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NEW UNDERESTIMATOR FOR MULTIVARIATE GLOBAL OPTIMIZATION WITH BOX CONSTRAINTS

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Abstract: The paper is concerned with the multivariate global optimization with box constraints. A new underestimator is investigated for twice continuously differentiable function on a box which is an extension of the approach developed in [5] for univariate global optimization.

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1. Introduction

We consider the following problem

$$(P)\begin{cases} \alpha := \min f(s) \\ s \in H \end{cases}$$

where H is a box in R^n and $f : O \subset R^n \to R$ is twice continuously differentiable on an open convex set O containing H. Even if constraints are simple, the problem remains very difficult to find the global optimum solution. In [1] the αBB method consists in a construction of convex lower bound function of the functions of class C^2 which combines the objective function and the quadratic term, and uses the branch and bound algorithm. The review of principal methods

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are presented in [2]. Efficient diagonal partitions are discussed in [6]. In [4] the interval method is used to compute global optimum. Basing on the idea developed in [5] for univariate global optimization we propose in this work a convex underestimating function for f and a branch and bound algorithm for solving (P) . In our method we combine a multi-linear term with quadratic term to find a convex lower bound function. In our branch and bound algorithm, we solve at each iteration a convex problem with objective function as a polynomial. For branching, we use the exhaustive $w - subdivision$ which has been shown to be efficient (see e.g. [5]). The paper is organized as follows: In Section 2 we present the main results. The algorithm and its convergence are presented in Section 3. Numerical examples found in the literature are treated in Section 4.

2. Main Results

Let $S := [s_0, s_1]$ is a bounded closed interval in R and f is a continuously twice differentiable function on S on which their second derivative is bounded, $h = s_1 - s_0$ and $w_j : \ R \to R \ (j = 0, 1)$ the functions defined by

$$w_0(s) = \begin{cases} \frac{s_1 - s}{s_1 - s_0} & \text{if } \ s_0 \leq s \leq s_1, \\ 0 & \text{otherwise,} \end{cases}$$
$$w_1(s) = \begin{cases} \frac{s - s_0}{s_1 - s_0} & \text{if } \ s_0 \leq s \leq s_1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Clearly,

$$\sum_{i=1}^{1} w_i\left(s\right) = 1, \forall s \in [s_0, s_1] \text{ and } w_i(s_j) = 0 \text{ if } i \neq j, \ 1, \text{ otherwise.}$$

Let $L_h f$ be the piecewise linear interpolant to f at points s_0, s_1.

$$L_h f(s) = \sum_{i=0}^{1} f\left(s_i\right) w_i\left(s\right). \tag{2.2}$$

In [5] we proposed a quadratic underestimator of f on the interval

$$LB(s) = L_h f(s) - \frac{1}{2} K\left(s - s_0\right)\left(s_1 - s\right)$$

where K is positive number such that $|f"(s)| \leq K$, $\forall s \in [s_0, s_1]$.

We can generalize this result to multivariate global optimization problem (P). Let H be the box $\Pi_{i=1}^{n}\left[s_i^0, s_i^1\right]$ whose vertex set is denoted $V(H)$. An element in $V(H)$ is denoted as $s := (s_1^{i_1}, ..., s_n^{i_n})$ with $i_k = 0$ or 1, for $k = 1, ..., n$. We express (P) in the form

$$(P)\begin{cases} \min f(s_1, ..., s_n) \\ (s_1, ..., s_n) \in H \end{cases}$$

and define the next function $\varphi : R^n \to R$ as

$$\varphi(s_1, ..., s_n) = L_{h_n}(...(L_{h_1} f(s_1, ..., s_n))...) - \frac{1}{2}K(\sum_{i=1}^{n}(s_i - s_i^0)(s_i^1 - s_i))$$

$$= \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n) - \frac{1}{2}K(\sum_{i=1}^{n}(s_i - s_i^0)(s_i^1 - s_i))$$

where K is a positive number such that

$$K \geq ||H_f(s_1, ..., s_n)|| \ , \ \forall (s_1, ..., s_n) \in H,$$

and $H_f(s_1, ..., s_n)$ is the Hessian matrix of the function f. Here the matrix norm of a $A = (a_{ij})$ is defined by

$$||A|| := \max_{i=1,...,n} \sum_{j=1}^{n} |a_{ij}| \, .$$

Theorem 2.1. *i) The functions φ and f agree on the vertex set $V(H)$ of H.*

ii) φ is a minorization of f on H, say $\varphi(s) \leq f(s)$, $\forall s \in H$ if $K \geq \max_{s\in H} |f''_{s_i s_i}(s)|$.

Proof. i) By the definition

$$\varphi(s) := \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n) - \frac{1}{2}K(\sum_{i=1}^{n}(s_i - s_i^0)(s_i^1 - s_i)).$$

Let $(s_1^{j_1}, ..., s_n^{j_n})$ be a vertex of H. We have

$$\varphi(s_1^{j_1}, ..., s_n^{j_n}) =$$

$$\sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1},...,s_n^{i_n})w_{i_1}(s_1^{j_1}))...)w_{i_n}(s_n^{j_n}) - \frac{1}{2}K(\sum_{i=1}^{n}(s_i^{j_i}-s_i^0)(s_i^1-s_i^{j_i})).$$

Clearly,

$$\frac{1}{2}K(\sum_{i=1}^{n}(s_i^{j_i}-s_i^0)(s_i^1-s_i^{j_i}))=0. \tag{2.3}$$

On the other hand, from the definition of w_{i_j} $(i_j=0,1)$ it follows that

$$w_0(s_1^0)=w_1(s_1^1)=1 \text{ and } w_0(s_1^1)=w_1(s_1^0)=0,$$

therefore

$$\sum_{i_1=0}^{1} f(s_1^{i_1},...,s_n^{i_n})w_{i_1}(s_1^{j_1}) = f(s_1^{j_1},s_2^{i_2},...,s_n^{i_n}).$$

Likewise

$$\sum_{i_2=0}^{1} f(s_1^{j_1},...,s_n^{i_n})w_{i_2}(s_2^{j_2}) = f(s_1^{j_1},s_2^{j_2},s_3^{i_3},...,s_n^{i_n}),$$

and so on

$$\sum_{i_n=0}^{1} f(s_1^{j_1},...,s_n^{i_n})w_{i_n}(s_n^{j_n}) = f(s_1^{j_1},...,s_n^{j_n}).$$

Hence

$$\varphi(s_1^{j_1},...,s_n^{j_n}) = f(s_1^{j_1},...,s_n^{j_n}).$$

ii) We have

$$\begin{aligned}
& |f(s_1,s_2,...,s_n) - \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1},s_2^{i_2},...,s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n)| \\
= \; & |(f(s_1,s_2,...,s_n) - \sum_{i_1=0}^{1} f(s_1^{i_1},s_2,...,s_n)w_{i_1}(s_1)) + \\
& (\sum_{i_1=0}^{1} f(s_1^{i_1},s_2,...,s_n)w_{i_1}(s_1) - \sum_{i_2=0}^{1}(\sum_{i_1=0}^{1} f(s_1^{i_1},s_2^{i_2},...,s_n)w_{i_1}(s_1))w_{i_2}(s_2)) + \\
& (\sum_{i_2=0}^{1}(\sum_{i_1=0}^{1} f(s_1^{i_1},s_2^{i_2},...,s_n)w_{i_1}(s_1))w_{i_2}(s_2) - ...) + ... + \\
& (\sum_{i_{n-1}=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1},s_2^{i_2},...,s_{n-1}^{i_{n-1}},s_n)w_{i_1}(s_1))...)w_{i_{n-1}}(s_{n-1}) -
\end{aligned}$$

$$
\begin{aligned}
&\sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, s_2^{i_2}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n))| \\
\leq\ & |(f(s_1, s_2, ..., s_n) - \sum_{i_1=0}^{1} f(s_1^{i_1}, s_2, ..., s_n)w_{i_1}(s_1))| + \\
& |\sum_{i_1=0}^{1} f(s_1^{i_1}, s_2, ..., s_n)w_{i_1}(s_1) - \sum_{i_2=0}^{1}(\sum_{i_1=0}^{1} f(s_1^{i_1}, s_2^{i_2}, ..., s_n)w_{i_1}(s_1))w_{i_2}(s_2)| + \\
& |\sum_{i_2=0}^{1}(\sum_{i_1=0}^{1} f(s_1^{i_1}, s_2^{i_2}, ..., s_n)w_{i_1}(s_1))w_{i_2}(s_2) - ...| + ... + \\
& |\sum_{i_{n-1}=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, s_2^{i_2}, ..., s_{n-1}^{i_{n-1}}, s_n)w_{i_1}(s_1))...)w_{i_{n-1}}(s_{n-1}) - \\
& \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, s_2^{i_2}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n))| \\
\leq\ & \frac{1}{2}K(s_1 - s_1^0)(s_1^1 - s_1) + \frac{1}{2}K(s_2 - s_2^0)(s_2^1 - s_2) + ... + \frac{1}{2}K(s_n - s_n^0)(s_n^1 - s_n) \\
=\ & \frac{1}{2}K(\sum_{i=1}^{n}(s_i - s_i^0)(s_i^1 - s_i))
\end{aligned}
$$

Thus

$$\varphi(s) \leq f(s)\ ,\ \forall s \in H.$$

As an immediate consequence of this theorem we have

$$
\begin{aligned}
&|\sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n) - f(s_1, ..., s_n)| \\
&\qquad \leq \frac{1}{2}K\left(h_1^2 + ... + h_n^2\right) \quad \textit{with} \quad h_i = s_i^1 - s_i^0. \quad (2.4)
\end{aligned}
$$

□

Theorem 2.2. *The function φ is convex on H if*

$$K \geq \max_{s\in H} \max_{i=1,...,n} \sum_{j=1, j\neq i}^{n} |f''_{s_i s_j}(s)|.$$

Proof. We express φ in the form

$$\varphi(s) = \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n) - \frac{1}{2}K(\sum_{i=1}^{n}(s_i - s_i^0)(s_i^1 - s_i))$$
$$= \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n)$$
$$+ \frac{1}{2}Ks_i^2 - \frac{1}{2}K\sum_{i=1}^{n}((s_i^1 + s_i^0)s_i - s_i^0 s_i^1).$$

Since the part

$$\frac{1}{2}K\sum_{i=1}^{n}((s_i^1 + s_i^0)s_i - s_i^0 s_i^1)$$

is linear, it suffices to prove that the function Φ defined by

$$\Phi(s_1, ..., s_n) := \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n) + \frac{1}{2}Ks_i^2$$

is convex. This amounts to show that the Hessian matrix of Φ, denoted H_Φ, is semi-definite positive. Let

$$L_h f(s_1, ..., s_n) := \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n).$$

From the definition of w_{i_j}, it is easy to see that all elements $(L_h f)''_{s_i s_i}$ are zero, for $i = 1, ..., n$. Hence H_Φ takes the form

$$H_\Phi = \begin{pmatrix} K & L_{12} & . & . & L_{1n} \\ L_{21} & K & L_{23} & . & L_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ L_{n1} & . & . & . & K \end{pmatrix}$$

with $L_{ij} := (L_h f)''_{s_i s_j}(s_1, ..., s_n)$ is the second mixed derivatives of $L_h f$ with respect to the variables s_i and s_j.

The second mixed derivatives of $L_h f$, for example the second mixed derivative with respect to variables s_1 and s_2, can be computed as follows. First, the derivative with respect to one variable, for example, s_1, can be expressed as

$$L_h f'_{s_1}(s_1, ..., s_n) = (\sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1}, ..., s_n^{i_n})w_{i_1}(s_1))...)w_{i_n}(s_n))'_{s_1}$$

$$= \sum_{i_n=0}^{1}(...(\sum_{i_2=0}^{1}(f(s_1^0, s_2^{i_2}, ..., s_n^{i_n})w_0'(s_1) + f(s_1^1, s_2^{i_2}, ..., s_n^{i_n})w_1'(s_1))w_{i_2}(s_2))...)$$
$$w_{i_n}(s_n)$$
$$= \sum_{i_n=0}^{1}(...(\sum_{i_2=0}^{1}(f(s_1^0, s_2^{i_2}, ..., s_n^{i_n})\frac{-1}{s_1^1 - s_1^0} + f(s_1^1, s_2^{i_2}, ..., s_n^{i_n})\frac{1}{s_1^1 - s_1^0})w_{i_2}(s_2))...)$$
$$w_{i_n}(s_n)$$
$$= \sum_{i_n=0}^{1}(...(\sum_{i_2=0}^{1}(\frac{f(s_1^1, s_2^{i_2}, ..., s_n^{i_n}) - (f(s_1^0, s_2^{i_2}, ..., s_n^{i_n})}{s_1^1 - s_1^0})w_{i_2}(s_2))...)w_{i_n}(s_n)$$
$$= \sum_{i_n=0}^{1}(...(\sum_{i_2=0}^{1}(f_{s_1}'(\xi, s_2^{i_2}, ..., s_n^{i_n}))w_{i_2}(s_2))...)w_{i_n}(s_n)$$

with $s_1^0 < \xi < s_1^1$.

Afterwards the second derivative with respect to the first two variables is computed as:

$$L_h f_{s_1 s_2}''(s_1, ..., s_n) = \sum_{i_n=0}^{1}(...(\sum_{i_2=0}^{1}(f_{s_1}'(\xi, s_2^{i_2}, ..., s_n^{i_n}))w_{i_2}(s_2))...)w_{i_n}(s_n))_{s_2}'$$
$$= \sum_{i_n=0}^{1}(...(\sum_{i_3=0}^{1}(f_{s_1}'(\xi, s_2^0, s_3^{i_3}, ..., s_n^{i_n})w_0'(s_2)$$
$$+f_{s_1}'(\xi, s_1^1, s_2^{i_2}, ..., s_n^{i_n})w_1'(s_2))w_{i_3}(s_3))...)w_{i_n}(s_n)$$
$$= \sum_{i_n=0}^{1}(...(\sum_{i_3=0}^{1}(f_{s_1}'(\xi, s_2^0, s_3^{i_3}, ..., s_n^{i_n})\frac{-1}{s_2^1 - s_2^0}$$
$$+f_{s_1}'(\xi, s_1^1, s_2^{i_2}, ..., s_n^{i_n})\frac{1}{s_2^1 - s_2^0})w_{i_3}(s_3))...)w_{i_n}(s_n)$$
$$= \sum_{i_n=0}^{1}(...(\sum_{i_3=0}^{1}(\frac{f_{s_1}'(\xi, s_1^1, s_2^{i_2}, ..., s_n^{i_n}) - f_{s_1}'(\xi, s_2^0, s_3^{i_3}, ..., s_n^{i_n})}{s_2^1 - s_2^0})w_{i_3}(s_3))...)w_{i_n}(s_n)$$
$$= \sum_{i_n=0}^{1}(...(\sum_{i_3=0}^{1}(f_{s_1 s_2}''(\xi, \eta, s_3^{i_3}, ..., s_n^{i_n}))w_{i_3}(s_3))...)w_{i_n}(s_n)$$

with $s_1^0 < \xi < s_1^1$ and $s_2^0 < \eta < s_2^1$.

Since

$$w_0(x) + w_1(x) = 1 \text{ for all } x \in \left[s_i^0, s_i^1\right], i = 1, ..., n$$

we have

$$\sum_{i_3=0}^{1} f''_{s_1s_2}(\xi,\eta,s_3^{i_3},...,s_n^{i_n})w_{i_3}(s_3) \le \max_{s_3\in\left[s_3^0,s_3^1\right]} f''_{s_1s_2}(\xi,\eta,s_3,...,s_n^{i_n}),$$

and so on

$$\sum_{i_n=0}^{1}(...(\sum_{i_3=0}^{1} f''_{s_1s_2}(\xi,\eta,s_3^{i_3},...,s_n^{i_n})w_{i_3}(s_3))...)w_{i_n}(s_n)) \le \max_{(s_1,...,s_n)\in H} f''_{s_1s_2}(s_1,...,s_n).$$

Hence

$$|L_h f''_{s_1s_2}(s_1,...,s_n)| \le \max_{(s_1,...,s_n)\in H} |f''_{s_1s_2}(s_1,...,s_n)|.$$

Likewise, for any pair $i \neq j$ we have

$$\left|(L_h f)''_{s_is_j}\right| \le \max_{(s_1,...,s_n)\in H} |f''_{s_is_j}(s_1,...,s_n)|.$$

Consequently, if

$$K \ge \max_{s\in H}\max_{i=1,...,n}\sum_{j=1,j\neq i}^{n} |f''_{s_is_j}(s)|,$$

then

$$K \ge \max_{s\in H}\max_{i=1,...,n}\sum_{j=1,j\neq i}^{n} |L_h f''_{s_isj}(s_1,...,s_n)|$$

and therefore H_Φ is semi-definite positive. □ □

Notes and Comments

The inequality $K \ge \max_{s\in H}\max_{i=1,...,n}\sum_{j=1,j\neq i}^{n}|f''_{s_is_j}(s)|$ implies that $K \ge \max_{s\in H}|f''_{s_is_i}(s)|$, a sufficient condition for φ to be a underestimator of f.

3. A Branch and Bound Algorithm and its Convergence

We can now describe the branch and bound algorithm for solving (P). Denote by, LB_k, UB_k and s^k respectively the best lower bound, the best upper bound and the best solution to (P) at iteration k.

Algorithm BB:

- **Initialization:** Let ε be a given sufficiently small number. Compute K, a upper bound of $\|H_f\|$ on H. Set $T^0 = H$, solve the convex program

$$\min\left\{\varphi(s) : s \in T^k\right\} \tag{3.1}$$

to obtain an optimal solution $\widetilde{\mathtt{s}}^0$.

Set $UB_0 := \min\{\min_{s\in V(H)} f(s), f(\widetilde{\mathtt{s}}^0)\}$, $LB(T^0) := \varphi(\widetilde{\mathtt{s}}^0)$.

Let s^0 such that $UB_0 = f(s^0)$ and $LB_0 = LB(T^0)$.

If $UB_0 - LB_0 \leq \varepsilon$ then STOP s^0 is an ε - optimal solution,

else set $M \leftarrow \{T^0\}$, $\quad k \leftarrow 1$, and go to iteration k.

- **Iteration k**

k1. Let $T^k = \prod_{k=1}^{n} [a_k, b_k] \in M$ be the rectangle such that $LB_k = LB(T^k)$,and $\widetilde{\mathtt{s}}^k$ be the solution of Problem (3.1).
Set K_k such that $\|H_f(s)\| \leq K_k$, $\forall s \in T^k$.

k2. Bisect T^k into two subrectangles T^{k1}, T^{k2} by w-subdivision procedure via $\widetilde{\mathtt{s}}^k$.

k3. For $i = 1, 2$ do
Solve the convex program (3.1) where k is replaced by k_i to obtain an optimal solution $\widetilde{\mathtt{s}}^{ki}$.

k4. Update the upper bound $UB_k = \min\left\{UB_k, f(\widetilde{\mathtt{s}}^{k1}), f(\widetilde{\mathtt{s}}^{k2})\right\}$. Let s^k be the best current solution, i.e. $f(s^k) = UB_k$.
Set $M \leftarrow M \cup \{T_i^k : LB(T_i^k) < UB^k - \varepsilon, i = 1, 2\} \backslash \{T^k\}$.
Update the lower bound: $LB_k = \min\{LB(T) : T \in M\}$.

k5. If $M = \emptyset$ then STOP, s^k is an optimal solution.
else set $k \leftarrow k + 1$, and return to **k1**.

Theorem 3.1. *Either the algorithm is finite or it generates a bounded sequence* $\{s^k\}$ *which converges to an optimal solution of* (P).

Proof. If the algorithm stops after a finite number of iterations then by the stopping rule the solution is optimal and exact.

If it generates an infinite sequence it suffices to show that $lim_{k\to\infty}(UB_k - LB_k) = 0$.

We have

$$0 \leq UB_k - LB_k = f(s^k) - \varphi(\widetilde{\mathtt{s}}^k) \leq f(\widetilde{\mathtt{s}}^k) - \varphi(\widetilde{\mathtt{s}}^k)$$

$$\leq f(\widetilde{\mathtt{s}}^k) - \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1},...,s_n^{i_n}) w_{i_1}(\widetilde{\mathtt{s}_1}^k))...) w_{i_n}(\widetilde{\mathtt{s}_n}^k)$$

$$+\frac{1}{2}K(\sum_{i=1}^{n}(\widetilde{\mathtt{s}_i}^k - s_i^{0k})(s_i^{1k} - \widetilde{\mathtt{s}_i}^k))$$

$$\leq f(\widetilde{\mathtt{s}}^k) - \min \sum_{i_n=0}^{1}(...(\sum_{i_1=0}^{1} f(s_1^{i_1},...,s_n^{i_n}) w_{i_1}(\widetilde{\mathtt{s}_1}^k))...) w_{i_n}(\widetilde{\mathtt{s}_n}^k)$$

$$+\frac{1}{2}K(\sum_{i=1}^{n}(\widetilde{\mathtt{s}_i}^k - s_i^{0k})(s_i^{1k} - \widetilde{\mathtt{s}_i}^k)) \leq$$

$$f(\widetilde{\mathtt{s}}^k) - \min_{s\in V(T^k)} f(s) + \frac{1}{2}K(\sum_{i=1}^{n}(\widetilde{\mathtt{s}_i}^k - s_i^{0k})(s_i^{1k} - \widetilde{\mathtt{s}_i}^k)) \leq$$

$$\delta||\widetilde{\mathtt{s}}^k - \overline{s}^k|| + \frac{1}{2}K\left(h_{1k}^2 + ... + h_{nk}^2\right) \leq \delta((h_{1k}^2 + ... + h_{nk}^2))^{\frac{1}{2}} + \frac{1}{2}K\left(h_{1k}^2 + ... + h_{nk}^2\right)$$

(i.e. by the continuity of f $(\delta > 0)$ and by using the lengths of the edges of the hyperrectangleT^k).

The exhaustive $w - subdivision$ implies $lim_{k\to\infty} h_{ik} = 0, \forall i = 1, ..., n$.

Hence $Lim_{k\to\infty}(UB_k - LB_k) = 0$ and the theorem is proved. □

4. Numerical Examples

Example 1. (see [3]) $f(s_1, s_2 = -sin(s_1)sin(s_1 s_2), [0, 4] \times [0, 4], K = 12, \varepsilon = 10^{-5}$.

The solution found by our method is $(1.5588, 0.99655)$ and

$$f(1.5588, 0.99655) = -0.99978),$$

with practically the same precision as in [3], the number of function evaluations is 889 which is less than that of [3] which is equal to 1013.

Example 2. (see [3]) $f(s_1, s_2) = (s_1 - 2)^2 + (s_2 - 1)^2 + 0.04(1 - \frac{s_1^2}{4} - s_2^2) + \frac{(s_1 - 2s_2 + 1)^2}{0.2}, [1, 2] \times [1, 2], K = 80, \varepsilon = 10^{-5}$.

The solution found by our method is $(1.7936, 1.3772)$ and $f(1.7936, 1.3772) = 0.16905$, with practically the same precision as in [3], the number of function evaluations is 805 which is significantly less than that of [3] which is equal to 2613.

5. Conclusion

We have proposed in this paper a new convex underestimator for twice continuously differentiable functions and a branch and bound algorithm for minimizing this functions with box constraints. The convergence of the algorithm is shown and numerical examples found in the literature are treated efficiently.

References

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