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PERTURBATION OF A TEMPERATURE FIELD BY AN INHOMOGENEITY

Nkem Ogbonna
Department of Mathematics
Michael Okpara University of Agriculture
Umudike, P.M.B. 7267, Umuahia 440001, Abia State, NIGERIA

Abstract: The perturbation of a temperature field by an inhomogeneity is investigated using the method of potentials. The remarkable features of this method include its independence of the coordinate system in which a problem is posed and a reduction of mathematical labour. It is shown that the method of potentials reduces the problem of steady heat flow past an arbitrarily- shaped body to the solution of an integral equation. In the special case of an ellipsoid in a linear temperature field, the problem is reduced to the solution of a mere algebraic equation.

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1. Introduction

Various physical phenomena involve the presence of an inhomogeneity in an external field. For example, a flying aircraft and a swimmer constitute inhomogeneities in the flow fields of the surrounding air and water, respectively. The introduction of an inhomogeneity naturally perturbs the existing field and it is often desired to determine the nature of the perturbation. For situations that can be modeled by linear partial differential equations, a method of obtaining

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simple solutions is the method of separation of variables. The basic assumption is that the multi-dimensional solution of a linear partial differential equation can be expressed as a product of one-dimensional functions, each of which is a function of a distinct independent variable of the problem. In effect, a linear partial differential equation involving n independent variables is reduced to a set of n ordinary differential equations which are presumably easier to solve than the original partial differential equation.

A major drawback of the separation of variable method is its dependence on the coordinate system in which the problem is posed. This creates considerable difficulty when the geometry of the problem demands that one works with general curvilinear coordinates. Except in simple cases, the orthogonal curvilinear coordinate system used in the implementation of the separation of variable method gives rise to unwieldy expressions which increase the mathematical labour and obscure direct insight into the overall nature of the solution.

The purpose of this paper is to explore the effectiveness of the method of potentials for the solution of a boundary value problem that models heat flow past an inhomogeneity. This approach is inspired by the fact that the method of potentials is independent of the coordinate system used and may, therefore, reduce some of the complexity encountered in using the separation of variable method. Of particular interest is the case of steady heat flow which falls into the class of physical problems that are governed by Laplace's equation. Such problems arise in many areas of continuum mechanics, such as electrostatics, electromagnetics and hydrodynamics. We shall develop the method of potentials from Newton's law of universal gravitation and apply it to the problem of steady heat flow past a body of arbitrary shape.

2. The Newtonian Potential

Newton's law of universal gravitation asserts that, between any two particles of matter located at two distinct points in space, there exists a force of attraction which acts along the line joining the masses with a magnitude that is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. If the attracting mass m is located at a fixed point $(\zeta_1, \zeta_2, \zeta_3)$ and the attracted mass M occupies the point (x_1, x_2, x_3) in a cartesian coordinate system, Newton's law gives

$$F_i = GmM\frac{(x_i - \zeta_i)}{R^3}, \qquad (i = 1, 2, 3), \tag{1}$$

where F_i is the force between the masses, G is the gravitational constant and the distance R between the masses is given by

$$R^2 = (x_i - \zeta_i)(x_i - \zeta_i). \tag{2}$$

For brevity, the indicial notation will be adopted in this section and the usual convention of summation over repeated indices is assumed. Also, a comma followed by a subscript (or subscripts) indicates differentiation with respect to the corresponding cartesian coordinate (or coordinates).

With an appropriate choice of units, the force of attraction which a particle of mass m located at a fixed point (ζ_i) exerts on a particle of unit mass located at a point (x_i) may be expressed as

$$F_i = -\left(\frac{m}{R}\right)_{,i}. \tag{3}$$

The quantity, $\frac{m}{R}$, is the gravitational potential which an attracting mass m induces at a point (x_i) in space, and it satisfies Laplace's equation everywhere except at the point (ζ_i) where the mass m is located. Thus,

$$\left(\frac{m}{R}\right)_{,ii} = 0.$$

For a continuous distribution of matter of density ρ occupying a region V, the gravitational potential Φ at a point (x_i) is given by

$$\Phi(x_i) = \int\int\int \frac{\rho(\zeta_i)}{R}\, d\tau, \tag{4}$$

where (ζ_i) is a point of distribution of mass, $d\tau = d\zeta_1 d\zeta_2 d\zeta_3$ is an element of volume, and R is given by equation (2). The function Φ possesses the following properties:

(i) Φ and $\Phi_{,i}$ are continuous across the boundary surface S of the region V.

(ii)

$$\Phi_{,ii} = \begin{cases} 0 & \text{(outside V)} \\ -4\pi\rho & \text{(inside V)} \end{cases}$$

(iii) $\Phi_{,ij}$ has a discontinuity across the boundary surface S given by

$$\Big[\Phi_{,ij}\Big]_{in}^{out} = -4\pi n_i n_j \Big[\rho\Big]_{in}^{out}, \tag{5}$$

where n_i is the outward drawn normal to the surface S and the superscripts $^{(in)}$ and $^{(out)}$ refer to the value of the associated quantity inside and outside the region V, respectively. Any function which admits a representation of the form (4) and has the associated properties (i) - (iii) shall be regarded as a harmonic potential.

By equation (2),

$$R_{,ij} = \frac{1}{R}\delta_{ij} - \frac{(x_i - \zeta_i)(x_i - \zeta_i)}{R^3},$$

where

$$\delta_{ij} = \begin{cases} 1 & (\text{if } i = j) \\ 0 & (\text{if } i \neq j) \end{cases}$$

In particular,

$$R_{,ii} = \frac{2}{R},$$

Consequently,

$$R_{,iijj} = \left(\frac{2}{R}\right)_{,jj} = 0.$$

It follows that

$$\Phi_2(x_i) = \int\int\int \rho(\zeta_i) R \, d\tau, \tag{6}$$

is a biharmonic potential and has the following properties:

(i) $\Phi_{2,ii} = 2\Phi$

(ii)

$$\Phi_{2,iijj} = \begin{cases} 0 & (\text{outside V}) \\ -8\pi\rho & (\text{inside V}) \end{cases}$$

(iii) Φ_2 and its derivatives up to the third order are continuous across the boundary surface S of the region V while its fourth order derivative has a discontinuity given by

$$\left[\Phi_{2,ijkl}\right]_{in}^{out} = -8\pi n_i n_j n_k n_l \left[\rho\right]_{in}^{out} \tag{7}$$

By examining the behaviour of $R^{2m-1}, (m = 1, 2, ...)$, the following generalization is obtained:

A polyharmonic potential of order m for matter of density ρ filling a volume V is given by

$$\Phi_m(x_i) = \int\int\int \rho(\zeta_i) R^{2m-3} \, d\tau, \qquad (m = 1, 2, ...) \tag{8}$$

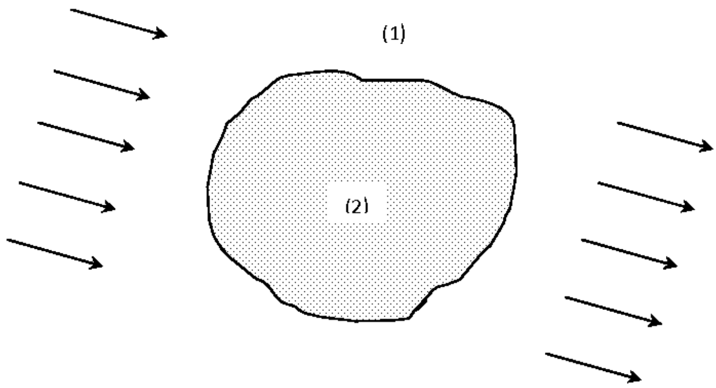

Figure 1: An arbitrary body in a temperature field.

and has the following properties:

(i) $\Phi_{m,ii} = (2m-3)(2m-2)\Phi_{m-1}\ , \qquad (m>1)$

(ii)

$$\Phi_{m,iijj...(2mplaces)} = \begin{cases} 0 & (outside\ V) \\ -4\pi\rho(2m-2)! & (inside\ V) \end{cases}$$

(iii) Φ_m *and its derivatives up to order* $2m-1$ *are continuous across the boundary surface* S *of the region* V *while the derivative of order* $2m$ *has a discontinuity given by*

$$\Big[\Phi_{m,ijkl...(2mplaces)}\Big]_{in}^{out} = -4\pi(2m-2)!n_i n_j n_k ... \Big[\rho\Big]_{in}^{out} \tag{9}$$

3. Inhomogeneity in a Temperature Field

Consider the problem of heat flow past a body of arbitrary shape which is introduced into an otherwise undisturbed temperature field. Let the space external to the body be referred to as region 1 while the space occupied by the body is region 2 (see Figure 1). In what follows, the corresponding subscripts $_1$ and $_2$ on material constants as well as superscripts $^{(1)}$ and $^{(2)}$ on field quantities shall refer to these two regions.

In the absence of heat sources in the field of flow, the steady-state temperature field, T, is governed by Laplace's equation,

$$\nabla^2 T = 0. \tag{10}$$

For any orthogonal curvilinear system of coordinates (u_1, u_2, u_3) ,

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right], \tag{11}$$

where

$$h_1 = \left|\frac{\partial \bar{r}}{\partial u_1}\right|, \qquad h_2 = \left|\frac{\partial \bar{r}}{\partial u_2}\right|, \qquad h_3 = \left|\frac{\partial \bar{r}}{\partial u_3}\right|, \tag{12}$$

are the scale factors while $\bar{r}$ is the position vector. The physical requirement for the maintenance of thermal stability is the continuity of both the temperature field and the rate of heat flow across the boundary surface S of the body. Thus, on the surface S of the inhomogeneity, the continuity conditions are

$$T^{(1)} = T^{(2)}, \qquad k_1 \frac{\partial T^{(1)}}{\partial n} = k_2 \frac{\partial T^{(2)}}{\partial n}, \quad \text{on} \quad S. \tag{13}$$

In addition, the exterior temperature field, $T^{(1)}$, must satisfy the far field condition,

$$T^{(1)} \longrightarrow T^{\infty} \qquad \text{as} \quad R \longrightarrow \infty, \tag{14}$$

where T^{∞} is the undisturbed temperature field. Equations (10) - (14) consitutute the governing equations for steady state heat flow past an inhomogeneity, provided there are no heat singularities in the field of flow.

3.1. The Case of an Ellipsoidal Inhomogeneity

The case of an ellipsoidal inhomogeneity is of special interest since the ellipsoid is a versatile shape that may be specialized to a sphere, a cylinder, an ellipse or a circle [2]. The ellipsoidal coordinates (λ, μ, ν) are related to the cartesian coordinates (x, y, z) through the equations [3],

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2 = \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)}, \tag{15}$$

$$z^2 = \frac{\left(c^2+\lambda\right)\left(c^2+\mu\right)\left(c^2+\nu\right)}{\left(c^2-a^2\right)\left(c^2-b^2\right)},$$

where a, b, and c are constants. Equation (15) holds, provided that

$$\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s} = 1, \tag{16}$$

where s is a parameter and λ, μ, ν are the roots of the cubic equation in s that results when the terms in equation (16) are cleared of all denominators. If $s = 0$, equation (16) defines an ellipsoid with principal axes a, b, c. In general, for any fixed s, it represents a central quadric of a confocal system. Conventionally, the quadric is an ellipsoid when the root, λ, is a constant. On setting $(u_1, u_2, u_3) \equiv (\lambda, \mu, \nu)$ and using equation (15) in (12), we obtain

$$h_1 = \frac{1}{2}\left[\frac{(\lambda-\mu)(\lambda-\nu)}{\mathcal{K}(\lambda)}\right]^{\frac{1}{2}}, \quad h_2 = \frac{1}{2}\left[\frac{(\mu-\lambda)(\mu-\nu)}{\mathcal{K}(\mu)}\right]^{\frac{1}{2}},$$

$$h_3 = \frac{1}{2}\left[\frac{(\nu-\lambda)(\nu-\mu)}{\mathcal{K}(\nu)}\right]^{\frac{1}{2}}, \tag{17}$$

where

$$\mathcal{K}(q) = \left(a^2+q\right)\left(b^2+q\right)\left(c^2+q\right). \tag{18}$$

By inserting the relations (17) into equation (11) and using the result in (10), we obtain the steady state heat equation in ellipsoidal coordinates as

$$(\mu-\nu)\left(\mathcal{K}(\lambda)\frac{\partial}{\partial\lambda}\right)^2 T + (\nu-\lambda)\left(\mathcal{K}(\mu)\frac{\partial}{\partial\mu}\right)^2 T$$

$$+ (\lambda-\mu)\left(\mathcal{K}(\nu)\frac{\partial}{\partial\nu}\right)^2 T = 0. \tag{19}$$

In principle, equation(19) may be solved by the separation of variable method but, in practice, it is too laborious a task for meaningful analysis. Here, we consider the method of the potential as an alternative way of finding a solution to the problem posed.

3.2. Application of the Potential Method

Let the undisturbed temperature field before the introduction of the inhomogeneity be T^{∞}. Suppose that, after the inhomogeneity is introduced, the temperatures in regions 1 and 2 are $T^{(1)}$ and $T^{(2)}$, respectively. Then, $T^{(1)} =$

$T^{\infty} + T^{\square}$, where $T^{\square}$ is the perturbation of the external field due to the introduction of the inhomogeneity. Since T^{∞} is known, the problem reduces to finding $T^{\square}$ and $T^{(2)}$ such that the conditions (13) and (14) are satisfied.

The method of the potential is based on the fact that the temperature fields in regions 1 and 2 must be harmonic functions and are, therefore, derivable from a harmonic potential of the form (4) to which the properties (5) are applicable. Guided by the continuity conditions stated in equation (13), and noting that the gradient of a harmonic function is harmonic, we choose $T^{\square}$ in the form:

$$T^{\square} = \frac{\partial \Phi}{\partial x_i},$$

where Φ is the harmonic potential given by equation (4). This choice ensures the automatic satisfaction of the first of the continuity conditions (13) while the second continuity condition takes the form,

$$k_1 n_j \left(\frac{\partial T^{\infty}}{\partial x_j} + \frac{\partial^2 \Phi^{(1)}}{\partial x_i \partial x_j} \right) = k_2 n_j \frac{\partial^2 \Phi^{(2)}}{\partial x_i \partial x_j}. \tag{20}$$

Substituting equations (4) and (5) into equation (20), we obtain

$$4\pi k_1 \rho(\zeta_i) n_i = -k_1 n_j \frac{\partial T^{\infty}}{\partial x_j} + (k_2 - k_1) n_j \frac{\partial^2}{\partial x_i \partial x_j} \int\int\int \frac{\rho(\zeta_i)}{R}\, d\tau. \tag{21}$$

The solution to the present problem is complete once the density, ρ, is determined. Thus, the original problem is reduced to finding the solution to an integral equation (21).

For an ellipsoidal inhomogeneity defined by,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

its interior harmonic potential is [4]

$$\Phi = \pi abc\rho \int_0^{\infty} \frac{U(s)ds}{\Delta}, \tag{22}$$

where

$$U(s) = 1 - \frac{x_1^2}{a^2+s} - \frac{x_2^2}{b^2+s} - \frac{x_3^2}{c^2+s}, \tag{23}$$

$$\Delta = [(a^2+s)(a^2+s)(a^2+s)]^{\frac{1}{2}}, \tag{24}$$

and ρ is uniform density. Equations (22) - (24) show that Φ is quadratic in x_i, and we may write

$$\frac{\partial^2\Phi}{\partial x_i \partial x_j} = \beta\rho_i\delta_{ij},$$

where β is a known geometric constant and $\rho_i = \rho(\zeta_i)$ is to be determined.

If the undisturbed temperature field has a linear character, then

$$T^{\infty} = \alpha_i x_i, \tag{25}$$

where α_i are constants. On substituting equations (22) and (25) into equation (21), we obtain the algebraic equation

$$k_1\alpha_j n_j + (k_1 - k_2)\beta\rho_i\delta_{ij}n_j + 4\pi k_1\rho_i n_i = 0,$$

which yields

$$\rho_i = \frac{-k_1\alpha_i}{4\pi k_1 + (k_1 - k_2)\beta}. \tag{26}$$

We deduce that the perturbation is also linear, and the temperature field retains a linear character throughout the domain of consideration.

4. Concluding Remarks

The method of the potential is presented as an alternative way of analyzing boundary value problems of partial differential equations, especially when the traditional separation of variable method is cumbersome to apply. The effectiveness of the method is illustrated using the problem of steady heat flow past an ellipsoidal inhomogeneity. Whereas the method of separation of variables would require the laborious process of solving a second order partial differential equation in ellipsoidal coordinates and could produce unwieldy results, the method of the potential cuts out such labour by using the properties of known potential functions to construct appropriate solutions to the boundary value problem posed. The method of the Newtonian potential has been used to calculate the scattering of a plane wave from an anisotropic dielectric ellipsoid in anistropic medium [5].

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