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# SOME IDENTITIES ON THE BERNSTEIN POLYNOMIALS AND TWISTED $q$-EULER POLYNOMIALS WITH WEAK WEIGHT $\alpha$

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**Abstract:** In this paper, by using fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, we give some interesting relationship between twisted $q$-Euler polynomials with weak weight $\alpha$ and Bernstein polynomials.

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**Key Words:** Euler numbers and polynomials, the twisted $q$-Euler numbers and polynomials with weak weight $\alpha$, Bernstein polynomials

## 1. Introduction

In this paper we investigate some relations between Bernstein polynomials and the twisted $q$-Euler numbers with weak weight $\alpha$. From these relations, we derive some interesting identities on the twisted $q$-Euler numbers with weak weight $\alpha$.

Let $p$ be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic

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rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The $p$-adic absolute value is defined by $|x|_p = \frac{1}{p^r}$, where $x = p^r \frac{s}{t}$( $r \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ with $(s,t) = (p,s) = (p,t) = 1$). In this paper we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ as an indeterminate. The $q$-number is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \text{ see [1, 2, 3, 4, 7].}$$

Note that $\lim_{q \to 1}[x]_q = x$. For

$$f \in UD(\mathbb{Z}_p) = \{f | f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [1, 2] .} \quad (1.1)$$

Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1\}$ is the cyclic group of order $p^N$. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$(see [5, 6]).

For $\alpha \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, and $w \in T_p$, the twisted $q$-Euler polynomials $\widetilde{E}_{n,q,w}^{(\alpha)}$ with weak weight $\alpha$ are defined by

$$\widetilde{E}_{n,q,w}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} w^y (x+y)^n d\mu_{-q^\alpha}(y). \quad (1.2)$$

In the special case, $x = 0$, $\widetilde{E}_{n,q,w}^{(\alpha)}(0) = \widetilde{E}_{n,q,w}^{(\alpha)}$ are called the $n$-th the twisted $q$-Euler numbers with weak weight $\alpha$(see [7]).

## 2. The Twisted $q$-Euler Numbers with Weak Weight $\alpha$ and Bernstein Polynomials

By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we obtain,

$$\begin{aligned}
\sum_{n=0}^{\infty} \widetilde{E}_{n,q,w}^{(\alpha)} \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} w^x e^{xt} d\mu_{-q^\alpha}(x) \\
&= \lim_{N \to \infty} \frac{1}{[p^N]_{-q^\alpha}} \sum_{x=0}^{p^N-1} w^x e^{xt} (-q^\alpha)^x \\
&= \frac{[2]_{q^\alpha}}{wq^\alpha e^t + 1}.
\end{aligned} \quad (2.1)$$

By (2.1), we have

$$\widetilde{F}^{(\alpha)}_{q,w}(t) = \sum_{n=0}^{\infty} \widetilde{E}^{(\alpha)}_{n,q,w} \frac{t^n}{n!} = [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m w^m q^{\alpha m} e^{mt}. \tag{2.2}$$

Thus the twisted $q$-Euler numbers $\widetilde{E}^{(\alpha)}_{n,q,w}$ with weak weight $\alpha$ are defined by means of the generating function

$$\widetilde{F}^{(\alpha)}_{q,w}(t) = [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m w^m q^{\alpha m} e^{mt}. \tag{2.3}$$

Using similar method as above, we have

$$\sum_{n=0}^{\infty} \widetilde{E}^{(\alpha)}_{n,q,w}(x) \frac{t^n}{n!} = \frac{[2]_{q^\alpha}}{wq^\alpha e^t + 1} e^{xt}. \tag{2.4}$$

By using (2.3) and (2.4), we obtain

$$\widetilde{F}^{(\alpha)}_{q,w}(t,x) = \sum_{n=0}^{\infty} \widetilde{E}^{(\alpha)}_{n,q,w}(x) \frac{t^n}{n!} = [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m w^m q^{\alpha m} e^{(m+x)t}. \tag{2.5}$$

By the above definition, we obtain

$$\begin{aligned} \sum_{l=0}^{\infty} \widetilde{E}^{(\alpha)}_{l,q,w}(x) \frac{t^l}{l!} &= \frac{[2]_{q^\alpha}}{wq^\alpha e^t + 1} e^{xt} = \sum_{l=0}^{\infty} \left( \sum_{n=0}^{l} \widetilde{E}^{(\alpha)}_{n,q,w} \frac{t^n}{n!} x^{l-n} \frac{t^{l-n}}{(l-n)!} \right) \\ &= \sum_{l=0}^{\infty} \left( \sum_{n=0}^{l} \binom{l}{n} \widetilde{E}^{(\alpha)}_{n,q,w} x^{l-n} \right) \frac{t^l}{l!}. \end{aligned}$$

By using comparing coefficients $\frac{t^l}{l!}$, we have the following theorem.

**Theorem 1.** *For any positive integer $n$ and $\alpha \in \mathbb{Z}$, we have*

$$\begin{aligned} \widetilde{E}^{(\alpha)}_{n,q,w}(x) &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \widetilde{E}^{(\alpha)}_{k,q,w} \\ &= \left( x + \widetilde{E}^{(\alpha)}_{q,w} \right)^n \\ &= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m w^m q^{\alpha m} (x+m)^n. \end{aligned}$$

By (2.5), we have

$$\widetilde{F}^{(\alpha)}_{q^{-1},w^{-1}}(1-t,-x)=\sum_{n=0}^{\infty}(-1)^n w^{-1}\widetilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}(1-x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\widetilde{E}^{(\alpha)}_{n,q,w}(x)\frac{t^n}{n!}.$$

Thus we have the following theorem.

**Theorem 2.** *For any positive integer $n$, we have*

$$\widetilde{E}^{(\alpha)}_{n,q,w}(x)=(-1)^n w^{-1}\widetilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}(1-x) \tag{2.6}$$

For the twisted $q$-Euler numbers with weak weight $\alpha$, we obtain the following theorem.

**Theorem 3.** *For $n\in\mathbb{Z}_+$, we have*

$$wq^{\alpha}\widetilde{E}^{(\alpha)}_{n,q,w}(1)+\widetilde{E}^{(\alpha)}_{n,q,w}=\begin{cases}[2]_{q^{\alpha}}, & \text{if } n=0,\\ 0, & \text{if } n\neq 0.\end{cases}$$

*with the usual convention about replacing $\left(\widetilde{E}^{(\alpha)}_{q,w}\right)^n$ by $\widetilde{E}^{(\alpha)}_{n,q,w}$ in the binomial expansion.*

By Theorem 1 and Theorem 3, we have the following corollary.

**Corollary 4.** *For $n\in\mathbb{Z}_+$, we have*

$$wq^{\alpha}(\widetilde{E}^{(\alpha)}_{q,w}+1)^n+\widetilde{E}^{(\alpha)}_{n,q,w}=\begin{cases}[2]_{q^{\alpha}}, & \text{if } n=0,\\ 0, & \text{if } n\neq 0,\end{cases}$$

*with the usual convention of replacing $(\widetilde{E}^{(\alpha)}_{q,w})^n$ by $\widetilde{E}^{(\alpha)}_{n,q,w}$.*

By Theorem 1 and Theorem 3, we obtain

$$\begin{aligned}\widetilde{E}^{(\alpha)}_{n,q,w}(2)&=\sum_{l=0}^{n}\binom{n}{l}2^{n-l}\widetilde{E}^{(\alpha)}_{l,q,w}\\&=\left(\widetilde{E}^{(\alpha)}_{q,w}+1+1\right)^n\\&=\widetilde{E}^{(\alpha)}_{0,q,w}(1)+\frac{1}{wq^{\alpha}}\sum_{l=1}^{n}\binom{n}{l}wq^{\alpha}\widetilde{E}^{(\alpha)}_{l,q,w}(1)\\&=\frac{[2]_{q^{\alpha}}}{wq^{\alpha}}+\frac{1}{w^2q^{2\alpha}}\widetilde{E}^{(\alpha)}_{n,q,w}\text{ if } n>0.\end{aligned}$$

Therefore, we obtain the following theorem.

**Theorem 5.** *For $n \in \mathbb{N}$, we have*

$$wq^{\alpha}\widetilde{E}^{(\alpha)}_{n,q,w}(2) = [2]_{q^{\alpha}} + \left(\frac{1}{wq^{\alpha}}\right)\widetilde{E}^{(\alpha)}_{n,q,w}.$$

By Theorem 5, we have the following corollary.

**Corollary 6.** *For $n \in \mathbb{N}$, we have*

$$\widetilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}(2) = w[2]_{q^{\alpha}} + w^2q^{2\alpha}\widetilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}.$$

As well known definition, Bernstein polynomials of degree $n$ are given by

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \text{ where } x \in [0,1], n,k \in \mathbb{Z}_+, \text{ see [4]}. \tag{2.7}$$

By (2.7), we get the symmetry of Bernstein polynomials as follows:

$$B_{k,n}(x) = B_{n-k,n}(1-x). \tag{2.8}$$

By Theorem 2 and Corollary 6, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} w^x(1-x)^n d\mu_{-q^{\alpha}}(x) &= (-1)^n \int_{\mathbb{Z}_p} w^x(x-1)^n d\mu_{-q^{\alpha}}(x) \\ &= \widetilde{E}^{(\alpha)}_{n,q,w^{-1}}(2) \\ &= [2]_{q^{\alpha}} + wq^{2\alpha}\widetilde{E}^{(\alpha)}_{n,q^{-1},w^{-1}}. \end{aligned} \tag{2.9}$$

Let us take the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ for the Bernstein polynomials of degree $n$ as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} w^x B_{k,n}(x) d\mu_{-q^{\alpha}}(x) &= \binom{n}{k}\int_{\mathbb{Z}_p} w^x x^k(1-x)^{n-k} d\mu_{-q^{\alpha}}(x) \\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\widetilde{E}^{(\alpha)}_{l+k,q,w}. \end{aligned} \tag{2.10}$$

Let $n,k \in \mathbb{Z}_+$ with $n > k$. Then we get

$$\begin{aligned} \int_{\mathbb{Z}_p} w^x B_{k,n}(x) d\mu_{-q^{\alpha}}(x) &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_{\mathbb{Z}_p} w^x(1-x)^{n-l} d\mu_{-q^{\alpha}}(x) \\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left([2]_{q^{\alpha}} + wq^{2\alpha}\widetilde{E}^{(\alpha)}_{n-l,q^{-1},w^{-1}}\right). \end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

**Theorem 7.** *Let $n, k \in \mathbb{Z}_+$ with $n > k$. Then we have*

$$\int_{\mathbb{Z}_p} w^x B_{k,n}(x) d\mu_{-q^\alpha}(x) = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( [2]_{q^\alpha} + wq^{2\alpha} \widetilde{E}^{(\alpha)}_{n-l,q^{-1},w^{-1}} \right).$$

*Moreover,*

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \widetilde{E}^{(\alpha)}_{l+k,q,w} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( [2]_{q^\alpha} + wq^{2\alpha} \widetilde{E}^{(\alpha)}_{n-l,q^{-1},w^{-1}} \right).$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} w^x B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-q^\alpha}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left( [2]_{q^\alpha} + wq^{2\alpha} \widetilde{E}^{(\alpha)}_{n_1+n_2-l,q^{-1},w^{-1}} \right). \end{aligned} \tag{2.12}$$

Therefore, by (2.12), we have the following theorem.

**Theorem 8.** *For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have*

$$\begin{aligned} &\int_{\mathbb{Z}_p} w^x B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-q^\alpha}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left( [2]_{q^\alpha} + wq^{2\alpha} \widetilde{E}^{(\alpha)}_{n_1+n_2-l,q^{-1},w^{-1}} \right). \end{aligned}$$

From the binomial theorem, we can derive the following equation.

$$\begin{aligned} &\int_{\mathbb{Z}_p} w^x B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-q^\alpha}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \widetilde{E}^{(\alpha)}_{2k+l,q,w}. \end{aligned} \tag{2.13}$$

Thus, by (2.13) and Theorem 8, we obtain the following corollary.

**Corollary 9.** *Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have*

$$\begin{aligned} &\sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \widetilde{E}^{(\alpha)}_{2k+l,q,w} \\ &= \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left( [2]_{q^\alpha} + wq^{2\alpha} \widetilde{E}^{(\alpha)}_{n_1+n_2-l,q^{-1},w^{-1}} \right). \end{aligned}$$

For $x \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we take the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ for the Bernstein polynomials of degree $n$ as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} w^x \underbrace{B_{k,n_1}(x) \cdots B_{k,n_s}(x)}_{s-\text{times}} d\mu_{-q^\alpha}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \int_{\mathbb{Z}_p} w^x x^{sk} (1-x)^{n_1+\cdots+n_s-sk} d\mu_{-q^\alpha}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left( [2]_{q^\alpha} + wq^{2\alpha} \widetilde{E}^{(\alpha)}_{n_1+\cdots+n_s-l,q^{-1},w^{-1}} \right).
\end{aligned} \tag{2.14}$$

Therefore, by (2.14), we have the following theorem.

**Theorem 10.** *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we get*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} w^x \underbrace{B_{k,n_1}(x) \cdots B_{k,n_s}(x)}_{s-times} d\mu_{-q^\alpha}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left( [2]_{q^\alpha} + wq^{2\alpha} \widetilde{E}^{(\alpha)}_{n_1+\cdots+n_s-l,q^{-1},w^{-1}} \right).
\end{aligned} \tag{2.15}$$

By the definition of Bernstein polynomials and the binomial theorem, we easily get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} w^x \underbrace{B_{k,n_1}(x) \cdots B_{k,n_s}(x)}_{s-\text{times}} d\mu_{-q^\alpha}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} \widetilde{E}^{(\alpha)}_{sk+l,q,w}.
\end{aligned} \tag{2.16}$$

Therefore, by (2.15) and (2.16), we have the following corollary.

**Corollary 11.** *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with*

$n_1+\cdots+n_s > sk$. *Then we have*

$$\sum_{l=0}^{n_1+\cdots+n_s-sk}(-1)^l\binom{n_1+\cdots+n_s-sk}{l}\widetilde{E}^{(\alpha)}_{sk+l,q,w}$$
$$=\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\left([2]_{q^\alpha}+wq^{2\alpha}\widetilde{E}^{(\alpha)}_{n_1+\cdots+n_s-l,q^{-1},w^{-1}}\right).$$

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