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YET ANOTHER X-RANK CHARACTERIZATION OF RATIONAL NORMAL CURVES

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Abstract: Fix positive integers s, k_i, $1 \le i \le s$, such that $k_1 \ge 2$ and $2k < n$, where $k := k_1 + \cdots + k_s$. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. We prove that X is not a rational normal curve if and only if the following condition holds: fix s general points $P_1, \dots, P_s \in X_{reg}$ and set $Z := \sum_{i=1}^{s} k_i P_i$; then there is some $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$ and $r_X(P) \le n + 1 - k$.

Moreover, if X is not a rational normal curve and we fix a finite set $E \subset X$, then we may find a set $S \subset X \setminus E$ with $\sharp(S) \le n + 1 - k$ and $P \in \langle S \rangle$.

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1. Introduction

Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\ \rangle$ denote the linear span. In characteristic zero we have $r_X(P) \le n + 1 - \dim(X)$ for any X and any P (see [5], Proposition 4.1). In positive characteristic this is true, except

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for at most one P (see [1]). In this short note we prove the following result.

Theorem 1. *Fix positive integers s, k_i, $1 \le i \le s$, such that $k_1 \ge 2$ and $2k < n$, where $k := k_1 + \cdots + k_s$. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. X is not a rational normal curve if and only if the following condition ♠ holds:*

Condition ♠ : Fix s general points $P_1, \ldots, P_s \in X_{reg}$ and set $Z := \sum_{i=1}^{s} k_i P_i$. Then there is some $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$ and $r_X(P) \le n + 1 - k$.

Moreover, if X is not a rational normal curve and we fix a finite set $E \subset X$, then we may find a set $S \subset X \setminus E$ with $\sharp(S) \le n + 1 - k$ and $P \in \langle S \rangle$.

In the case $k_i = 1$ for all i we prove the following result.

Theorem 2. *Fix integers $k \ge 2$ and $n > 2k$. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. Fix a general $Z \subset X$ such that $\sharp(Z) = s$. X is not a rational normal curve if and only if there is some $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$ and $W \supseteq Z$ for every set $W \subset X$ such that $\sharp(W) \le n + 1 - k$, $W \cap Z = \emptyset$ and $P \in \langle W \rangle$. Moreover, if X is not a rational normal curve for any fixed finite set $E \subset X$ we may find W as above with $W \cap E = \emptyset$.*

Instead of $W \cap Z = \emptyset$ we may assume $W \not\supseteq Z$.

Question 1. For general Z is it possible to take as P a general element of $\langle Z \rangle$? Or, even, every $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$?

2. The Proofs

Lemma 1. *Let $C \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Fix a general $P \in C$ and a finite set $E \subset C$ with $P \notin E$. There is a finite set $S \subset C \setminus E$ such that $P \in \langle S \rangle$, $P \notin S$ and $\sharp(S) \le r$ if and only if C is not a rational normal curve.*

Proof. If C is a rational normal, then no such set exists, because any $r+1$ points of C are linearly independent. Now assume that C is a rational normal curve. Take a general hyperplane $H \subset \mathbb{P}^r$ containing P. For general H we have $H \cap E = \emptyset$. Since P is general in C, H may be seen as a general hyperplane. Hence $C \cap H$ is a general hyperplane section of C. Hence $C \cap H$ is formed by $\deg(C) > n$ distinct points and $C \cap H$ spans H. If C is not very strange in the sense of [6], then we may take as S any r points of $C \cap H \setminus \{P\}$. If C is very strange, then we need to check that not all r-ples of point of $C \cap H$

spanning H contains P. Fix $P_1, P_2 \in C \cap H \setminus \{P\}$ such that $P_1 \neq P_2$. If $r = 2$ we take $S = \{P_1, P_2\}$. Assume $r > 2$. For every integer $t \in \{2, \dots, r-2\}$ all t-dimensional linear subspaces of H spanned by points of $C \cap H$ contain the same number of points of $C \cap H$. Hence there is $P_3 \in C \cap H \setminus (\langle P_1, P_2 \rangle \cup \{P\})$. And so on if $r > 3$. □

Proof of Theorems 1 and 2. In characteristic zero the " if part of Theorem 1 is the easy part of theorem of Sylvester (see [4], [3], [5]). In arbitrary characteristic it is just [2], Lemma 1, and the observation that if X is a rational normal curve, then every zero-dimensional scheme $A \subset X$ with $\deg(A) \leq n+1$ is linearly independent. Hence it is sufficient to prove the " only if " part. In the set-up of Theorem 2 take $s := k$ and write $Z = \{P_1, \dots, P_s\}$. Fix a finite set $E \subset X$. Let $\ell : \mathbb{P}^n \setminus \{P_s\} \to \mathbb{P}^{n-1}$ denote the linear projection from P_s. Since P_s is general in X, a dimensional count gives that for a general $Q \in X$ the line $\{Q, P_s\}\rangle$ meets X only at Q and P_s and that a general tangent line of X does not contain P_s. Hence $\ell | X \setminus \{P_s\}$ is birational onto its image. Let $C \subset \mathbb{P}^{n-1}$ denote the closure of $\ell(X \setminus \{P_s\})$ in $\mathbb{P}^{n-1}$. Since $\deg(C) = \deg(X) - 1$, X is a rational normal curve if and only if C is a rational normal curve. Assume that C is not a rational normal curve. Set $Q_i := \ell(P_i)$, $1 \leq i \leq s-1$. Let $Q_s \in C$ be the only point corresponding to the tangent line of X at P_s. Set $E' := \ell(E \setminus \{P_s\}) \cup \{Q_1, \dots, Q_s\}$. Set $B := \sum_{i=1}^{s-1} k_i Q_i + (k_s - 1)Q_s$ with the convention that $0Q_s$ is the zero divisor C. Since $P_1, \dots, P_s$ are general in X, $Q_1, \dots, Q_s$ are general in C. We use induction on k. First assume $k = 2$. In this case $B = Q_1$. Lemma 1 with the set E' gives the existence of a set $S' \subset C \setminus (E_1 \cup Z_{red})$ such that $\sharp(S') \leq n + 1 - k$ and $Q_1 \in \langle S' \rangle$. Since $Q_s \notin S$, there is a unique set $S \subset X$ such that $\ell(S) = S'$. By construction we have $S \cap E = \emptyset$. Since $Q_1 \in \langle S' \rangle$ and ℓ is the linear projection from P_1, we have $\langle Z \rangle \cap \langle S \rangle$. Since $P_s \notin S$ and $P_1 \notin S$, the set $\langle Z \rangle \cap \langle S \rangle$ is a unique point, P, and $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Now assume $k > 2$. We apply the inductive assumption the integer $k-1$. Take $S' \subset C \setminus S'$ such that $\sharp(S') \leq n - 1 - (k-1) = n - k$ and $\langle S' \rangle \cap \langle B \rangle$ contains a point P' such that $P' \notin \langle B' \rangle$ for any $B' \subsetneq B$. Then as in the case $k = 2$ we may take as S the only subset of X with $\ell(S) = S'$. □

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