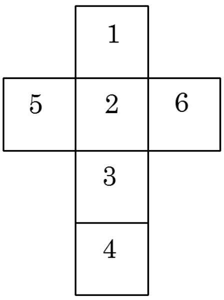

Figure 3: The numbers cor

No.	1	2	3	4	5	6	B	Y	R		
1	B	B	B	B	B	B	6				
2	Y	Y	Y	Y	Y	Y		6		6	1 color
3	R	R	R	R	R	R			6		
4	B	Y	B	B	B	B	5	1			
5	B	R	B	B	B	B	5		1		
6	Y	B	Y	Y	Y	Y	1	5		5+1	
7	Y	R	Y	Y	Y	Y		5	1		
8	R	B	R	R	R	R	1		5		
9	R	Y	R	R	R	R		1	5		
10	B	Y	Y	B	B	B	4	2			
11	B	R	R	B	B	B	4		2		
12	Y	B	B	Y	Y	Y	2	4			
13	Y	R	R	Y	Y	Y		4	2		
14	R	B	B	R	R	R	2		4		
15	R	Y	Y	R	R	R		2	4	4+2	2 colors
16	B	Y	B	Y	B	B	4	2			
17	B	R	B	R	B	B	4		2		
18	Y	B	Y	B	Y	Y	2	4			
19	Y	R	Y	R	Y	Y		4	2		
20	R	B	R	B	R	R	2		4		
21	R	Y	R	Y	R	R		2	4		
22	B	Y	Y	B	B	Y	3	3			
23	Y	R	R	Y	Y	R		3	3		
24	R	B	B	R	R	B	3		3	3+3	
25	B	Y	Y	Y	B	B	3	3			
26	Y	R	R	R	Y	Y		3	3		
27	R	B	B	B	R	R	3		3		
28	B	Y	R	B	B	B	4	1	1		
29	Y	R	B	Y	Y	Y	1	4	1		
30	R	B	Y	R	R	R	1	1	4	4+1+1	
31	B	Y	B	R	B	B	4	1	1		
32	Y	R	Y	B	Y	Y	1	4	1		
33	R	B	R	Y	R	R	1	1	4		
34	B	Y	Y	R	B	B	3	2	1		
35	Y	R	R	B	Y	Y	1	3	2		
36	R	B	B	Y	R	R	2	1	3		
37	B	R	R	Y	B	B	3	1	2		
38	Y	B	B	R	Y	Y	2	3	1		
39	R	Y	Y	B	R	R	1	2	3		
40	B	Y	R	B	B	Y	3	2	1		
41	Y	R	B	Y	Y	R	1	3	2		
42	R	B	Y	R	R	B	2	1	3	3+2+1	3 colors
43	B	R	Y	B	B	R	3	1	2		
44	Y	B	R	Y	Y	B	2	3	1		
45	R	Y	B	R	R	Y	1	2	3		
46	B	Y	R	Y	B	B	3	2	1		
47	Y	R	B	R	Y	Y	1	3	2		
48	R	B	Y	B	R	R	2	1	3		
49	B	R	Y	R	B	B	3	1	2		
50	Y	B	R	B	Y	Y	2	3	1		
51	R	Y	B	Y	R	R	1	2	3		
52	Y	B	R	B	R	Y	2	2	2		
53	R	Y	B	Y	B	R	2	2	2		
54	B	R	Y	R	Y	B	2	2	2	2+2+2	
55	B	Y	R	R	B	Y	2	2	2		
56	B	Y	R	R	Y	B	2	2	2		
57	B	R	B	R	Y	Y	2	2	2		

Figure 4: Discriminating the die faces using 3 colors (B: blue, Y: yellow, R: red)

The number of orbits means the number of things which are equivalent.

It was shown above that there are $3^6 = 729$ different ways of partitioning the 6 faces of a dice using 3 colors. This set is denoted X. There are 4 types of rotation group G which can be considered with respect to X.

(1) Rotation by 90° about an axis through two parallel faces (this can be performed in 6 different ways). In the case of a rotation of 90° about the axis through faces ABFE and DCGH shown in Figure 5(1), the parallel faces ABFE and DCGH may be different colors, so there are 3^2 different colorings of these faces, but the four faces which are moved by 90°, ABCD, BFGC, EFGH, and AEHD must all be the same color, so there are 3 colorings. For each axis there are thus 3^3 possibilities, which gives a total of 6×3^3 possibilities.

(2) Rotation by 180° about an axis through two parallel faces (this can be performed in 3 different ways). In the case of a rotation of 180 about the axis through faces ABFE and DCGH shown in Figure 5(1), the parallel faces ABFE and DCGH may be different colors, so there are 3^2 colorings of these faces. Since the rotation is by 180°, it is necessary for the corresponding faces, among the four which remain, to have the same color. For example, faces ABCD and EFGH must be the same, as well as BFGC and AEHD. This gives 3^2 possible colorings. There are thus 3^4 possibilities for each axis, which gives a total of 3×3^4 possibilities.

(3) Rotation by 120° about an axis through two opposite vertices (this can be performed in 8 possible ways). In the case of a rotation of 120° about the axis through vertices B and H shown in Figure 5(2), the 3 faces adjacent to vertex B (ABCD, BFGC and ABFE) must all be the same color. Likewise, the 3 faces adjacent to vertex H (DCGH, EFGH and AEHD) must all be the same color. For each axis there are 3^2 combinations of colors, which gives a total of 8×3^2 possibilities.

(4) Rotation by 180° about an axis through two opposite edges (this can be performed in 6 possible ways). In the case of a rotation of 180° about the axis through edges BF and DH shown in Figure 5(3), the 2 faces adjacent to the edge BF (BFGC and ABFE) must be the same color, and the 2 faces adjacent to DH (DCGH and AEHD) must be the same color. Also, the two opposite faces, ABCD and EFGH, which are shifted through 180° must also be the same color. For each axis there are 3^3 combinations of colors, which gives a total of 6×3^3 possibilities.

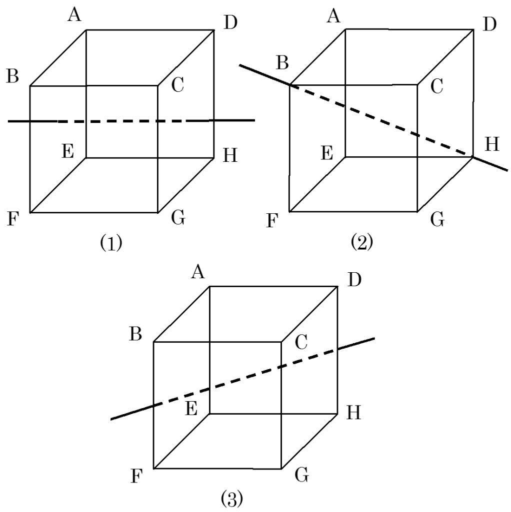

Figure 5: Rotation axes and rotation groups

The number of elements in the rotation group G, including the identity transformation e, is $1 + 6 + 3 + 8 + 6 = 24$. Applying the information above yields the equation

$$\frac{1}{24}(3^6 + 6 \times 3^3 + 3 \times 3^4 + 8 \times 3^2 + 6 \times 3^3) = 57,$$

and there are thus 57 different patterns.

4. Group Theory, Permutation Groups, and Equivalence Classes

Considering a set X, and a permutation group G which acts on the set X, we'd like to obtain the number of equivalence classes in X according to the equivalence relation on X derived from G. This problem can be solved directly by finding the equivalence relation, and then counting the number of equivalence classes. However, when the set X has a particularly large number of elements, such a counting method may be sufficiently awkward as to be beyond human capability.

The number of equivalence classes can be found with Burnside's theory, by counting the numbers of elements (permutations) of X that are invariant under the group. If a given permutation transforms a given element onto itself, then the element is described as 'invariant' under the permutation (see Liu, translated by Narishima and Akiyama, 1995).[2]

The number of elements (permutations) included in the permutation group G, is denoted by $|G|$. For a permutation $\pi \in G$, the elements which are mapped by π onto themselves are known as 'invariant', i.e., they do not vary from their original values, and the number of invariant elements is denoted by $n(\pi)$ (see Oyama, 1997).[4]

Theorem. (Burnside) *For a set X and permutation group G, the number of equivalence classes in X under the equivalence relation imposed by G, written $N(X)$, is given by the following formula*

$$N(X) = \frac{1}{|G|} \sum_{\pi \in G} n(\pi) \tag{1}$$

A simple example is shown below. Denote the 3 vertices of an equilateral triangle such as that shown in Figure 6, by A, B and C, and consider the cases when these vertices are colored either red or white. The total number of ways of coloring the vertices, as shown by $P = \{P_1, P_2, \cdots, P_8\}$ in Figure 7, is $2^3 = 8$. This triangle may, by way of example, be subjected to rotations of 120° in a clockwise direction about an axis perpendicular to the triangle and passing through its centre. This transforms P_2 in Figure 7 to P_3 , and P_3 to P_4. Sets like P_2, P_3, P_4 can thus be considered "equivalent".

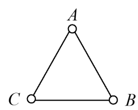

Figure 6: Equilateral triangle

The permutations of $P_1, \cdots, P_8$ that result when the equilateral triangle is rotated by 120° or 240° in a clockwise direction about a perpendicular axis passing through its center can be expressed as shown below.

$$\pi_1 = (P_1)(P_2P_3P_4)(P_5P_6P_7)(P_8),$$

$$\pi_2 = \pi_1\pi_1 = (P_1)(P_2P_4P_3)(P_5P_7P_6)(P_8).$$

The number of invariant elements for each permutation π , written $n(\pi)$, is given by the following equation.

$$n(\pi_1) = n(\pi_2) = 2. \tag{2}$$

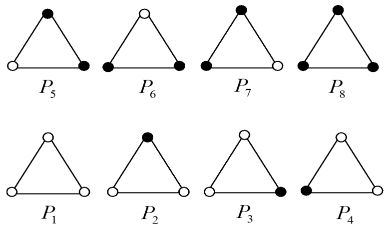

Figure 7: Coloring an equilateral triangle

On the other hand, when the triangle is rotated by 180° about an axis from one vertex to the mid-point of the opposite edge, P_5 becomes P_7, or alternatively, P_5 becomes P_6. This reveals that these are indeed "equivalent". For these cases the permutations of $P = \{P_1, P_2, \cdots, P_8\}$ may be expressed as shown below.

$$\pi_3 = (P_1)(P_2)(P_3P_4)(P_5P_7)(P_6)(P_8),$$

$$\pi_4 = (P_1)(P_2P_4)(P_3)(P_5P_6)(P_7)(P_8),$$

$$\pi_5 = (P_1)(P_2P_3)(P_4)(P_5)(P_6P_7)(P_8).$$

The number of invariant elements $n(\pi)$, for each permutation π, is as follows.

$$n(\pi_3) = n(\pi_4) = n(\pi_5) = 4 \tag{3}$$

Considering the permutations $\{\pi_1, \pi_2, \cdots, \pi_5\}$ expressed above, and in addition, the identity permutation which maps every element to itself,

$$\pi_0 = (P_1)(P_2)(P_3)(P_4)(P_5)(P_6)(P_7)(P_8),$$

it can be seen that they constitute a group. In this way, the number of equivalence classes $N(X)$ imposed by the permutation group on the set $P = \{P_1, P_2, \cdots, P_8\}$, is given by the following formula, based on Equation 1, and using Equations (2) and (3), and π_0.

$$N(P) = \frac{1}{|G|}\sum_{i=0}^{5} n(\pi_i) = \frac{1}{6}(8 + 2 \times 2 + 3 \times 4) = 4$$

The number of equivalence classes is thus 4, and it can be seen that the equivalence classes are $\{P_1\}$,$\{P_2, P_3, P_4\}$,$\{P_5, P_6, P_7\}$, and $\{P_8\}$. The transformations of the elements are shown in Table 1, the invariant elements are shown in Table 2, and Table 3 shows the equivalence relations.

	π_0	π_1	π_2	π_3	π_4	π_5
P_1	P_1	P_1	P_1	P_1	P_1	P_1
P_2	P_2	P_3	P_4	P_2	P_4	P_3
P_3	P_3	P_4	P_2	P_4	P_3	P_2
P_4	P_4	P_2	P_3	P_3	P_2	P_4
P_5	P_5	P_6	P_7	P_7	P_6	P_5
P_6	P_6	P_7	P_5	P_6	P_5	P_7
P_7	P_7	P_5	P_6	P_5	P_7	P_6
P_8	P_8	P_8	P_8	P_8	P_8	P_8

Table 1. Transformations of the elements according to the permutations

	π_0	π_1	π_2	π_3	π_4	π_5
P_1	=	=	=	=	=	=
P_2	=			=		
P_3	=				=	
P_4	=					=
P_5	=					=
P_6	=			=		
P_7	=				=	
P_8	=	=	=	=	=	=
	8	2	2	4	4	4

Table 2. Invariant elements

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
P_1	$\surd$							
P_2		$\surd$	$\surd$	$\surd$				
P_3		$\surd$	$\surd$	$\surd$				
P_4		$\surd$	$\surd$	$\surd$				
P_5					$\surd$	$\surd$	$\surd$	
P_6					$\surd$	$\surd$	$\surd$	
P_7					$\surd$	$\surd$	$\surd$	
P_8								$\surd$

Table 3. Equivalence relations

References

[1] Burnside's lemma, from Wikipedia.

[2] C.L. Liu, *Elements of Discrete Mathematics*, Second Edition (Translated by H. Narishima, J. Akiyama), Tokyo, Ohmsha (1995), 450-457.

[3] Y. Nishiyama, Elegant na Kaito wo Motomu (Seeking Elegant Solutions), *Sugaku Semina (Mathematics Seminar)*, **45**, No. 9 (2006), 95-100.

[4] T. Oyama, *Power-Up Risan Sugaku (Power-Up Discrete Mathematics)*, Tokyo, Kyoritsu (1997), 70-76.