International Journal of Pure and Applied Mathematics
Volume 82 No. 4 2013, 513-519
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v82i4.1
ijpam.eu

TUPLE OF OPERATORS WITH THE PROPERTY OF HYPERCYCLICITY CRITERION

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Abstract: In this paper we give the conditions under which a tuple of operators holds in the Hypercyclicity Criterion.

AMS Subject Classification: 47B37, 47B33
Key Words: tuple, hypercyclic vector, hypercylicity criterion, hereditarily hypercyclicity

1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let $\mathcal{F} = \{{T_1}^{k_1} {T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \in \mathbb{Z}_+, \ \ i = 1, ..., n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e. $\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}$.

Received: December 12, 2011

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We say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, ..., x_k \in X$ such that $\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$ is dense in the k copies of X, $X \oplus ... \oplus X$.

Note that if $T_1, T_2, ..., T_n$ are commutative bounded linear operators on a Banach space X, and $\{m_j(i)\}_j$, is a sequence of natural numbers for $i = 1, ..., n$, then we say $\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)} : j \geq 0\}$ is hypercyclic if there exists $x \in X$ such that $\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)}x : j \geq 0\}$ is dense in X.

Definition 1.2. We say that a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is topologically transitive with respect to a tuple of nonnegative integer sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j),$$

if for every nonempty open subsets U, V of X there exists $j_0 \in \mathbb{N}$ such that $T_1^{k_{j_0(1)}} T_2^{k_{j_0(2)}} ... T_n^{k_{j_0(n)}}(U) \cap V \neq \emptyset$. Also, we say that an n-tuple $\mathcal{T}$ is topologically transitive if it is topologically transitive with respect an n-tuple of nonnegative integer sequences.

Definition 1.3. We say that a pair $\mathcal{T} = (T_1, T_2, ...T_n)$ is hereditarily hypercyclic with respect to a tuple of nonnegative increasing sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$$

of integers provided for all tuple of subsequences $(\{k_{j_i(1)}\}_i, \{k_{j_i(2)}\}_i, ..., \{k_{j_i(n)}\}_i)$ of $(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$, the sequence $\{T_1^{k_{j_i(1)}} T_2^{k_{j_i(2)}} ... T_n^{k_{j_i(n)}} : i \geq 1\}$ is hypercyclic. We say that an n-tuple $\mathcal{T}$ is hereditarily hypercyclic, if it is hereditarily hypercyclic with respect to an n-tuple of nonnegative increasing sequences of integers.

The formulation of the Hypercyclicty Criterion in the next section was given by N. S. Feldman ([5]). Here, we want to extend some properties of hypercyclic operators to a pair of commuting operators, and although the techniques work for any n-tuple of operators but for simplicity we prove our results only for the case $n = 2$. For some other topics we refer to [1–18].

2. Main Results

In this section we characterize the equivalent conditions for a tuple of operators, satisfying the Hypercyclicity Criterion.

Theorem 2.1. (The Hypercyclicity Criterion) *Suppose that X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple* *of operators T_1, T_2, ..., T_n acting on X. If there exist two dense subsets Y and Z in X, and strictly increasing sequences $\{m_j(i)\}_j$ for $i = 1, ..., n$ such that:*

1. $T_1^{m_j(1)}...T_n^{m_j(n)}y \to 0$ for all $y \in Y$,

2. there exist a sequence of functions $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_j(1)}...T_n^{m_j(n)}S_j z \to z$ as $j \to \infty$.
Then $\mathcal{T}$ is a hypercyclic tuple.

In the proof of the following lemma, we use a method of the proof of Theorem 1.2 in [2] to extend the results for tuples . We will use HC($\mathcal{T}$) for the collection of hypercyclic vectors for a tuple of operator $\mathcal{T}$.

Lemma 2.2. *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the tuple of operators $T_1, T_2, ..., T_n$. Then the followings are equivalent:*

(i) $\mathcal{T}$ is hypercyclic.

(ii) for all nonempty open subsets U, V in X, there exists a tuple of sequences $(\{k_j(1)\}_j, \{k_j(2)\}_j, ..., \{k_j(n)\}_j)$ of integers such that for all $j \geq 0$,

$$T_1^{k_j(1)}{T_2}^{k_j(2)}...{T_n}^{k_j(n)}(U) \cap V \neq \emptyset.$$

(iii) $\mathcal{T}$ is topologically transitive.

Proof. First note that if x is a hypercyclic vector for $\mathcal{T}$, then for all natural numbers $m_1, ..., m_n$, the orbit $Orb(\mathcal{T}, T_1^{m_1}{T_2}^{m_2}...{T_n}^{m_n}x)$ is equal to

$$Orb(\mathcal{T}, x)\backslash\{{T_1}^{j_1}{T_2}^{j_2}...{T_n}^{j_n}x : j_i = 0, ..., m_i - 1; i = 1, ..., n\}.$$

Since X has no isolated point, $Orb(\mathcal{T}, x)$ remains dense after the removal of a finite number of points. Hence $T_1^{m_1}{T_2}^{m_2}...{T_n}^{m_n}x \in HC(\mathcal{T})$ for all $m_1, ..., m_n \in \mathbb{N}$ and so $Orb(\mathcal{T}, x) \subset HC(\mathcal{T})$. So $HC(\mathcal{T})$ is dense or empty.

(i) $\to$ (ii): Let $\mathcal{T}$ be hypercyclic and (U, V) be a pair of nonempty open subsets of X. Since $HC(\mathcal{T})$ is dense, thus the sets $U \cap HC(\mathcal{T})$ and $V \cap HC(\mathcal{T})$ are nonempty. Choose $x \in U \cap HC(\mathcal{T})$ and $y \in V \cap HC(\mathcal{T})$, and pick k_0 such that $B(y, \frac{1}{k_0}) \subset V$. Since $B(y, \frac{1}{k}) \cap Orb(\mathcal{T}, x)$ is nonempty, thus there there exists a tuple $(m'_k(1), m'_k(2), ..., m'_k(n))$ of integers such that $T_1^{m'_k(1)}{T_2}^{m'_k(2)}...{T_n}^{m'_k(n)}(x) \in B(y, \frac{1}{k}) \cap HC(\mathcal{T})$, and so

$$T_1^{m'_k(1)}{T_2}^{m'_k(2)}...{T_n}^{m'_k(n)}(U) \cap V \neq \emptyset$$

for all $k \geq k_0$. Define $m_k(i) = m'_{k_0+k}(i)$ for all $i = 1, ..., n$ and for all $k \geq 0$. Then $T_1^{m_k(1)} T_2{}^{m_k(2)} ... T_n{}^{m_k(n)}(U) \cap V \neq \emptyset$ for all $k \geq 0$.

(ii) $\rightarrow$ (iii): it is clear.

(iii) $\rightarrow$ (i): Fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of the open balls in X with rational radii, and centers in a countable dense subset of X. By the continuity of the operators $T_1, T_2, ..., T_n$, the sets

$$G_k = \bigcup \{T_1^{-m(1)} T_2^{-m(2)} ... T_n^{-m(n)}(B_k) : m(i) \geq 0, \ \ i = 1, ..., n\}$$

are open. Clearly HC($\mathcal{T}$) is equal to $\bigcap \{G_k : k \in \mathbb{N}\}$. Now let $\mathcal{T}$ be topologically transitive and let W be an arbitrary nonempty open set in X. Then for all $i = 1, ..., n$ and $k \in \mathbb{N}$, there exists $m_k(i)$ in $\mathbb{N}$ such that

$$T_1^{m_k(1)} T_2{}^{m_k(2)} ... T_n{}^{m_k(n)}(W) \cap B_k \neq \emptyset$$

which implies that $W \cap G_k \neq \emptyset$ for all k. Thus each G_k is dense in X and so by the Bair Category Theorem $HC(\mathcal{T})$ is also dense in X. In particular, $HC(\mathcal{T})$ is nonempty and so (i) holds. $\square$

Corollary 2.3. *$\mathcal{T}_d^{(2)}$ is hypercyclic if and only if for given four nonempty open subsets U_1, U_2, V_1, V_2 of X, there exists a tuple of integers $(m(1), ..., m(n))$ such that*

$$T_1^{m(1)} T_2^{m(2)} ... T_n^{m(n)}(U_1) \cap V_1 \neq \emptyset; \ \ T_1^{m(1)} T_2^{m(2)} ... T_n^{m(n)}(U_2) \cap V_2 \neq \emptyset.$$

Theorem 2.4. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators acting on a separable infinite dimensional Banach space X. Then the followings are equivalent:*

(i) $\mathcal{T}$ satisfies the Hypercyclicity Criterion.

(ii) $\mathcal{T}$ is hereditarily hypercyclic.

(iii) $\mathcal{T}_d^{(2)}$ is hypercyclic.

Proof. (i) $\rightarrow$ (ii): Let X_0, Y_0, $(\{m_j(1)\}_j, \{m_j(2)\}_j, ..., \{m_j(n)\}_j)$, and $S_j : Y_0 \longrightarrow X$ be as given in the Hypercyclicity Criterion. Notice that Hypercyclicity Criterion will also be satisfied by any tuple of subsequences

$$(\{m_{j_k}(1)\}_k, \{m_{j_k}(2)\})_k, ..., \{m_{j_k}(n)\}_k)$$

of $(\{m_j(1)\}_j, \{m_j(2)\}_j, ..., \{m_j(n)\}_j)$. Hence, it suffices to check that

$$\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)} : j \geq 1\}$$

is hypercyclic that is equivalent to that $\mathcal{T}$ is hereditarily hypercyclic with respect to the tuple $(\{m_j(1)\}_j, \{m_j(2)\}_j, ..., \{m_j(n)\}_j)$. Now let U and V be nonempty open subsets of X and pick $x \in X_0$, $y \in Y_0$ and $\epsilon > 0$ so that $B(x, \epsilon) \subset U$ and $B(y, 2\epsilon) \subset V$. By the conditions of Hypercyclicity Criterion, there exist r arbitrary large satisfying:

$$T_1^{m_r(1)}T_2^{m_r(2)}...T_n^{m_r(n)}x \in B(0, \epsilon), \quad S_r y \in B(0, \epsilon),$$

and

$$T_1^{m_r(1)}T_2^{m_r(2)}...T_n^{m_r(n)}S_r y - y \in B(0, \epsilon).$$

So we get $x + S_r y \in B(x, \epsilon) \subset U$ and

$$||T_1^{m_r(1)}T_2^{m_r(2)}...T_n^{m_r(n)}S_r y - y|| + ||T_1^{m_r(1)}T_2^{m_r(2)}...T_n^{m_r(n)}x|| < 2\epsilon.$$

Thus $T_1^{m_r(1)}T_2^{m_r(2)}...T_n^{m_r(n)}S_r y - y \in B(y, 2\epsilon) \subset V$. Hence

$$T_1^{m_r(1)}T_2^{m_r(2)}...T_n^{m_r(n)}(U) \cap V$$

is nonempty and so by the Lemma 2.2, $\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)} : j \geq 1\}$ is hypercyclic which implies that $\mathcal{T}$ is hereditarily hypercyclic with respect to $(\{m_j(1)\}_j, \{m_j(2)\}_j, ..., \{m_j(n)\}_j)$.

(ii) $\rightarrow$ (iii): Suppose $\mathcal{T}$ is hereditarily hypercyclic with respect to the tuple of sequences $(\{m_j(1)\}_j, \{m_j(2)\}_j, ..., \{m_j(n)\}_j)$ and for $i = 1, 2$, let (U_i, V_i) be a pair of nonempty open subsets of X. We want to show that there are arbitrary large positive integers $m_1, ..., m_n$ satisfying $T_1^{m_1}T_2^{m_2}...T_n^{m_n}(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2$. Since $\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)} : j \geq 1\}$ is hypercyclic, by Lemma 2.2., there exists a tuple of subsequences $(\{m_{j_k}(1)\}_k, \{m_{j_k}(2)\})_k, ..., \{m_{j_k}(n)\}_k)$ of $(\{m_j(1)\}_j, \{m_j(2)\}_j, ..., \{m_j(n)\}_j)$ with $T_1^{m_{j_k}(1)}T_2^{m_{j_k}(2)}...T_n^{m_{j_k}(n)}(U_1) \cap V_1 \neq \emptyset$ for $j \geq 1$. But $\mathcal{T}$ is hereditarily hypercyclic with respect to $(\{m_k\}, \{n_k\})$, thus $\{T_1^{m_{j_k}(1)}T_2^{m_{j_k}(2)}...T_n^{m_{j_k}(n)} : k \geq 1\}$ is also hypercyclic. Hence, by Lemma 2.2, for $i = 1, ..., n$, there exists $m_i \in \{m_{j_k}(i)\}$ arbitrary large so that

$$T_1^{m_1}T_2^{m_2}...T_n^{m_n}(U_2) \cap V_2 \neq \emptyset.$$

Hence we get $T_1^{m_1}T_2^{m_2}...T_n^{m_n}(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2$, and so $\mathcal{T}_d^{(2)}$ is hypercyclic and (iii) holds.

(iii) $\rightarrow$ (i): Let $x \oplus y$ be a hypercyclic vector for $\mathcal{T}_d^{(2)}$. In particular, x and y are hypercyclic for $\mathcal{T}$. Thus for all tuple of nonnegative integers $(m_1, m_2, ..., m_n)$, the vector $T_1^{m_1} T_2^{m_2} ... T_n^{m_n} y$ is hypercyclic for $\mathcal{T}$ and so $(x, T_1^{m_1} T_2^{m_2} ... T_n^{m_n} y)$ is also a hypercyclic vector for $\mathcal{T}_d^{(2)}$. This implies that for all nonempty open subset U of X, there is $u \in U$ such that (x, u) is a hypercyclic vector for $\mathcal{T}_d^{(2)}$. Fix now $\{U_k\}_{k \geq 1}$ a decreasing 0-basis in X. Proceeding by induction we find $u_j \in U_j$ for all $j \in \mathbb{N}$, and increasing sequences $\{m_j(i)\}_j$ $(i = 1, ..., n)$ of natural numbers satisfying ${T_1}^{m_j(1)} {T_2}^{m_j(2)} ... {T_n}^{m_j(n)} x \in U_j$ and

$${T_1}^{m_j(1)} {T_2}^{m_j(2)} ... {T_n}^{m_j(n)} u_j \in x + U_j$$

for all $j \in \mathbb{N}$. Let $X_0 = Orb(\mathcal{T}, x)$ which is dense in X. Then we have that ${T_1}^{m_j(1)} {T_2}^{m_j(2)} ... {T_n}^{m_j(n)} x \rightarrow 0$ and so ${T_1}^{m_j(1)} {T_2}^{m_j(2)} ... {T_n}^{m_j(n)} v \rightarrow 0$ for all $v \in X$. Define $S_j(T_1^{m_1} T_2^{m_2} ... T_n^{m_n}) x = T_1^{m_1} T_2^{m_2} ... T_n^{m_n} u_j$ for all m_i, $i = 1, ..., n$, and all j in $\mathbb{N}$. Then $S_j v \rightarrow 0$ for all $v \in X_0$. Finally, given $m_0(1), m_0(2), ..., m_0(n) \in \mathbb{N}$, we get that

$${T_1}^{m_j(1)} {T_2}^{m_j(2)} ... {T_n}^{m_j(n)} S_j (T_1^{m_0(1)} T_2^{m_0(2)} ... T_n^{m_0}(n) x)$$

is equal to

$$T_1^{m_0(1)} T_2^{m_0(2)} ... T_n^{m_0}(n) ({T_1}^{m_j(1)} {T_2}^{m_j(2)} ... {T_n}^{m_j(n)} u_j)$$

which tends to $T_1^{m_0(1)} T_2^{m_0(2)} ... T_n^{m_0}(n) x$ as $j \rightarrow \infty$. This completes the proof. $\square$

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