International Journal of Pure and Applied Mathematics
Volume 82 No. 5 2013, 747-762
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v82i5.8
ijpam.eu

HERMITE-HADAMARD-LIKE TYPE INEQUALITIES FOR (α, m)-CONVEX AND S-CONVEX FUNCTIONS

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Abstract: In this article, a generalized lemma which help us to formulate some integral inequalities of Hermite-Hadamard-like type for functions whose q-powers of the absolute values of second derivatives are (α, m)-convex and s-convex in the second sense is given.

AMS Subject Classification: 26A24, 26A51, 26D15
Key Words: convexity, (α, m)-convexity, s-convexity, Hadamard inequality

1. Introduction

Recall that a function $f : I \subset [0, b^*] \to R, b^* > 0$, is said to be convex on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the reversed inequality in the inequality (1.1), then f is concave.

The following inequality is well-known in the literature as Hadamard's inequality: Let $f : I \subset [0, b^*] \to R$ be a convex mapping defined on an interval I in R, where $a, b \in I$ with $a < b$ and $b^* > 0$. Then the following inequality

Received: December 3, 2012

holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

If f is concave, then both inequalities in (1.2) hold in reversed direction [10, 12, 20].

Some basic definitions can be given as followings: In [6], Dragomir defined m-convex functions as following:

Definition 1.1. A mapping $f : I \subseteq [0, b^*] \to R, b^* > 0$, is said to be m-convex on I for $m \in [0, 1]$ if the following inequality

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

For recent results on m-convex functions we refer the interested reader to [7, 13, 15, 16, 21].

As generalizations of the previous notions, we have the following definition [5, 7, 11, 13, 14]:

Definition 1.2. A mapping $f : I \subset [0, b^*] \to R$ is said to be (α, m)-convex on I for $(\alpha, m) \in [0, 1]^2$ if the following inequality

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

holds for $a, b \in I$ with $a < b$ and $t \in [0, 1]$.

Denote by K_m^α the class of all (α, m)-convex functions on $[0, b^*]$ for which $f(0) \leq 0$. Note that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$, one obtains the following classes of functions: increasing, α-star-shaped, star-shaped, m-convex, convex and α-convex. For recent results on (α, m)-convex functions we refer the interested reader to [13, 14, 15].

In [18], Park established the following theorem for (α, m)-convex functions:

Theorem 1.3. *Suppose that $f : I \subset [0, b^*] \to R$, $b^* > 0$, is an (α, m)-convex function on I for $(\alpha, m) \in [0, 1]^2$. If $f \in L[a, b]$, then the following inequality*

$$(a)\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a)+\alpha m f(\frac{b}{m})}{\alpha+1}, \frac{\alpha m f(\frac{a}{m})+f(b)}{\alpha+1}\right\},$$

$$(b)\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\{f(a)+f(b)\}+\alpha m\{f(\frac{a}{m})+f(\frac{b}{m})\}}{2(\alpha+1)} \qquad (2)$$

holds , where $a, b \in I$ with $a < b$ and $t \in [0, 1]$.

In [9], Hudzik and Maligranda considered among others the class of functions which are s-convex functions in the second sense. This class is defined in the following ways [17, 21]:

Definition 1.4. A function $f : I \subset R^+ \to R$, where $R^+ = [0, \infty)$, is said to be s-convex in the second sense for some fixed $s \in (0, 1]$ if the following inequality

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{3}$$

holds for $x, y \in I$ and $t \in [0, 1]$. The above inequality (1.3) holds in reversed direction if f is s-concave in the second sense.

Obviously one can see that if we choose $s = 1$, the above definition reduces to ordinary concept of convexity.

For recent results on co-ordinated s-convex mappings we refer the interested reader to [2, 3, 4, 8, 11, 17, 19]. In [8], Dragomir and Fitzpatrik proved the following variant of Hermite-Hadamard inequality which holds for s-convex functions in the second sense:

Theorem 1.5. *Suppose that $f : I \subset R^+ = [0, \infty) \to R$ is an s-convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequalities hold:*

$$2^{s-1} f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}. \tag{4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.4).

The main purpose of the present article is to give a generalized lemma which help us to formulate some integral inequalities of Hermite-Hadamard-like type for functions whose q-powers of the absolute values of second derivatives are (α, m)-convex and s-convex in the second sense.

2. Inequalities for Differentiable (α, m)-Convex Functions

To prove our main results, we will give a generalized lemma:

Lemma 1. Let $f : I \subseteq R \to R$ be a twice differentiable function on I^0, the interior of I, where $a, b \in I$ with $a < b$. If $f'' \in L[a, b]$, then the following identity

$$I(f, n, [a, b])$$

$$
\begin{aligned}
&=^{\text{put}} \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\Big\{f(\frac{(n-1)a+b}{n}) + f(\frac{a+(n-1)b}{n})\Big\} \\
&\quad - (\frac{n-4}{8n})\Big\{f'(\frac{(n-1)a+b}{n}) - f'(\frac{a+(n-1)b}{n})\Big\}(b-a) \\
&= \Big(\frac{b-a}{4}\Big)^2\Big\{\Big(\frac{2}{n}\Big)^3(I_1+I_4) + \Big(\frac{n-2}{n}\Big)^3\big(I_2+I_3\big)\Big\} \qquad (5)
\end{aligned}
$$

holds for $n \geq 3$, where

$$
\begin{aligned}
I_1 &= \int_0^1 t^2 f''(t\frac{(n-1)a+b}{n} + (1-t)a)dt, \\
I_2 &= \int_0^1 (1-t)^2 f''(t\frac{a+b}{2} + (1-t)\frac{(n-1)a+b}{n})dt, \\
I_3 &= \int_0^1 t^2 f''(t\frac{a+(n-1)b}{n} + (1-t)\frac{a+b}{2})dt, \\
I_4 &= \int_0^1 (1-t)^2 f''(tb + (1-t)\frac{a+(n-1)b}{n})dt.
\end{aligned}
$$

Proof. By integrating by parts and using the substitutions:

$$
\begin{aligned}
&x = t\frac{(n-1)a+b}{n} + (1-t)a, \qquad x = t\frac{a+b}{2} + (1-t)\frac{(n-1)a+b}{n}, \\
&x = t\frac{a+(n-1)b}{n} + (1-t)\frac{a+b}{2}, \quad x = tb + (1-t)\frac{a+(n-1)b}{n},
\end{aligned}
$$

we can write

$$
\begin{aligned}
I_1 =& \{\frac{n}{b-a}\}f'(\frac{(n-1)a+b}{n}) - 2\{\frac{n}{b-a}\}^2 f(\frac{(n-1)a+b}{n}) \\
&+ 2\{\frac{n}{b-a}\}^3\int_a^{\frac{(n-1)a+b}{n}} f(x)dx, \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
I_2 =& -\{\frac{2n}{(n-2)(b-a)}\}f'(\frac{(n-1)a+b}{n}) - 2\{\frac{2n}{(n-2)(b-a)}\}^2 \\
&\times f(\frac{(n-1)a+b}{n}) + 2\{\frac{2n}{(n-2)(b-a)}\}^3\int_{\frac{(n-1)a+b}{n}}^{\frac{a+b}{2}} f(x)dx, \qquad (7) \\
I_3 =& \{\frac{2n}{(n-2)(b-a)}\}f'(\frac{a+(n-1)b}{n}) - 2\{\frac{2n}{(n-2)(b-a)}\}^2 \\
&\times f(\frac{a+(n-1)b}{n}) + 2\{\frac{2n}{(n-2)(b-a)}\}^3\int_{\frac{a+b}{2}}^{\frac{a+(n-1)b}{n}} f(x)dx, \qquad (8)
\end{aligned}
$$

$$I_4 = -\{\frac{n}{b-a}\}f'(\frac{a+(n-1)b}{n}) - 2\{\frac{n}{b-a}\}^2 f(\frac{a+(n-1)b}{n}) + 2\{\frac{n}{b-a}\}^3 \int_{\frac{a+(n-1)b}{n}}^{b} f(x)dx. \tag{9}$$

By the simple calculations we get the desired result (5) by using the identities (6)-(9).

Theorem 2.1. *Let $f : I \subseteq R^+ \to R$ be a twice differentiable function on I^0, the interior of I, where $a, b \in I$ with $a < b$ and $f'' \in L[a,b]$. For some fixed $q > 1$, if $|f''|^q$ is an (α, m)-convex function on $[a,b]$, then the following inequality*

$$\begin{aligned} &\Big|I(f,n,[a,b])\Big| \\ &\le \Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}} \\ &\times \Big[\Big(\frac{2}{n}\Big)^3\Big\{\mu_{11}^{\frac{1}{q}} + \mu_{12}^{\frac{1}{q}}\Big\} + (\frac{n-2}{n})^3\Big\{\mu_{21}^{\frac{1}{q}} + \mu_{22}^{\frac{1}{q}}\Big\}\Big]. \end{aligned} \tag{10}$$

where

$$\mu_{11} = \min\Big\{\frac{|f''(a)|^q + \alpha m|f''(\frac{(n-1)a+b}{mn})|^q}{\alpha+1}, \frac{\alpha m|f''(\frac{a}{m})|^q + |f''(\frac{(n-1)a+b}{n})|^q}{\alpha+1}\Big\},$$

$$\mu_{12} = \min\Big\{\frac{|f''(\frac{a+(n-1)b}{n})|^q + \alpha m|f''(\frac{b}{m})|^q}{\alpha+1}, \frac{\alpha m|f''(\frac{a+(n-1)b}{mn})|^q + |f''(b)|}{\alpha+1}\Big\},$$

$$\mu_{21} = \min\Big\{\frac{|f''(\frac{(n-1)a+b}{n})|^q + \alpha m|f''(\frac{a+b}{2m})|^q}{\alpha+1}, \frac{\alpha m|f''(\frac{(n-1)a+b}{mn})|^q + |f''(\frac{a+b}{2})|^q|}{\alpha+1}\Big\},$$

$$\mu_{22} = \min\Big\{\frac{|f''(\frac{a+b}{2})|^q + \alpha m|f''(\frac{a+(n-1)b}{mn})|^q}{\alpha+1}, \frac{\alpha m|f''(\frac{a+b}{2m})|^q + |f''(\frac{a+(n-1)b}{n})|^q}{\alpha+1}\Big\}.$$

Proof. From Lemma 2.1 and by using the property of modulus, we have

$$\begin{aligned}
&\Big|I(f,n,[a,b])\Big| \\
&= \Big|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\Big\{f(\frac{(n-1)a+b}{n}) + f(\frac{a+(n-1)b}{n})\Big\} \\
&\quad - (\frac{n-4}{8n})\Big\{f'(\frac{(n-1)a+b}{n}) - f'(\frac{a+(n-1)b}{n})\Big\}(b-a)\Big| \\
&\le \Big(\frac{b-a}{4}\Big)^2\Big\{\Big(\frac{2}{n}\Big)^3(|I_1|+|I_4|) + \Big(\frac{n-2}{n}\Big)^3(|I_2|+|I_3|)\Big\} \\
&\le \Big(\frac{b-a}{4}\Big)^2\Big[\Big(\frac{2}{n}\Big)^3 \\
&\quad \times \Big\{\Big(\int_0^1 t^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f''(t\frac{(n-1)a+b}{n} + (1-t)a)\Big|^q dt\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\int_0^1 (1-t)^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f''(tb + (1-t)\frac{a+(n-1)b}{n})\Big|^q dt\Big)^{\frac{1}{q}}\Big\} \\
&\quad + \Big(\frac{n-2}{n}\Big)^3 \\
&\quad \times \Big\{\Big(\int_0^1 (1-t)^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f''(t\frac{a+b}{2} + (1-t)\frac{(n-1)a+b}{n})\Big|^q dt\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\int_0^1 t^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f''(t\frac{a+(n-1)b}{n} + (1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}\Big] \\
&= \Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}} \\
&\quad \times \Big[\Big\{\Big(\frac{2}{n}\Big)^3\Big\{\Big(\int_0^1 \Big|f''(t\frac{(n-1)a+b}{n} + (1-t)a)\Big|^q dt\Big)^{\frac{1}{q}} \\
&\qquad\qquad + \Big(\int_0^1 \Big|f''(tb + (1-t)\frac{a+(n-1)b}{n})\Big|^q dt\Big)^{\frac{1}{q}}\Big\} \\
&\quad + \Big\{\Big(\frac{n-2}{n}\Big)^3\Big\{\Big(\int_0^1 \Big|f''(t\frac{a+b}{2} + (1-t)\frac{(n-1)a+b}{n})\Big|^q dt\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\int_0^1 \Big|f''(t\frac{a+(n-1)b}{n} + (1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}\Big]. \qquad (11)
\end{aligned}$$

By the definition of (α, m)-convex functions for $|f''|^q$ for some fixed $q > 1$ and Theorem 1.3, we have

$$(i) \int_0^1 \Big|f''(t\frac{(n-1)a+b}{n} + (1-t)a)\Big|^q dt$$

$$= \frac{1}{\frac{(n-1)a+b}{n} - a} \int_a^{\frac{(n-1)a+b}{n}} \left|f(x)\right|^q dx$$

$$\leq \min\Big\{ \frac{\left|f''(a)\right|^q + \alpha m \left|f''(\frac{(n-1)a+b}{mn})\right|^q}{\alpha+1}, \frac{\alpha m\left|f''(\frac{a}{m})\right|^q + \left|f''(\frac{(n-1)a+b}{n})\right|^q}{\alpha+1}\Big\}, \tag{12}$$

(ii) $$\int_0^1 \left|f''(tb + (1-t)\frac{a+(n-1)b}{n})\right|^q dt$$

$$\leq \min\Big\{ \frac{\left|f''(\frac{a+(n-1)b}{n})\right|^q + \alpha m\left|f''(\frac{b}{m})\right|^q}{\alpha+1}, \frac{\alpha m\left|f''(\frac{a+(n-1)b}{mn})\right|^q + \left|f''(b)\right|}{\alpha+1}\Big\}, \tag{13}$$

(iii) $$\int_0^1 \left|f''(t\frac{a+b}{2} + (1-t)\frac{(n-1)a+b}{n})\right|^q dt$$

$$\leq \min\Big\{ \frac{\left|f''(\frac{(n-1)a+b}{n})\right|^q + \alpha m\left|f''(\frac{a+b}{2m})\right|^q|}{\alpha+1}, \frac{\alpha m\left|f''(\frac{(n-1)a+b}{mn})\right|^q + \left|f''(\frac{a+b}{2})\right|^q|}{\alpha+1}\Big\}, \tag{14}$$

(iv) $$\int_0^1 \left|f''(t\frac{a+(n-1)b}{n} + (1-t)\frac{a+b}{2})\right|^q dt$$

$$\leq \min\Big\{ \frac{\left|f''(\frac{a+b}{2})\right|^q + \alpha m\left|f''(\frac{a+(n-1)b}{mn})\right|^q}{\alpha+1}, \frac{\alpha m\left|f''(\frac{a+b}{2m})\right|^q + \left|f''(\frac{a+(n-1)b}{n})\right|^q}{\alpha+1}\Big\}. \tag{15}$$

Substituting the inequalities (12)-(15) in (11), we get the desired result (10). Another variant of previous theorem is the following:

Theorem 2.2. *Let $f : I \subseteq R^+ \to R$ be a twice differentiable function on I^0, the interior of I, where $a, b \in I$ with $a < b$ and $f'' \in L[a,b]$. For some fixed $q > 1$, if $\left|f''\right|^q$ is an (α, m)-convex function on $[a,b]$, then the following inequality*

$$\left|I(f,n,[a,b])\right|$$

$$\leq \Big(\frac{b-a}{4}\Big)^2 \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}} \Big[\Big(\frac{2}{n}\Big)^3 \Big\{\Big(\frac{1}{q+\alpha+1}\{\left|f''(\frac{(n-1)a+b}{n})\right|^q$$

$$+\frac{\alpha m}{q+1}|f''(\frac{a}{m})|^q\}\Big)^{\frac{1}{q}}+\Big(B(\alpha+1,q+1)m|f''(\frac{b}{m})|^q$$
$$+(\frac{1}{q+1}-B(\alpha+1,q+1))\Big|f''(\frac{a+(n-1)b}{n})\Big|^q\Big)^{\frac{1}{q}}\Big\}$$
$$+\Big(\frac{n-2}{n}\Big)^3\Big\{\Big(\frac{1}{q+\alpha+1}\{|f''(\frac{(n-1)a+b}{n})|^q$$
$$+\frac{\alpha m}{q+1}|f''(\frac{a+b}{2m})|^q\}\Big)^{\frac{1}{q}}+\Big(B(\alpha+1,q+1)m|f''(\frac{a+b}{2m})|^q$$
$$+(\frac{1}{q+1}-B(\alpha+1,q+1))\Big|f''(\frac{(n-1)a+b}{n})\Big|^q\Big)^{\frac{1}{q}}\Big\}\Big].$$

Proof. From Lemma 2.1 and by using the property of modulus, we have

$$\Big|I(f,n,[a,b])\Big|$$
$$=\Big|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\Big\{f(\frac{(n-1)a+b}{n})+f(\frac{a+(n-1)b}{n})\Big\}$$
$$-(\frac{n-4}{8n})\Big\{f'(\frac{(n-1)a+b}{n})-f'(\frac{a+(n-1)b}{n})\Big\}(b-a)\Big|$$
$$\leq\Big(\frac{b-a}{4}\Big)^2\Big[\Big\{\Big(\frac{2}{n}\Big)^3$$
$$\times\Big\{\Big(\int_0^1 t^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^q\Big|f''(t\frac{(n-1)a+b}{n}+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1(1-t)^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1(1-t)^q\Big|f''(tb+(1-t)\frac{a+(n-1)b}{n})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}$$

$$+\Big\{\Big(\frac{n-2}{n}\Big)^3$$
$$\times\Big\{\Big(\int_0^1(1-t)^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1(1-t)^q\Big|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 t^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^q\Big|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}\Big]$$
$$=\Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\Big\{\Big(\frac{2}{n}\Big)^3\Big\{\Big(\int_0^1 t^q\Big|f''(t\frac{(n-1)a+b}{n}+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1(1-t)^q\Big|f''(tb+(1-t)\frac{a+(n-1)b}{n})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}$$
$$+\Big\{\Big(\frac{n-2}{n}\Big)^3\Big\{\Big(\int_0^1(1-t)^q\Big|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t^q\Big|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}\Big]. \tag{16}$$

By the definition of (α, m)-convex functions in the second sense for $|f''|^q$ for some fixed $q > 1$, we have

$$(i)\int_0^1 t^q\Big|f''(t\frac{(n-1)a+b}{n}+(1-t)a)\Big|^q dt$$
$$\leq \frac{1}{q+\alpha+1}\Big\{\Big|f''(\frac{(n-1)a+b}{n})\Big|^q+\frac{\alpha m}{q+1}\Big|f''(a)\Big|^q\Big\}, \tag{17}$$
$$(ii)\int_0^1 (1-t)^q\Big|f''(tb+(1-t)\frac{a+(n-1)b}{n})\Big|^q dt$$
$$\leq B(\alpha+1,q+1)m\Big|f''(\frac{b}{m})\Big|^q$$
$$+\Big(\frac{1}{q+1}-B(\alpha+1,q+1)\Big)\Big|f''(\frac{a+(n-1)b}{n})\Big|^q, \tag{18}$$
$$(iii)\int_0^1 (1-t)^q\Big|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\Big|^q dt$$
$$\leq B(\alpha+1,q+1)m\Big|f''(\frac{a+b}{2m})\Big|^q$$
$$+\Big(\frac{1}{q+1}-B(\alpha+1,q+1)\Big)\Big|f''(\frac{(n-1)a+b}{n})\Big|^q, \tag{19}$$
$$(iv)\int_0^1 t^q\Big|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\Big|^q dt$$

$$\leq \frac{1}{q+\alpha+1}\Big\{\Big|f''(\frac{a+(n-1)b}{n})\Big|^q+\frac{\alpha m}{q+1}\Big|f''(\frac{a+b}{2m})\Big|^q\Big\}. \tag{20}$$

Substituting the inequalities (17)-(20) in (16), we get the desired result.

3. Inequalities for Differentiable s-Convex Functions

For functions with the power of the second derivative in absolute value s-convex we will obtain the following theorem:

Theorem 3.1. *Let $f : I \subseteq R^+ \to R$ be a twice differentiable function on I^0, the interior of I, where $a, b \in I$ with $a < b$ and $f'' \in L[a, b]$. For some* *fixed* $q > 1$, *if* $\left|f''\right|^{\frac{p}{p-1}}$ *is* s*-convex in the second sense on* $[a,b]$ *for some fixed* $s \in (0,1]$, *then the following inequality*

$$\begin{aligned}
&\left|I(f,n,[a,b])\right| \\
&\le \Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{s+1}\Big)^{\frac{1}{q}} \\
&\quad\times\Big[\Big(\frac{2}{n}\Big)^3\Big\{\Big(\big|f''(\frac{(n-1)a+b}{n})\big|^q+\big|f''(a)\big|^q\Big)^{\frac{1}{q}} \\
&\qquad\qquad+\Big(\big|f''(b)\big|^q+\big|f''(\frac{a+(n-1)b}{n})\big|^q\Big)^{\frac{1}{q}}\Big\} \\
&\quad+(\frac{n-2}{n})^3\Big\{\Big(\big|f''(\frac{a+b}{2})\big|^q+\big|f''(\frac{(n-1)a+b}{n})\big|^q\Big)^{\frac{1}{q}} \\
&\qquad\qquad+\Big(\big|f''(\frac{a+(n-1)b}{n})\big|^q+\big|f''(\frac{a+b}{2})\big|^q\Big)^{\frac{1}{q}}\Big\}\Big]
\end{aligned} \tag{21}$$

holds for $n \ge 3$, *where* $p > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and using the property of modulus, we have

$$\begin{aligned}
&\left|I(f,n,[a,b])\right| \\
&=\Big|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\Big\{f(\frac{(n-1)a+b}{n})+f(\frac{a+(n-1)b}{n})\Big\} \\
&\quad-(\frac{n-4}{8n})\Big\{f'(\frac{(n-1)a+b}{n})-f'(\frac{a+(n-1)b}{n})\Big\}(b-a)\Big|
\end{aligned}$$

$$\begin{aligned}
&\le\Big(\frac{b-a}{4}\Big)^2\Big[\Big(\frac{2}{n}\Big)^3 \\
&\quad\times\Big\{\Big(\int_0^1 t^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f''(t\frac{(n-1)a+b}{n}+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}} \\
&\quad+\Big(\int_0^1(1-t)^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f''(tb+(1-t)\frac{a+(n-1)b}{n})\Big|^q dt\Big)^{\frac{1}{q}}\Big\} \\
&\quad+\Big(\frac{n-2}{n}\Big)^3 \\
&\quad\times\Big\{\Big(\int_0^1(1-t)^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\Big|^q dt\Big)^{\frac{1}{q}} \\
&\quad+\Big(\int_0^1 t^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}\Big]
\end{aligned}$$

$$= \Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}$$

$$\times \Big[\Big(\frac{2}{n}\Big)^3\Big\{\Big(\int_0^1 \Big|f''(t\frac{(n-1)a+b}{n}+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 \Big|f''(tb+(1-t)\frac{a+(n-1)b}{n})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}$$

$$+\Big(\frac{n-2}{n}\Big)^3\Big\{\Big(\int_0^1 \Big|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 \Big|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}\Big]. \tag{22}$$

By the definition of s-convex functions for $\big|f''\big|^q$, we have

$$(i) \int_0^1 \Big|f''(t\frac{(n-1)a+b}{n}+(1-t)a)\Big|^q dt$$

$$\leq \frac{\big|f''(\frac{(n-1)a+b}{n})\big|^q+\big|f''(a)\big|^q}{s+1}, \tag{23}$$

$$(ii) \int_0^1 \Big|f''(tb+(1-t)\frac{a+(n-1)b}{n})\Big|^q dt$$

$$\leq \frac{\big|f''(b)\big|^q+\big|f''(\frac{a+(n-1)b}{n})\big|^q}{s+1}, \tag{24}$$

$$(iii) \int_0^1 \Big|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\Big|^q dt$$

$$\leq \frac{\big|f''(\frac{a+b}{2})\big|^q+\big|f''(\frac{(n-1)a+b}{n})\big|^q}{s+1}, \tag{25}$$

$$(iv) \int_0^1 \Big|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\Big|^q dt$$

$$\leq \frac{\big|f''(\frac{a+(n-1)b}{n})\big|^q+\big|f''(\frac{a+b}{2})\big|^q}{s+1}. \tag{26}$$

Substituting the inequalities (23)-(26) in (22), we get the desired result (21).

Theorem 3.2. *Let $f : I \subseteq R^+ \to R$ be a twice differentiable function on I^0, the interior of I, where $a, b \in I$ with $a < b$ and $f'' \in L[a, b]$. For some fixed $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, if $\big|f''\big|^{\frac{p}{p-1}}$ is s-concave in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequality*

$$\Big|I(f, n, [a, b])\Big|$$

$$\leq \Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}2^{\frac{s-1}{q}}\Big[\Big(\frac{2}{n}\Big)^3\Big\{\Big|f''(\frac{(2n-1)a+b}{2n})\Big| + \Big|f''(\frac{a+(2n-1)b}{2n})\Big|\Big\} + (\frac{n-2}{n})^3\Big\{\Big|f''(\frac{(3n-2)a+(n+2)b}{4n})\Big| + \Big|f''(\frac{(n+2)a+(3n-2)b}{4n})\Big|\Big\}\Big] \tag{27}$$

holds for $n \geq 3$.

Proof. From Lemma 2.1 and using the property of modulus, we have

$$\begin{aligned}
&\Big|I(f,n,[a,b])\Big| \\
&= \Big|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\Big\{f(\frac{(n-1)a+b}{n}) + f(\frac{a+(n-1)b}{n})\Big\} \\
&\quad - (\frac{n-4}{8n})\Big\{f'(\frac{(n-1)a+b}{n}) - f'(\frac{a+(n-1)b}{n})\Big\}(b-a)\Big| \\
&\leq \Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}} \\
&\quad \times \Big[\Big(\frac{2}{n}\Big)^3\Big\{\Big(\int_0^1 \Big|f''(t\frac{(n-1)a+b}{n} + (1-t)a)\Big|^q dt\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\int_0^1 \Big|f''(tb + (1-t)\frac{a+(n-1)b}{n})\Big|^q dt\Big)^{\frac{1}{q}}\Big\} \\
&\quad + \Big(\frac{n-2}{n}\Big)^3\Big\{\Big(\int_0^1 \Big|f''(t\frac{a+b}{2} + (1-t)\frac{(n-1)a+b}{n})\Big|^q dt\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\int_0^1 \Big|f''(t\frac{a+(n-1)b}{n} + (1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}\Big].
\end{aligned} \tag{28}$$

By the definition of s-concave functions in the second sense for $\big|f''\big|^q$ and using Theorem 1.5, we have

$$(i) \int_0^1 \Big|f''(t\frac{(n-1)a+b}{n} + (1-t)a)\Big|^q dt \leq 2^{s-1}\Big|f''(\frac{(2n-1)a+b}{2n})\Big|^q, \tag{29}$$

$$(ii) \int_0^1 \Big|f''(tb + (1-t)\frac{a+(n-1)b}{n})\Big|^q dt \leq 2^{s-1}\Big|f''(\frac{a+(2n-1)b}{2n})\Big|^q, \tag{30}$$

$$(iii) \int_0^1 \left|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\right|^q dt \leq 2^{s-1}\left|f''(\frac{(3n-2)a+(n+2)b}{4n})\right|^q, \tag{31}$$

$$(iv) \int_0^1 \left|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\right|^q dt \leq 2^{s-1}\left|f''(\frac{(n+2)a+(3n-2)b}{4n})\right|^q. \tag{32}$$

Substituting the inequalities (29)-(32) in (28), we get the desired result (27).

Theorem 3.3. *Let $f : I \subseteq R^+ \to R$ be a twice differentiable function on I^0, the interior of I, where $a, b \in I$ with $a < b$ and $f'' \in L[a, b]$. For some fixed $s \in (0, 1]$ and $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, if $|f''|^q$ is s-convex in the second sense on $[a, b]$, then, for $n \geq 3$ the following inequality holds:*

$$\begin{aligned}
&\left|I(f,n,[a,b])\right| \\
&\leq \Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{3}\Big)^{\frac{1}{p}}\Big[\Big(\frac{2}{n}\Big)^3 \\
&\times\Big\{\Big(\frac{1}{s+3}\left|f''(\frac{(n-1)a+b}{n})\right|^q+\frac{2}{s^3+6s^2+11s+6}\left|f''(a)\right|^q\Big)^{\frac{1}{q}} \\
&+\Big(\frac{2}{s^3+6s^2+11s+6}\left|f''(b)\right|^q+\frac{1}{s+3}\left|f''(\frac{a+(n-1)b}{n})\right|^q\Big)^{\frac{1}{q}}\Big\} \\
&+(\frac{n-2}{n})^3 \\
&\times\Big\{\Big(\frac{2}{s^3+6s^2+11s+6}\left|f''(\frac{a+b}{2})\right|^q+\frac{1}{s+3}\left|f''(\frac{(n-1)a+b}{n})\right|^q\Big)^{\frac{1}{q}} \\
&+\Big(\frac{1}{s+3}\left|f''(\frac{a+(n-1)b}{n})\right|^q+\frac{2}{s^3+6s^2+11s+6}\left|f''(\frac{a+b}{2})\right|^q\Big)^{\frac{1}{q}}\Big\}\Big].
\end{aligned} \tag{33}$$

Proof. From Lemma 2.1 and by using the property of modulus, we have

$$\begin{aligned}
&\left|I(f,n,[a,b])\right| \\
&=\Big|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\Big\{f(\frac{(n-1)a+b}{n})+f(\frac{a+(n-1)b}{n})\Big\} \\
&-(\frac{n-4}{8n})\Big\{f'(\frac{(n-1)a+b}{n})-f'(\frac{a+(n-1)b}{n})\Big\}(b-a)\Big| \\
&\leq\Big(\frac{b-a}{4}\Big)^2\Big(\frac{1}{3}\Big)^{\frac{1}{p}}\Big[\Big(\frac{2}{n}\Big)^3\Big\{\Big(\int_0^1 t^2\left|f''(t\frac{(n-1)a+b}{n}+(1-t)a)\right|^q dt\Big)^{\frac{1}{q}}
\end{aligned}$$

$$+\Big(\int_0^1(1-t)^2\Big|f''(tb+(1-t)\frac{a+(n-1)b}{n})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}$$
$$+\Big(\frac{n-2}{n}\Big)^3\Big\{\Big(\int_0^1(1-t)^2\Big|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 t^2\Big|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}\Big]. \tag{34}$$

By the definition of s-convex functions for $\big|f''\big|^q$, we have

$$(i)\int_0^1 t^2\Big|f''(t\frac{(n-1)a+b}{n}+(1-t)a)\Big|^q dt$$
$$\le\int_0^1 t^2\Big\{t^s\Big|f''(\frac{(n-1)a+b}{n})\Big|^q+(1-t)^s\Big|f''(a)\Big|^q\Big\}dt$$
$$=\frac{1}{s+3}\Big|f''(\frac{(n-1)a+b}{n})\Big|^q+\frac{2}{s^3+6s^2+11s+6}\Big|f''(a)\Big|^q, \tag{35}$$
$$(ii)\int_0^1(1-t)^2\Big|f''(tb+(1-t)\frac{a+(n-1)b}{n})\Big|^q dt$$
$$\le\frac{2}{s^3+6s^2+11s+6}\Big|f''(b)\Big|^q+\frac{1}{s+3}\Big|f''(\frac{a+(n-1)b}{n})\Big|^q, \tag{36}$$

$$(iii)\int_0^1(1-t)^2\Big|f''(t\frac{a+b}{2}+(1-t)\frac{(n-1)a+b}{n})\Big|^q dt$$
$$\le\frac{2}{s^3+6s^2+11s+6}\Big|f''(\frac{a+b}{2})\Big|^q+\frac{1}{s+3}\Big|f''(\frac{(n-1)a+b}{n})\Big|^q, \tag{37}$$
$$(iv)\int_0^1 t^2\Big|f''(t\frac{a+(n-1)b}{n}+(1-t)\frac{a+b}{2})\Big|^q dt$$
$$\le\frac{1}{s+3}\Big|f''(\frac{a+(n-1)b}{n})\Big|^q+\frac{2}{s^3+6s^2+11s+6}\Big|f''(\frac{a+b}{2})\Big|^q. \tag{38}$$

Substituting the inequalities (35)-(38) in (34), we get the desired result (33).

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